Math 235 Solutions for Practice Exam I Spring 2005

1. Consider the subset $S = \{x^3 - 2x^2 + 1, 4x^2 - x + 3, 3x - 2\}$ of $\mathsf{P}_3(\mathbb{R})$.
a) Explain how you know that S does not generate $\mathsf{P}_3(\mathbb{R})$.

Since S has 3 vectors and the dimension of $\mathsf{P}_3(\mathbb{R})$ is 4, S cannot generate $\mathsf{P}_3(\mathbb{R})$.

b) Can you add a vector v to S so that $S \cup \{x\}$ is a basis of $\mathsf{P}_3(\mathbb{R})$? Justify and find such a vector if possible.

As long as S is linearly independent we know that S can be extended to a basis. To see S is linearly independent suppose that $a(x^3-2x^2+1)+b(4x^2-x+3)+c(3x-2) = 0$. This clearly implies that $a = 0$ since only one term has an x^3. So now $b(4x^2 - x + 3) + c(3x - 2) = 0$, and again we see that $b = 0$. Clearly c must also be 0.
We see that we can add $v = 1$ as the last vector using a similar arguement.

2. Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$. So β is an ordered basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$ and γ is an ordered basis of $\mathsf{P}_2(\mathbb{R})$.
Define $\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ by $\mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Compute $[\mathsf{T}]_\gamma^\beta$.

We let $B = [\mathsf{T}]_\gamma^\beta$. First we see that $\mathsf{T}(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. So $B_{1\,1} = 0$, $B2\ 1 = 2$, $B3\ 1 = 0$, and $B4\ 1 = 0$.
Next $\mathsf{T}(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. So $B_{1\,2} = 1$, $B2\ 2 = 2$, $B3\ 2 = 0$, and $B4\ 2 = 0$.
Finally $\mathsf{T}(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. So $B_{1\,3} = 0$, $B2\ 3 = 2$, $B3\ 3 = 0$, and $B4\ 3 = 2$.

So in total we get $B = [\mathsf{T}]_\gamma^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

3. Let V and W be vector spaces with Z a subspace of W and $\mathsf{T}:\mathsf{V}\to\mathsf{W}$ a linear transformation. Prove that $\{x\in\mathsf{V}|\mathsf{T}(x)\in\mathsf{Z}\}$ is a subspace of V.

Let $U=\{x\in\mathsf{V}|\mathsf{T}(x)\in\mathsf{Z}\}$. Since $T(0)=0$ and $0\in\mathsf{Z}$, $0\in U$.
Let $a\in F$ and $x,y\in U$. By linearity, $\mathsf{T}(ax+y)=a\mathsf{T}(x)+\mathsf{T}(y)$. Since $\mathsf{T}(x),\mathsf{T}(y)\in\mathsf{Z}$, we see that $\mathsf{T}(ax+y)\in\mathsf{Z}$. So $ax+y\in U$. Therefore U is a subspace of V.

4. Let $\mathsf{T}:\mathsf{P}_2(\mathbb{R})\to\mathsf{P}_3(\mathbb{R})$ be given by $\mathsf{T}(f(x))=x\cdot f(x)+f'(x)$.
a) Show that T is linear.

Let $a\in\mathbb{R}$ and $f(x),g(x)\in\mathsf{P}_2(\mathbb{R})$. $\mathsf{T}(af(x)+g(x))=x\cdot(af(x)+g(x)+(af(x)+g(x))'=ax\cdot f(x)+x\cdot f(x)+af'(x)+g'(x)=a\mathsf{T}(f(x))+\mathsf{T}(g(x))$.

b) Find $\mathsf{N}(\mathsf{T})$ (the nullspace of T).

Suppose that $f(x)\in\mathsf{P}_2(\mathbb{R})$ such that $\mathsf{T}(f(x))=0$. In this case, $x\cdot f(x)-f'(x)=0$ but there is not such non-zero polynomial. So $\mathsf{N}(\mathsf{T})=\{0\}$.
c) Find the rank of T and give a basis for RT.

By the dimension theorem we knwo that $\text{rank}(\mathsf{T})=\dim(\mathsf{P}_2(\mathbb{R}))=3$. We also know that a basis for RT is the image of the standard basis of $\mathsf{P}_2(\mathbb{R})$. So our basis is $\{\mathsf{T}(1),\mathsf{T}(x),\mathsf{T}(x^2)\}=\{x,x^2-1,x^3-2x\}$.

5. Let S_1 and S_2 be subsets of a vector space V.
a) Prove that $\text{span}(S_1\cap S_2)\subseteq\text{span}(S_1)\cap\text{span}(S_2)$.

Let $x\in\text{span}(S_1\cap S_2)$. So there exists vectors $v_1,\ldots,v_n\in S_1\cap S_2$ and $a_1,\ldots a_n\in F$ such that $x=a_1v_1+\cdots a_nv_n$. But since $v_1,\ldots,v_n\in S_1$, we see that $x\in\text{span}(S_1)$. Simalarly since $v_1,\ldots,v_n\in S_2$, we see that $x\in\text{span}(S_2)$. So we have that $x\in\text{span}(S_1)\cap\text{span}(S_2)$.

b) Give an example where $S_1\neq S_2$ but $\text{span}(S_1\cap S_2)=\text{span}(S_1)\cap\text{span}(S_2)$.

There are many such examples but I will give you one. Let $\mathsf{V}=\mathbb{R}^3$, $S_1=\{(1,0,0),(0,1,0),(1,1,0)\}$, and $S_2=\{(0,1,0),(1,2,0),(1,0,0)\}$.

6. Let W_1 and W_2 be finite dimensional subspaces of a vector space V. Recall that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V which contains both W_1 and W_2. Complete the following arguement to show that $\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2) - \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$.
a) Let $k = \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$ and write down a basis β for $\mathsf{W}_1 \cap \mathsf{W}_2$.

Let $\beta = \{v_1, \ldots, v_k\}$.

b) Write down a basis for W_1 where $m = \dim(\mathsf{W}_1)$ containing all of the vectors in β. (Explain why we can do this.)

This basis is $\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}\}$. This can be done since β is linearly independent and any linearly independent subset of a vector space can be expanded to a basis.

c) Write down a basis for W_2 where $n = \dim(\mathsf{W}_1)$ containing all of the vectors in β. (The same reason as part (b) should hold.)

This basis is $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}$.
d) Write down a basis for $\mathsf{W}_1 + \mathsf{W}_2$, prove it is a basis, and verify the equality above.

A basis for $\mathsf{W}_1 + \mathsf{W}_2$ is then $\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}, w_1, \ldots, w_{n-k}\}$, which has size $m+n-k$. To prove it is a basis we need to show that it is linearly independent and that it generates. Notice that $\mathsf{W}_1 + \mathsf{W}_2 = \text{span}(\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}\} + \text{span}(\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}) = \text{span}(\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}, w_1, \ldots, w_{n-k}\})$, so we have a generating set. Suppose that this set was not linearly independent. Then there are elements $a_1, \ldots a_k, b_1, \ldots b_{m-k}, c_1, \ldots, c_{n-k} \in F$ not all zero such that
$a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_{m-k} u_{m-k} + c_1 w_1 + \cdots + c_{n-k} w_{n-k} = 0$. Notice that at least one $c_i \neq 0$ (or else $\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}\}$ is not linearly independent). So now
$a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_{m-k} u_{m-k} = -c_1 w_1 + \cdots - c_{n-k} w_{n-k}$. The right hand side cannot be zero, since then $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}$ would be linearly dependent. Now the right hand side is in W_1 and the left hand side is in W_2; therefore both sides must be in $\mathsf{W}_1 \cap \mathsf{W}_2$. This means that $-c_1 w_1 + \cdots - c_{n-k} w_{n-k} \in \text{span}(\beta)$ which contradicts that $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}$ is linearly independent since some c_i is non-zero. We now

have that $\{v_1, \ldots, v_k, u_1, \ldots, u_{m-k}, w_1, \ldots, w_{n-k}\}$ is linearly independent and so a basis of $\mathsf{W}_1 + \mathsf{W}_2$.