Practice problems for mathematical contests to be discussed in
MTH 190 "Topics in Problem Solving" (Fall 2014)

Combinatorics, number theory, and analytic geometry

Introductory problems

1. For an integer $n \geq 3$ consider the sets

$$S_n = \{(x_1, x_2, \ldots, x_n); \forall i, x_i \in \{0, 1, 2\}\},$$

$$A_n = \{(x_1, x_2, \ldots, x_n) \in S_n; \forall i \leq n-2, |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\},$$

and

$$B_n = \{(x_1, x_2, \ldots, x_n) \in S_n; \forall i \leq n-1, (x_i = x_{i+1} \Rightarrow x_i \neq 0)\}.$$

Prove that $|A_{n+1}| = 3 \cdot |B_n|$. (Note: $|A|$ denotes the number of elements of the set A.)

2. Let V be a convex polygon with n vertices.

(a) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.

(b) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.

3. Find the number of positive integers x satisfying the following two conditions:

i) $x < 10^{2006}$;

ii) $x^2 - x$ is divisible by 10^{2006}.

4. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

5. Let x, y, and z be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that S is divisible by 29^4.

6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$19f(x) - 17f(f(x)) = 2x, \qquad (\forall)x \in \mathbb{Z}.$$

7. Prove that the number

$$2^{2^k - 1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$.

8. Let a and b be given positive coprime integers. Then for every integer n there exist integers x and y such that

$$n = ax + by.$$

Prove that $n = ab$ is the greatest integer for which $xy \leq 0$ in all such representations of n.

9. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

10. Let l be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to l is greater than or equal to two times the distance between X and P. If the distance from P to l is $d > 0$, find the volume of S.

More challenging problems

1. We say a triple (a_1, a_2, a_3) of nonnegative reals is "better" than another triple (b_1, b_2, b_3) if two out of the three following inequalities $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ are satisfied. We call a triple (x, y, z) "special" if x, y, z are nonnegative and $x + y + z = 1$. Find all natural numbers n for which there is a set S of n special triples such that for any given special triple we can find at least one better triple in S.

2. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x)$, $g(x) \in \mathbb{Z}[x]$ be non-constant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

3. In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

4. An alien race has three genders: male, female, and emale. A married triple consists of three persons, one from each gender who all like each other. Any person is allowed to belong to at most one married triple. The feelings are always mutual (i.e., if x likes y then y likes x).

The race wants to colonize a planet and sends n males, n females and n emales. Every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples so that the colony could grow.

a) Prove that if n is even and $k = n/2$ then there might be no married triple.

b) Prove that if $k \geq 3n/4$ then there can be formed n married triples (i.e., everybody is in a triple).

5. Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a real matrix such that $a_{ii} = 0$ for all $1 \leq i \leq n$. Prove that there exists a set $J \subset \{1, 2, \ldots, n\}$ of indices such that

$$\sum_{i \in J,\, j \notin J} a_{ij} + \sum_{i \notin J,\, j \in J} a_{ij} \geq \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}.$$

6. Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k = 2$, each player has two possible moves: either replace the number k by $k + 1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n?

7. Alice has got a circular key ring with n keys, $n \geq 3$. When she takes it out of her pocket, she does not know whether it got rotated and/or flipped. The only way she can distinguish the keys is by colouring them (a colour is assigned to each key). What is the minimum number of colors needed?

8. Let S be a finite set with n elements and F a family of subsets of S with the following property:

$$A \in F, \quad A \subseteq B \subseteq S \Rightarrow B \in F.$$

Prove that the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(t) = \sum_{A \in F} t^{|A|}(1 - t)^{|S \setminus A|}$$

is nondecreasing (Note: $|A|$ denotes the number of elements of A).

9. The numbers of the set $\{1, 2, \ldots, n\}$ are colored with 6 colors. Let

$$S = \{(x, y, z) \in \{1, 2, \ldots, n\}^3 : x + y + z \equiv 0 \,(\mathrm{mod}\, n) \text{ and } x, y, z \text{ have the same color}\}$$

and

$$D = \{(x, y, z) \in \{1, 2, \ldots, n\}^3 : x + y + z \equiv 0 \,(\mathrm{mod}\, n) \text{ and } x, y, z \text{ have three different colors}\}.$$

Prove that

$$|D| \leq 2|S| + \frac{n^2}{2}.$$

(Note: $|A|$ denotes the number of elements of A.)

10. Let k and n be positive integers such that $k \leq n-1$. Let $S = \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of S. Prove that it is possible to color some elements of S using two colors, red and blue, such that the following conditions are satisfied:

(i) Each element of S is either left uncolored or is colored red or blue.

(ii) At least one element of S is colored.

(iii) Each set A_i $(1 \leq i \leq k)$ is either completely uncolored or it contains at least one red and at least one blue element.

11. Let k, m, n be positive integers such that $1 \leq m \leq n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are m-element subsets of S with the following property: for every $1 \leq i \leq k$ there exists a partition $S = S_{1,i} \cup S_{2,i} \cup \ldots \cup S_{m,i}$ (into pairwise disjoint subsets) such that

(i) A_i has precisely one element in common with each member of the above partition.

(ii) Every A_j, $j \neq i$ is disjoint from at least one member of the above partition.

Show that $k \leq \binom{n-1}{m-1}$.

12. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$(x+m)^3 = nx$$

has three distinct integer roots.

13. Let P be a polynomial with integer coefficients and let $a_1 < a_2 < \ldots < a_k$ be integers.

a) Prove that there exists $a \in \mathbb{Z}$ such that $P(a_i)$ divides $P(a)$ for all $1 \leq i \leq k$.

b) Does there exist an $a \in \mathbb{Z}$ such that the product $P(a_1) \cdot P(a_2) \cdot \ldots \cdot P(a_k)$ divides $P(a)$?

14. Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0 \leq k < n/2} \binom{n}{2k+1} 5^k.$$

15. Let p be a prime number and F_p be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in F_p such that

i) the polynomials $x+1$ and $x^{p-2} + x^{p-3} + \ldots + x^2 + 2x + 1$ are in W;

ii) for any polynomials $h_1(x)$ and $h_2(x)$ in W the polynomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$, is also in W.

How many polynomials are there in W?

16. Let a, b be two integers and suppose that n is a positive integer for which the set

$$\mathbb{Z}\backslash\{ax^n + by^n | \, x, y \in \mathbb{Z}\}$$

is finite. Prove that $n = 1$.

17. Let p be a prime number. Call a positive integer n "interesting" if

$$x^n - 1 \, = \, (x^p - x + 1) \, f(x) \, + \, p \, g(x)$$

for some polynomials f and g with integer coefficients.

a) Prove that the number $p^p - 1$ is interesting.

b) For which p is $p^p - 1$ the minimal interesting number?

18. Let f be a polynomial with real coefficients of degree n. Suppose that $\frac{f(x)-f(y)}{x-y}$ is an integer for all $0 \leq x < y \leq n$. Prove that $\frac{f(x)-f(y)}{x-y}$ is an integer for all distinct integers x and y.

19. Construct a set $A \subset [0,1] \times [0,1]$ such that A is dense in $[0,1] \times [0,1]$ and every vertical and every horizontal line intersects A in at most one point.

20. A positive integer m is called "self-descriptive" in base b, where $b \geq 2$ is an integer, if:

i) The representation of m in base b is of the form $(a_0 a_1 \ldots a_{b-1})_b$ (i.e., $m = a_0 b^{b-1} + a_1 b^{b-2} + \ldots + a_{b-2} b + a_{b-1}$, where $0 \leq a_i \leq b-1$ are integers).

ii) a_i is equal to the number of occurences of the number i in the sequence $(a_0 a_1 \ldots a_{b-1})$.

For example, $(1210)_4$ is self-descriptive in base 4, because it has four digits and contains one 0, two 1s, one 2 and no 3s.

a) Find all bases $b \geq 2$ such that no number is self-descriptive in base b.

b) Prove that if x is a self-descriptive number in base b then the last (least signicant) digit of x is 0.

21. For every positive integer n let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called "weird" if $\sigma(n) \geq 2n$ and there exists no representation

$$n = d_1 + d_2 + \ldots + d_r,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n. Prove that there are infinitely many weird numbers.

22. Let $F = A_0 A_1 \ldots A_n$ be a convex polygon in the plane. Define for all $1 \leq k \leq n-1$ the operation f_k which replaces F with a new polygon $f_k(F) = A_0 A_1 \ldots A_{k-1} A'_k A_{k+1} \ldots A_n$

where A'_k is the symmetric of A_k with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ f_3 \circ \ldots \circ f_{n-1})^n(F) = F$.

23. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define $p(0) = 1$. Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

24. Given an integer $n > 1$, let S_n be the group of permutations of the numbers 1, 2, ..., n. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

25. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}x^{2011} + \ldots + a_1x + a_0.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients a_0, ..., a_{2011} and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.

i) Which of the players has a winning strategy if $m(x) = x - 2012$?

ii) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

26. Is the set of positive integers n such that $n! + 1$ divides $(2012\,n)!$ finite or infinite?

27. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

28. Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

29. Suppose that $v_1, \dots, v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unit vector u such that, for all $1 \le i \le d$,

$$|u \cdot v_i| \le 1/\sqrt{d}.$$

(Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.)

30. Does there exist an infinite set M consisting of positive integers such that for any a, $b \in M$, with $a < b$, the sum $a + b$ is square-free?

(A positive integer is called square-free if no perfect square greater than 1 divides it.)

31. Let p and q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor} = \begin{cases} 0 & \text{if } pq \text{ is even,} \\ 1 & \text{if } pq \text{ is odd.} \end{cases}$$

(Here $\lfloor x \rfloor$ denotes the integer part of x.)

32. Find all positive integers n for which there exists a positive integer k such that the decimal representation of n^k starts and ends with the same digit.

33. Let S be a finite set of integers. Prove that there exists a number c depending on S such that, for each non-constant polynomial f with integer coefficients, the number of integers k satisfying $f(k) \in S$ does not exceed $\max(\deg f, c)$.

34. Let n and k be positive integers. Evaluate the following sum:

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n + 2k - j}{2k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

35. Let S_n denote the sum of the first n prime numbers. Prove that for any n there exists the square of an integer between S_n and S_{n+1}.

36. An n-dimensional cube is given. Consider all the segments connecting any two different vertices of the cube. How many distinct intersection points do these segments have (excluding the vertices)?

37. Prove that there is no polynomial P with integer coefficients such that $P(\sqrt[3]{5}+\sqrt[3]{25}) = 5 + \sqrt[3]{5}$.

38. We have a deck of $2n$ cards. Each shuffling changes the order from $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ to $a_1, b_1, a_2, b_2, \dots, a_n, b_n$. Determine all even numbers $2n$ such that after shuffling the deck 8 times the original order is restored.

39. Let P_1, P_2, P_3, P_4 be the graphs of four quadratic polynomials drawn in the coordinate plane. Suppose that P_1 is tangent to P_2 at the point q_2, P_2 is tangent to P_3 at the point q_3, P_3 is tangent to P_4 at the point q_4, and P_4 is tangent to P_1 at the point q_1. Assume that all the points q_1, q_2, q_3, q_4 have distinct x-coordinates. Prove that q_1, q_2, q_3, q_4 lie on a graph of an at most quadratic polynomial.

40. Let k be a positive even integer. Show that

$$\sum_{n=0}^{k/2} (-1)^n \binom{k+2}{n} \binom{2(k-n)+1}{k+1} = \frac{(k+1)(k+2)}{2}.$$

41. Given vectors $\bar{a}$, $\bar{b}$, $\bar{c} \in \mathbb{R}^n$, show that

$$\|\bar{a}\|^2 \langle \bar{b}, \bar{c} \rangle^2 + \|\bar{b}\|^2 \langle \bar{a}, \bar{c} \rangle^2 \leq \|\bar{a}\| \|\bar{b}\| \|\bar{c}\|^2 (\|\bar{a}\| \|\bar{b}\| + |\langle \bar{a}, \bar{b} \rangle|),$$

where $\langle \bar{x}, \bar{y} \rangle$ denotes the inner product of vectors $\bar{x}$ and $\bar{y}$ and $\|\bar{x}\|^2 = \langle \bar{x}, \bar{x} \rangle$.

42. Inside a square consider circles such that the sum of their circumferences is twice the perimeter of the square.

a) Find the minimum number of circles having this property.

b) Prove that there exist infinitely many lines which intersect at least 3 of these circles.

43. Let P_0, P_1, P_2, $\ldots$ be a sequence of convex polygons such that, for each $k \geq 0$, the vertices of P_{k+1} are the midpoints of all sides of P_k. Prove that there exists a unique point lying inside all these polygons.

44. Given $a \in (0,1) \cap \mathbb{Q}$, let $a = 0.a_1 a_2 a_3 \ldots$ be its decimal representation. Define

$$f_a(x) = \sum_{n \geq 1} a_n x^n, \qquad x \in (0,1).$$

Prove that f_a is a rational function of the form $f_a = \frac{P}{Q}$, where P and Q are polynomials with integer coefficients.

Conversely, if $a_k \in \{0, 1, 2, \ldots, 9\}$ for all positive integers k, and $f_a(x) = \sum_{n \geq 1} a_n x^n$ for $x \in (0,1)$ is a rational function of the form $f_a = \frac{P}{Q}$, where P and Q are polynomials with integer coefficients, show that $a = 0.a_1 a_2 a_3 \ldots$ is rational.

45. Let $n > 6$ be a *perfect* number, and let $n = p_1^{e_1} \ldots p_k^{e_k}$ be its prime factorization with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number (n is called *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the factors of n).

46. Let $A_1 A_2 \ldots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $1 \leq i \leq$

$3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60°$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3n^2}{2} - 2n + 1$.

47. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e., $d_n(x) \in \{0, 1, \ldots, 9\}$ and $x = \sum_{n\geq 1} d_n(x)10^{n-1}$. Suppose that for some sequence $(a_n)_{n\geq 1}$, there are only finitely many zeros in the sequence $(d_n(a_n))_{n\geq 1}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n\geq 1}$.

48. We say that a subset of $\mathbb{R}^n$ is $k-$*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers k and n, find the minimal number $d(k, n)$ such that every finite k-*generic* set in $\mathbb{R}^n$ contains a k-*generic* subset with at most $d(k, n)$ elements.

49. For every positive integer n, denote by D_n the number of permutations $(x_1, x_2, \ldots, x_n)$ of $(1, 2, \ldots, n)$ such that $x_j \neq j$ for every $1 \leq j \leq n$. For $1 \leq k \leq n/2$, denote by $\Delta(n, k)$ the number of permutations $(x_1, x_2, \ldots, x_n)$ of $(1, 2, \ldots, n)$ such that $x_i = k + i$ for every $1 \leq i \leq k$ and $x_j \neq j$ for every $1 \leq j \leq n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}.$$