Practice problems for mathematical contests to be discussed in
MTH 190 "Topics in Problem Solving" (Fall 2014)

Analysis

Introductory problems

1. Let $f(x) = x^2 + bx + c$, where b and c are real numbers, and let

$$M \; = \; \{x \in \mathbb{R} : |f(x)| < 1\}.$$

Clearly the set M is either empty or consists of disjoint open intervals. Denote the sum of their lengths by $|M|$. Prove that $|M| \leq 2\sqrt{2}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements:

a) If f is continuous and $f(\mathbb{R}) = \mathbb{R}$ then f is monotone.

b) If f is monotone and $f(\mathbb{R}) = \mathbb{R}$ then f is continuous.

c) If f is monotone and f is continuous then $f(\mathbb{R}) = \mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of $c\,f$ using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

4. Suppose that f and g are real-valued functions on the real line and $f(r) \leq g(r)$ for every rational r. Does this imply that $f(x) \leq g(x)$ for every real x if

a) f and g are nondecreasing?

b) f and g are continuous?

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and

$$f''(x) - 5f'(x) + 6f(x) \geq 0, \quad (\forall) x \geq 0.$$

Prove that

$$f(x) \geq 3e^{2x} - 2e^{3x}, \quad (\forall) x \geq 0.$$

6. Let $0 < a < b$. Prove that

$$\int_a^b (x^2 + 1)e^{-x^2}\, dx \; \geq e^{-a^2} - e^{-b^2}.$$

7. (i) A sequence $(x_n)_{n \geq 1}$ of real numbers satisfies

$$x_{n+1} \; = \; x_n \cos x_n, \quad \forall n \geq 1.$$

Does it follow that this sequence converges for all initial values x_1?

(ii) A sequence $(y_n)_{n\geq 1}$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n, \quad \forall n \geq 1.$$

Does it follow that this sequence converges for all initial values y_1?

8. Let $S_0 = \{z \in \mathbb{C} : |z| = 1, z \neq -1\}$ and $f(z) = \mathrm{Im}z/(1 + \mathrm{Re}z)$. Prove that f is a bijection between S_0 and $\mathbb{R}$. Find f^{-1}.

9. Let $\triangle ABC$ be a non-degenerate triangle in the Euclidean plane. Define a sequence $(C_n)_{n\geq 0}$ of points as follows: $C_0 = C$ and C_{n+1} is the center of the incircle of the triangle $\triangle ABC_n$. Find $\lim_{n\to\infty} C_n$.

10. Let E be the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$. Define

$$J(f) = \int_0^1 (1 + x^2)(f'(x))^2\, dx$$

Prove that $\inf_{f\in E} J(f)$ is attained and find its value.

More challenging problems

1. Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that:

$$\left|\int_0^1 f^3(x)\, dx \;-\; f^2(0)\int_0^1 f(x)\, dx\right| \;\leq\; \max_{x\in[0,1]} |f'(x)| \left(\int_0^1 f(x)\, dx\right)^2.$$

2. Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \;\leq\; 1, \quad (\forall) x > 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

3. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that

$$f'''(\xi) \;=\; 3(f(1) - f(-1)) - 6f'(0).$$

4. Find all $r > 0$ such that whenever $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\nabla f(0,0)| = 1$ and

$$|\nabla f(u) - \nabla f(v)| \;\leq |u - v|, \quad (\forall) u, v \in \mathbb{R}^2,$$

then the maximum of f on the disk $\{u \in \mathbb{R}^2 : |u| \leq r\}$ is attained at exactly one point. (Note: $\nabla f(u) = (\partial_1 f(u), \partial_2 f(u))$ is the gradient vector of f at the point u, while for a vector $u = (a, b)$, $|u| = \sqrt{a^2 + b^2}$.)

5. Let $a, b, c, d, e > 0$ be real numbers such that

$$a^2 + b^2 + c^2 = d^2 + e^2, \quad a^4 + b^4 + c^4 = d^4 + e^4.$$

Compare the numbers $a^3 + b^3 + c^3$ and $d^3 + e^3$.

6. Find all sequences $a_0, a_1, \ldots a_n$ of real numbers where $n \geq 1$ and $a_n \neq 0$, for which the following statement is true:

If $f : \mathbb{R} \to \mathbb{R}$ is an n times differentiable function and $x_0 < x_1 < \ldots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \ldots = f(x_n) = 0$, then there exists an $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \ldots + a_n f^{(n)}(h) = 0$$

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.

8. Compare $\tan(\sin x)$ and $\sin(\tan x)$ for all $x \in (0, \frac{\pi}{2})$.

9. How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z)| \leq 2$ for any complex number z satisfying $|z| = 1$?

10. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$. (Note: A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \leq g(y)$ for all $x \leq y$.)

11. Let $f \neq 0$ be a polynomial with real coefficients. Dene the sequence $(f_n)_{n \geq 0}$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number N such that for every $n \geq N$, all roots of f_n are real.

12. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y with $x - y$ rational.

13. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \ldots \geq c_n \geq 0$ is a sequence of real numbers which is convex (i.e., $2c_k \leq c_{k-1} + c_{k+1}$ for every $1 \leq k \leq n - 1$), and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \ldots + c_n a_n z^n.$$

Prove that:

$$\max_{|z| \leq 1} |q(z)| \leq \max_{|z| \leq 1} |p(z)|.$$

14. Compute the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}$$

15. Define the sequence $(x_n)_{n\geq 1}$ by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Find

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot \ldots \cdot x_n}{x_{n+1}}$$

16. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.

17. Let a_0, a_1, ..., a_n, be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $0 \leq k \leq n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right)\cdots\left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_n}\right)$$

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow* point if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that $f(x) \leq f(b)$ for all $a < x < b$ and $f(a) = f(b)$.

19. Let $(a_n)_n \subset (1/2, 1)$ and define the sequence $(x_n)_{n\geq 0}$ by

$$x_0 = 0, \quad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n}, \ (\forall) n \geq 0.$$

Is this sequence convergent? If yes find the limit.

20. Calculate

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \ln\left(1 + \frac{1}{2n}\right) \ln\left(1 + \frac{1}{2n+1}\right).$$

21. Let $(x_n)_{n\geq 2}$ be a sequence of real numbers such that $x_2 > 0$ and

$$x_{n+1} = -1 + \sqrt[n]{1 + nx_n}, \quad (\forall) n \geq 2.$$

Find $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} nx_n$.

22. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \left(\frac{(2n+1)!}{(n!)^2} \right)^2 \int_0^1 \int_0^1 (xy(1-x)(1-y))^n \, f(x,y)\, dx\, dy.$$

23. Given real numbers $0 = x_1 < x_2 < \ldots < x_{2n} < x_{2n+1} = 1$ such that $x_{i+1} - x_i \leq h$ for all $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} \;<\; \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \;<\; \frac{1+h}{2}.$$

24. Suppose that $(a_n)_{n\geq 1}$ is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Prove that the sequence

$$b_n \;=\; \frac{\sum_{i=1}^{n} a_i}{n}$$

is convergent and find its limit.

25. For a function $f : [0,1] \to \mathbb{R}$, the secant of f at a and $b \in [0,1]$, $a < b$, is the line in $\mathbb{R}^2$ passing through the points $(a, f(a))$ and $(b, f(b))$. A function is said to intersect its secant at a and b if there exists a point $c \in (a,b)$ such that $(c, f(c))$ lies on the secant of f at a and b.

i) Find the set F of all continuous functions f such that for any a and $b \in [0,1]$, $a < b$, the function f intersects its secant at a and b.

ii) Does there exist a continuous function $f \notin F$ such that for any rational a and $b \in [0,1]$, $a < b$, the function f intersects its secant at a and b?

26. Let $f : [0,\infty) \to \mathbb{R}$ be a a strictly convex continuous function such that

$$\lim_{x\to\infty} \frac{f(x)}{x} \;=\; \infty.$$

Prove that the improper integral $\int_0^\infty \sin(f(x))\, dx$ is convergent but not absolutely convergent.

27. A function $f : [0,\infty) \to \mathbb{R}\backslash\{0\}$ is called slowly changing if for any $t > 1$,

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} \;=\; 1.$$

Is it true that every slowly changing function has, for sufficiently large x, a constant sign (i.e., there exists N such that for every x, $y > N$, we have $f(x)f(y) > 0$.)?

28. Let $f : [0,1] \to [0,\infty)$ be an arbitrary function satisfying

$$\frac{f(x)+f(y)}{2} \le f\left(\frac{x+y}{2}\right) + 1, \quad (\forall) x, y \in [0,1].$$

Prove that

$$\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) \le f(v) + 2, \quad (\forall) 0 \le u < v < w \le 1.$$

29. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that the Lebesgue measure of the set

$$A = \{h \in [0,1] : f(x+h) = f(x) \text{ for some } x \in [0,1]\}$$

is at least $1/2$.

30. Find all functions $f : (0,\infty) \to (0,\infty)$ such that

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x, \quad (\forall) x > 0.$$

31. Find all $c \in \mathbb{R}$ for which there exists an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that, for all positive integers n, we have:

$$f^{(n+1)}(x) > f^{(n)}(x) + c, \quad (\forall) x \in \mathbb{R}.$$

32. Find all continuously differentiable functions $f : [0,1] \to (0,\infty)$ such that $\frac{f(1)}{f(0)} = e$ and

$$\int_0^1 \frac{1}{f(x)^2} + f'(x)^2 \, dx \le 2.$$

33. We consider the following game for one person. The aim of the player is to reach a fixed capital $C > 2$. The player begins with capital $0 < x_0 < C$. In each turn let x be the player's current capital. Define

$$s(x) = \begin{cases} x & \text{if } x < 1 \\ C - x & \text{if } C - x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then a fair coin is tossed and the player's capital either increases or decreases by $s(x)$, each with probability $1/2$. Find the probability that in a finite number of turns the player wins by reaching the capital C.

34. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n\geq 1}$ covers the set of positive integers if for any positive integer m there exists a positive integer k such that

$$\sum_{n=1}^{\infty} a_n^k = m.$$

a) Does there exist a sequence of real positive numbers which covers the set of positive integers?

b) Does there exist a sequence of real numbers which covers the set of positive integers?

35. i) Is it true that for every bijection $f:\mathbb{N}\to\mathbb{N}$ the series

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)}$$

is convergent?

ii) Prove that there exists a bijection $f:\mathbb{N}\to\mathbb{N}$ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$$

is convergent. (Note $\mathbb{N}$ is the set of positive integers.)

36. Prove that there exist positive constants c_1 and c_2 with the following properties:

a) For all real $k>1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\cos kx\,dx\right| < \frac{c_1}{k^{3/2}}.$$

b) For all real $k>1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\sin kx\,dx\right| > \frac{c_2}{k}.$$

37. Prove or disprove that if a real sequence $(a_n)_n$ satisfies $a_{n+1}-a_n\to 0$ and $a_{2n}-2a_n\to 0$ as $n\to\infty$, then $a_n\to 0$.

38. Let $f:[0,1]\to\mathbb{R}$ be a function satisfying

$$|f(x)-f(y)| \leq |x-y|, \quad (\forall)x,y\in[0,1].$$

Show that for every $\epsilon>0$ there exists a countable family of rectangles $(R_n)_n$ of dimensions $a_n\times b_n$, $a_n\leq b_n$, respectively, in the plane such that

$$\{(x,f(x)) : x\in[0,1]\} \subset \bigcup_n R_n \quad \text{and} \quad \sum_n a_n < \epsilon.$$

(The edges of the rectangles are not necessarily parallel to the coordinate axes.)

39. Let $(a_n)_{n\geq 1}$ be an unbounded and strictly increasing sequence of positive reals such that the arithmetic mean of any four consecutive terms a_n, a_{n+1}, a_{n+2}, a_{n+3} belongs to the same sequence. Prove that the sequence $(a_{n+1}/a_n)_n$ converges and find all possible values of its limit.

40. Prove that

$$\sum_{n=0}^{\infty} x^n \frac{1+x^{2n+2}}{(1-x^{2n+2})^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(1-x^{n+1})^2}$$

for all $-1 < x < 1$.

41. Let k be a positive integer. Compute

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \ldots n_k (n_1 + \ldots + n_k + 1)}.$$

42. Let p and q be complex polynomials with $\deg p > \deg q$ and let $f(z) = \frac{p(z)}{q(z)}$. Suppose that all roots of p lie inside the unit circle $|z| = 1$ and that all roots of q lie outside the unit circle. Prove that:

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|.$$

43. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies

$$f'(t) > f(f(t)), \qquad (\forall)\, t \in \mathbb{R}.$$

Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

44. Define the sequence $(a_n)_{n\geq 0}$ inductively by $a_0 = 1$, $a_1 = 1/2$, and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n}, \qquad (\forall)\, n \geq 1.$$

Show that the series

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$$

converges and determine its value.

45. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

46. Let $n \geq 3$ and let $x_1, \ldots, x_n$ be nonnegative real numbers. Define $A = \sum_{i=1}^n x_i$, $B = \sum_{i=1}^n x_i^2$, and $C = \sum_{i=1}^n x_i^3$. Prove that

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC.$$

47. Does there exist a sequence $(a_n)_n$ of complex numbers such that for every positive integer p we have that $\sum_n a_n^p$ converges if and only if p is not a prime?

48. Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

49. Let $f:[0,1]\to[0,1]$ be a differentiable function such that $|f'(x)|\neq 1$ for all $x\in[0,1]$. Prove that there exist unique points α, $\beta\in[0,1]$ such that $f(\alpha)=\alpha$ and $f(\beta)=1-\beta$.

50. Determine the smallest real number C such that the inequality

$$\frac{x}{(x+1)\sqrt{yz}}+\frac{y}{(y+1)\sqrt{zx}}+\frac{z}{(z+1)\sqrt{xy}}\leq C$$

holds for all positive real numbers x, y and z with

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1.$$

51. Let $f:[1,\infty)\to(0,\infty)$ be a non-increasing function such that

$$\limsup_{n\to\infty}\frac{f(2^{n+1})}{f(2^n)}<\frac{1}{2}.$$

Prove that $\int_1^\infty f(x)\,dx<\infty$.

52. Let a, b, c, x, y, z, t be positive real numbers with $1\leq x,\ y,\ z\leq 4$. Prove that

$$\frac{x}{(2a)^t}+\frac{y}{(2b)^t}+\frac{z}{(2c)^t}\geq\frac{y+z-x}{(b+c)^t}+\frac{z+x-y}{(c+a)^t}+\frac{x+y-z}{(a+b)^t}.$$

53. Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable function with $|f(x)|\leq M$ and $f(x)f'(x)\geq\cos x$ for all $x\in[0,\infty)$, where $M>0$. Prove that $f(x)$ does not have a limit as $x\to\infty$.

54. Let $\mathcal{F}$ be the set of all continuous functions $f:[0,1]\to\mathbb{R}$ with the property

$$\left|\int_0^x\frac{f(t)}{\sqrt{x-t}}\,dt\right|\leq 1,\qquad(\forall)\,x\in(0,1].$$

Compute $\sup_{f\in\mathcal{F}}\left|\int_0^1 f(x)\,dx\right|$.

55. Find all complex numbers z such that $|z^3+2-2i|+z\bar{z}|z|=2\sqrt{2}$. ($\bar{z}$ is the conjugate of z.)

56. Let $n\geq 2$ be an integer and let $x>0$ be a real number. Prove that

$$\left(1-\sqrt{\tanh x}\right)^n+\sqrt{\tanh(nx)}<1.$$

Recall that $\tanh t=\frac{e^{2t}-1}{e^{2t}+1}$.

57. Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function. Assume that

$$\lim_{x\to\infty}f(x)+\frac{f'(x)}{x}=0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

58. Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $\int_a^b f(t)\,dt = 0$. Show that

$$\int_a^b \int_a^b f(x)\,f(y)\,\ln(x+y)\,dx\,dy \,\le\, 0.$$

59. Let n be a nonzero natural number and $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ be a function such that $f(2014) = 1 - f(2013)$. Let $x_1, x_2, x_3, \ldots, x_n$ be real numbers not equal to each other. If

$$\begin{pmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \ldots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \ldots & 1+f(x_n) \end{pmatrix}$$

is singular, show that f is not continous.

60. Consider the sequence $(x_n)_n$ given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \qquad n \ge 1.$$

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{x_n^2 - 1}$$

is convergent and find its sum.

61. i) Show that

$$\lim_{n\to\infty} n \int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,dx \,=\, \frac{\pi}{2}.$$

ii) Find the limit

$$\lim_{n\to\infty} n \left(n \int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,dx \,-\, \frac{\pi}{2} \right).$$

62. Find all continuous functions $f : [1, 8] \to \mathbb{R}$, such that

$$\int_1^2 f^2(t^3)\,dt \,+\, 2\int_1^2 f(t^3)\,dt \,=\, \frac{2}{3}\int_1^8 f(t)\,dt \,-\, \int_1^2 (t^2-1)^2\,dt.$$

63. Find the maximum value of

$$\int_0^1 |f'(x)|^2\,|f(x)|\,\frac{1}{\sqrt{x}}\,dx$$

over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and

$$\int_0^1 |f'(x)|^2\,dx \,\le\, 1.$$

64. a) Compute

$$\lim_{n\to\infty} n \int_0^1 \left(\frac{1-x}{1+x}\right)^n dx.$$

b) Let $k \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} n^{k+1} \int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k\, dx.$$

65. Let $a_n > 0$, $n \geq 1$. Consider the sequence of right triangles $\triangle A_0A_1A_2$, $\triangle A_0A_2A_3$, $\ldots$, $\triangle A_0A_{n-1}A_n$, $\ldots$, such that:

for every $n \geq 2$, the hypotenuse A_0A_n of $\triangle A_0A_{n-1}A_n$ is a leg in $\triangle A_0A_nA_{n+1}$ with right angle $\angle A_0A_nA_{n+1}$, and the vertices A_{n-1} and A_{n+1} lie on the opposite sides of the straight line A_0A_n. Moreover $|A_{n-1}A_n| = a_n$ for every $n \geq 1$.

Is it possible for the set of points $\{A_n | n \geq 0\}$ to be unbounded but the series

$$\sum_{n\geq 2} m(\angle A_{n-1}A_0A_n)$$

to be convergent? Here $m(\angle ABC)$ denotes the measure of $\angle ABC$.

66. For a given integer $n \geq 1$, let $f : [0,1] \to \mathbb{R}$ be a nondecreasing function. Prove that

$$\int_0^1 f(x)\, dx \;\leq\; (n+1) \int_0^1 x^n f(x)\, dx.$$

Find all non-decreasing continuous functions for which equality holds.

67. Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable increasing function. Define the sequences given by

$$L_n \;=\; \sum_{k=0}^{n-1} f(k/n), \qquad U_n \;=\; \sum_{k=1}^{n} f(k/n), \qquad n \geq 1.$$

The interval $[L_n, U_n]$ is divided into three equal segments. Prove that, for large enough n, the number $I = \int_0^1 f(x)\, dx$ belongs to the middle one of these three segments.

68. Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function. Define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) \;=\; \int_0^x f_{n-1}(t)\, dt, \qquad (\forall)\, n \geq 1.$$

Prove that the series $\sum_{n\geq 1} f_n(x)$ is convergent for every $x \in [0,1]$ and find an explicit formula for its sum.

69. a) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n x^k\, dx,$$

where k is a nonnegative integer.

b) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x)\,dx,$$

where $f:[0,1]\to\mathbb{R}$ is a continuous function.

70. Let $f:[1,\infty)\to(0,\infty)$ be a continuous function. Assume that, for every $a>0$, the equation $f(x)=ax$ has at least one solution in the interval $[1,\infty)$.

a) Prove that, for every $a>0$, the equation $f(x)=ax$ has infinitely many solutions.

b) Give an example of a strictly increasing continuous function f with these properties.

71. Let n be a positive integer and $f:[0,1]\to\mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,dx \;=\; 1, \qquad (\forall)\, 0\le k\le n-1.$$

Prove that

$$\int_0^1 f^2(x)\,dx \;\ge\; n^2.$$

72. For $x\in\mathbb{R}$, $y\ge 0$, and $n\in\mathbb{Z}$, denote by $w_n(x,y)\in[0,\pi)$ the angle in radians formed by the segments joining the point $(x,1)\in\mathbb{R}^2$ with the points $(n,0)$ and $(n+y,0)$, respectively.

a) Show that, for every $(x,y)\in\mathbb{R}\times[0,\infty)$, the series $\sum_{n=-\infty}^{\infty} w_n(x,y)$ converges. Moreover, if we set $w(x,y)=\sum_{n=-\infty}^{\infty} w_n(x,y)$, prove that

$$w(x,y) \;\le\; (\lfloor y\rfloor+1)\pi,$$

where $\lfloor y\rfloor$ is the floor function computed at y.

b) Prove that for every $\epsilon>0$ there exists $\delta>0$ such that

$$w(x,y) \;<\; \epsilon, \qquad (\forall)\, x\in\mathbb{R}, 0<y<\delta.$$

c) Prove that the function $w:\mathbb{R}\times[0,\infty)\to[0,\infty)$ defined in a) is continuous.

73. Consider the following sequence

$$(a_n)_{n=1}^{\infty} \;=\; (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1,\ldots).$$

Find all pairs (α,β) of positive real numbers such that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n^{\alpha}} \;=\; \beta.$$

74. Let $f(x)=\frac{\sin x}{x}$, for $x>0$, and let n be a positive integer. Prove that $|f^{(n)}(x)|<\frac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.