Math 235 Practice Final Spring 2005

PART I

1. Recall that $\mathsf{P}_3(\mathbb{R})$ is the vector space consisting of all polynomials of degree 3 or less. Let W be the subset of $\mathsf{P}_3(\mathbb{R})$ consisting of all $f(x) \in \mathsf{P}_3(\mathbb{R})$ such that $f(0) = f(1) = 0$.
a) Prove that W is a subspace of $\mathsf{P}_3(\mathbb{R})$.

First it is obvious that the zero polynomial is in W.
Let $f(x), g(x) \in \mathsf{W}$ and $c \in \mathbb{R}$. We want to show that $(cf + g)(x) \in \mathsf{W}$.
We evaluate at 0 and 1 recalling that $f(0) = g(0) = f(1) = g(1)$.
$(cf + g)(0) = c(f(0)) + g(0) = c(0) + 0 = 0$ and $(cf + g)(1) = c(f(1)) + g(1) = c(0) + 0 = 0$.
So W is a subspace.

b) Find a basis of W. (Be sure to prove that it is a basis.)

Notice that if $f(0) = f(1) = 0$, then $(x - 0) = x$ and $(x - 1)$ are factors of $f(x)$. Since f must have degree at most 3, $f(x) = ax(x - 1)$ for some $a \in \mathbb{R}$ or $f(x) = ax(x - 1)(x - b) = ax^2(x - 1) - abx(x - 1)$ for some $a, b \in \mathbb{R}$. So a basis for W is $\{x(x - 1), x^2(x - 1)\}$.

2. Let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation on a finite dimensional vector space V.
a) Prove that $\mathsf{R}(\mathsf{T}^2)$ is a subspace of $\mathsf{R}(\mathsf{T})$.

$\mathsf{R}(\mathsf{T}) = \{v \in \mathsf{V} | \exists z \in \mathsf{V} \text{ with } \mathsf{T}(z) = v\}$.
Let $v \in \mathsf{R}(\mathsf{T}^2)$. So there exists a $z \in \mathsf{V}$ with $\mathsf{T}^2(z) = v$. Since $\mathsf{T}(\mathsf{T}(z)) = v$, $v \in \mathsf{R}(\mathsf{T})$. So $\mathsf{R}(\mathsf{T}^2)$ is a subset of $\mathsf{R}(\mathsf{T})$. It is a subspace since both $\mathsf{R}(\mathsf{T}^2)$ and $\mathsf{R}(\mathsf{T})$ are subspaces of V.

b) Prove that if $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$, then $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$.

Since $\text{rank}(\mathsf{T}) = \dim(\mathsf{R}(\mathsf{T}))$ and $\text{rank}(\mathsf{T}^2) = \dim(\mathsf{R}(\mathsf{T}^2))$, $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$, then $\mathsf{R}(\mathsf{T}^2) = \mathsf{R}(\mathsf{T})$ by part (a). Assume that $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^2)$.
Define the transformation $\mathsf{U} : \mathsf{R}(\mathsf{T}) \to \mathsf{V}$ by $U(x) = T(x)$. Apply the dimension theorem to this transformation to see $\dim(\mathsf{R}(\mathsf{T}) = \text{nullity}(\mathsf{U}) + \text{rank}(\mathsf{U})$. Clearly the $\text{rank}(\mathsf{U}) = \dim(\mathsf{R}(\mathsf{U})) = \dim(\mathsf{R}(\mathsf{T}^2))$. So by the dimension theorem $\text{nullity}(\mathsf{U}) = \dim(\mathsf{N}(\mathsf{U})) = 0$ or $\mathsf{N}(\mathsf{U}) = \{0\}$. So for every $0 \neq x \in \mathsf{R}(\mathsf{T})$, $\mathsf{T}(x) \neq 0$. This implies that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$.

3. Let $\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ be the linear transformation defined by $\mathsf{T}(f(x)) = f(x) - xf'(x)$ (where $f'(x)$ is the derivative of $f(x)$).
a) Find a basis for $\mathsf{R}(\mathsf{T})$.

$\mathsf{R}(\mathsf{T}) = \text{span}\{\mathsf{T}(1), \mathsf{T}(x), \mathsf{T}(x^2)\}$
$= \text{span}\{1, x - x, x^2 - 2x^2\} = \text{span}\{1, -x^2\}$.
So a basis for $\mathsf{R}(\mathsf{T})$ is $\{1, -x^2\}$.

b) Is T one-to-one? Explain.

T cannot be one-to-one. By part (a), the rank of T is 2. By the dimension theorem, the nullity of T is 1. This means $\mathsf{N}(\mathsf{T}) \neq \{0\}$ and so T is not one-to-one.

4. Compute the determinants of the two following matrices. You may use any legal method as long as you explain yourself.

$$\text{a) } A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 & 7 & -9 \\ 0 & 0 & 5 & 14 & -5 & 0 \\ 0 & 2 & 2 & -3 & 11 & 1 \\ 1 & 18 & 0 & 6 & 9 & 17 \end{pmatrix} \qquad \text{b) } B = \begin{pmatrix} 3 & 1 & 73 & -11 & 16 & 82 \\ 18 & 3 & 7 & -24 & 23 & 11 \\ 18 & 3 & 7 & -24 & 23 & 11 \\ -5 & -4 & 7 & 26 & 36 & -45 \\ 5 & -6 & 13 & -53 & 17 & -4 \\ 18 & 0 & -3 & 0 & 14 & 75 \end{pmatrix}.$$

For A switch rows one and six, two and five, as well as three and four. Now you have an upper triangular matrix with diagonal entries 1, 2, 5, 3, 2, and 1. So $\det(A) = (-1)^3 2(5)(3)(2) = -60$.
$\det(B) = 0$ since the second and third rows are identical.

5. Find the general solution to the following system of equations or show that no solution exists.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & = & 1 \\ 2x_1 & + & 4x_2 & - & x_3 & = & 0 \\ x_1 & + & x_2 & - & 3x_3 & = & 4 \end{array}$$

The corresponding augmented matrix is $\left(\begin{array}{ccr|c} 1 & 3 & 2 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{array}\right)$.

Begin Gaussian elimination to get to reduced row echelon form.

We get $\left(\begin{array}{rrr|r} 1 & 3 & 2 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & -2 & -5 & 3 \end{array}\right)$ and then $\left(\begin{array}{rrr|r} 1 & 3 & 2 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right)$.

The last row of the last matrix gives the equation $0 = 5$ which of course is never true. So the system is inconsistent.

6. Let $A \in \mathsf{M}_{n\times n}(\mathbb{R})$ with the property that $A^2 = 3A$. Prove that the only numbers that could possibly be eigenvalues of A are 0 and 3.
(Hint: suppose that λ is an eigenvalue corresponding to eigenvector v. Consider A^2v.)

Suppose that λ is an eigenvalue corresponding to eigenvector v. So $Av = \lambda v$. Notice that $A^2v = A(Av) = A(\lambda v) = \lambda^2 v$. So $\lambda^2 v = 3Av = 3\lambda v$. So $\lambda^2 = 3\lambda$; therefore $\lambda = 0$ or $\lambda = 3$. Clearly these are the only choices for eigenvalues of A are 0 and 3.

7. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find an ordered basis β of $\mathbb{R}^3$ such that $[\mathsf{L}_A]_\beta$ is a diagonal matrix. Also write out $[\mathsf{L}_A]_\beta$.

$$f(t) = \det(A - tI) = \det \begin{pmatrix} 1-t & 0 & 0 \\ -1 & 1-t & 1 \\ 1 & 0 & -t \end{pmatrix} = -t(1-t)^2.$$

The eigenvalues of A are 0 and 1.

Now we row reduce $(A - 0I) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

The corresponding eigenspace is then span$\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\}$.

Next we row reduce $(A - 1I) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ and get $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The corresponding eigenspace is then span$\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$.

So let $\beta = \{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$.

Using this β, $[\mathsf{L}_A]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8. Let $A = \begin{pmatrix} 1 & 5 & 4 & 2 & -1 \\ 0 & -3 & -3 & 4 & 12 \\ 0 & 0 & -5 & 2 & 23 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

a) Find the eigenvalues of A.

$A - tI$ is still an upper triagular matrix so the eigenvalues are the diagonal entries 1, -3, -5, 4, and 2.

b) Explain why A is diagonalizable.

Since A is a 5×5 matrix with the 5 distinct eigenvalues it is diagonalizable.

c) True or False: Every upper triangular matrix is diagonalizable. (Explain.)

False. For example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable.

9. Determine an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 1, 0)$, $(1, 1, 1, 1)$, and $(-1, 2, 0, 1)$.

Use Gram-Schmidt to get an orthogonal basis.
I will leave out the details but the basis is: $\{(1, 0, 1, 0), (0, 1, 0, 1), (-1/2, 1/2, 1/2, -1/2)\}$.
Normalizing give you the orthonormal basis
$\{1/\sqrt{2}(1, 0, 1, 0), 1/\sqrt{(}2)(0, 1, 0, 1), (-1/2, 1/2, 1/2, -1/2)\}$.

10. Recall that for $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ for V an inner product space, the adjoint of T is given as $\mathsf{T}^* : \mathsf{V} \to \mathsf{V}$ such that for all $x, y \in \mathsf{V}$, $\langle \mathsf{T}(x), y \rangle = \langle x, \mathsf{T}^*(y) \rangle$. Prove that for such a T and V, $\mathsf{R}(\mathsf{T}^*)^\perp = \mathsf{N}(\mathsf{T})$.

Use the following arguement:
$x \in \mathsf{N}(\mathsf{T}) \Leftrightarrow \mathsf{T}(x) = 0 \Leftrightarrow \langle \mathsf{T}(x), y \rangle = 0 \ \forall y \in \mathsf{V}$
$\Leftrightarrow \langle x, \mathsf{T}^*(y) \rangle = 0 \ \forall y \in \mathsf{V} \Leftrightarrow \langle x, z \rangle = 0 \ \forall z \in \mathsf{R}(\mathsf{T}^*) \Leftrightarrow x \in \mathsf{R}(\mathsf{T}^*)^\perp$