

PART I

1. Recall that $\mathcal{P}_3(\mathbb{R})$ is the vector space consisting of all polynomials of degree 3 or less. Let W be the subset of $\mathcal{P}_3(\mathbb{R})$ consisting of all $f(x) \in \mathcal{P}_3(\mathbb{R})$ such that $f(0) = f(1) = 0$.

- Prove that W is a subspace of $\mathcal{P}_3(\mathbb{R})$.
- Find a basis of W . (Be sure to prove that it is a basis.)

2. Let $T : V \rightarrow V$ be a linear transformation on a finite dimensional vector space V .

- Prove that $R(T^2)$ is a subspace of $R(T)$.
- Prove that if $\text{rank}(T) = \text{rank}(T^2)$, then $R(T) \cap N(T) = \{0\}$.

3. Let $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f(x) - xf'(x)$ (where $f'(x)$ is the derivative of $f(x)$).

- Find a basis for $R(T)$.
- Is T one-to-one? Explain.

4. Compute the determinants of the two following matrices. You may use any legal method as long as you explain yourself.

$$\text{a) } A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 & 7 & -9 \\ 0 & 0 & 5 & 14 & -5 & 0 \\ 0 & 2 & 2 & -3 & 11 & 1 \\ 1 & 18 & 0 & 6 & 9 & 17 \end{pmatrix} \quad \text{b) } B = \begin{pmatrix} 3 & 1 & 73 & -11 & 16 & 82 \\ 18 & 3 & 7 & -24 & 23 & 11 \\ 18 & 3 & 7 & -24 & 23 & 11 \\ -5 & -4 & 7 & 26 & 36 & -45 \\ 5 & -6 & 13 & -53 & 17 & -4 \\ 18 & 0 & -3 & 0 & 14 & 75 \end{pmatrix}.$$

5. Find the general solution to the following system of equations or show that no solution exists.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 1 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 4 \end{aligned}$$

PART II

6. Let $A \in M_{n \times n}(\mathbb{R})$ with the property that $A^2 = 3A$. Prove that the only numbers that could possibly be eigenvalues of A are 0 and 3.

(Hint: suppose that λ is an eigenvalue corresponding to eigenvector v . Consider A^2v .)

7. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find an ordered basis β of \mathbb{R}^3 such that $[L_A]_\beta$ is a diagonal matrix. Also write out $[L_A]_\beta$.

8. Let $A = \begin{pmatrix} 1 & 5 & 4 & 2 & -1 \\ 0 & -3 & -3 & 4 & 12 \\ 0 & 0 & -5 & 2 & 23 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

a) Find the eigenvalues of A .

b) Explain why A is diagonalizable.

c) True or False: Every upper triangular matrix is diagonalizable. (Explain.)

9. Determine an orthonormal basis for the subspace of \mathbb{R}^4 spanned by the vectors $(1, 0, 1, 0)$, $(1, 1, 1, 1)$, and $(-1, 2, 0, 1)$.

10. Recall that for $T : V \rightarrow V$ for V an inner product space, the adjoint of T is given as $T^* : V \rightarrow V$ such that for all $x, y \in V$, $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$. Prove that for such a T and V , $R(T^*)^\perp = N(T)$.