Practice problems for mathematical contests to be discussed in

MTH 190 "Topics in Problem Solving" (Fall 2014)

Algebra

Introductory problems

1. Let A be the $n \times n$ matrix, whose $(i,j)^{\text{th}}$ entry is $i+j$ for all i, $j = 1, 2, \ldots, n$. What is the rank of A?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every integer $n \geq 2$. Does it follow that f is a polynomial?

3. Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

4. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

5. Let A, B and C be real square matrices of the same size, and assume that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

6. For an arbitrary square matrix M, define

$$\exp(M) = I + \frac{M}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots.$$

Construct 2×2 matrices A and B such that $\exp(A+B) \neq \exp(A)\exp(B)$.

7. Can the set of positive rationals be split into two nonempty disjoint subsets Q_1 and Q_2, such that both are closed under addition (i.e., $p + q \in Q_k$ for every p, $q \in Q_k$, $k = 1$, 2)? Can it be done when addition is exchanged for multiplication (i.e., $p \cdot q \in Q_k$ for every p, $q \in Q_k$, $k = 1$, 2)?

8. Find all complex roots (with multiplicities) of the polynomial

$$P(x) = \sum_{n=1}^{2008} (1004 - |1004 - n|)\, x^n.$$

9. Let A and B be two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.

10. (a) Is there a polynomial $P(x)$ with real coefficients such that

$$P\left(\frac{1}{k}\right) = \frac{k+2}{k},$$

for all positive integers k?

(b) Is there a polynomial $P(x)$ with real coefficients such that

$$P\left(\frac{1}{k}\right) = \frac{1}{2k+1},$$

for all positive integers k?

More challenging problems

1. Find all polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ $(a_n \neq 0)$ satisfying the following two conditions:

i) $(a_0, a_1, \ldots, a_n)$ is a permutation of the numbers $(0, 1, \ldots, n)$;

ii) all roots of $P(x)$ are rational numbers.

2. Given a group G, denote by $G(m)$ the subgroup generated by the m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that $G((m,n))$ is also commutative ((m,n) denotes the greatest common factor of m and n.).

3. In the linear space of all real $n \times n$ matrices, nd the maximum possible dimension of a linear subspace V such that

$$\operatorname{tr}(XY) = 0, \quad (\forall) X, Y \in V.$$

(Note: tr denotes the trace of a matrix, which the sum of its diagonal entries.)

4. Prove that if p and q are rational numbers and $r = p + q\sqrt{7}$, then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with integer entries and with $ad - bc = 1$ such that

$$\frac{ar+b}{cr+d} = r.$$

5. Let A be an $n \times n$-matrix with integer entries and $b_1, \ldots, b_k$ be integers satisfying $\det A = b_1 \cdot \ldots \cdot b_k$. Prove that there exist $n \times n$-matrices $B_1, \ldots, B_k$ with integer entries such that $A = B_1 \cdot \ldots \cdot B_k$ and $\det B_i = b_i$ for all $1 \leq i \leq k$.

6. Let f be a rational function (i.e., the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.

7. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \ldots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \leq i, j \leq n+1$. Prove that $v_1, v_2, \ldots, v_{n+1}$ are linearly dependent over the rationals.

8. Let $A_i, B_i, S_i (i = 1, 2, 3)$ be invertible real 2×2 matrices such that:

(1) not all A_i's have a common real eigenvector;

(2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;

(3) $A_1 A_2 A_3 = B_1 B_2 B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.

9. Call a polynomial $P(x_1, \ldots, x_k)$ "good" if there exist 2×2 real matrices $A_1, \ldots, A_k$ such that

$$P(x_1, \ldots, x_k) = \det \left(\sum_{i=1}^{k} x_i A_i \right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are "good". (Note: A polynomial is homogeneous if each term has the same total degree.)

10. Let G be a finite group. For arbitrary sets $U, V, W \subset G$, denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ for which xyz is the unity. Suppose that G is partitioned into three sets A, B, and C (i.e., sets A, B, and C are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

11. Let n be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum_{i=1}^{n} f(k + a_i l) = 0$ whenever k and l are integers and $l \neq 0$. Prove that $f = 0$.

12. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1,\ldots,n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Find $\det A$.

13. For each positive integer k, nd the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \ldots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \ldots = A_k^2 = 0$;

(2) $A_i A_j = A_j A_i$ for all $1 \le i, j \le k$;

(3) $A_1 A_2 \ldots A_k \neq 0$.

14. Denote by V the real vector space of all real polynomials in one variable, and let $P : V \to \mathbb{R}$ be a linear map. Suppose that for all f, $g \in V$ with $P(fg) = 0$, we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x and c such that $P(f) = cf(x)$ for all $f \in V$.

15. Does there exist a finite group G with a normal subgroup H such that $|\mathrm{Aut}\, H| > |\mathrm{Aut}\, G|$? (Note: $|\mathrm{Aut}\, K|$ represents the number of automorphisms of group K.)

16. For a permutation $\sigma = (i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ define

$$D(\sigma) = \sum_{k=1}^{n} |i_k - k|.$$

Let $Q(n, d)$ be the number of permutations σ of $(1, 2, \ldots, n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geq 2n$.

17. Let n be a positive integer, and consider the matrix $A = (a_{ij})_{i,j=1,\ldots,n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

18. Let H be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in H such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in H$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : x \in S \right\}$$

is an orthonormal system of vectors in H.

19. Let n be a positive integer. An n-simplex in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its vertices, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \ldots + v(S_n)C(S_n) = v(S)C(S).$$

20. Let A, $B \in M_n(\mathbb{C})$ be two $n \times n$ matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

21. Let M be the vector space of $m \times p$ real matrices. For a vector subspace $S \subseteq M$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S .

Say that a vector subspace $T \subseteq M$ is a covering matrix space if

$$\bigcup_{A \in T, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a T is minimal if it does not contain a proper vector subspace $S \subset T$ which is also a covering matrix space.

(a) Let T be a minimal covering matrix space and let $n = \dim T$. Prove that

$$\delta(T) \leq \binom{n}{2}.$$

(b) Prove that for every positive integer n we can find m and p, and a minimal covering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$.

22. Denote by S_n the group of permutations of the sequence $(1, 2, \ldots, n)$. Suppose that G is a subgroup of S_n, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $1 \leq k \leq n$ for which $\pi(k) = k$ (Here e is the unit element in the group S_n.). Show that this k is the same for all $\pi \in G \setminus \{e\}$.

23. Let A be a symmetric $m \times m$ matrix over the two-element field, all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

24. Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0,$$

for every prime number p and every real number y.

25. Does there exist a real 3×3 matrix A such that $\operatorname{tr}(A) = 0$ and $A^2 + A^t = I$? (Note: $\operatorname{tr}(A)$ denotes the trace of A, A^t the transpose of A, and I is the identity matrix.)

26. Let $A_1, A_2, \ldots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}|}$$

Prove that f is nondecreasing on $[0, 1]$. (Note: $|A|$ denotes the number of elements in A.)

27. Let n be a positive integer and let V be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \ldots, v_{4n-1} \in V$, there exists a sequence $1 \leq i_1 < \ldots < i_{2n} \leq 4n-1$ of indices such that $v_{i_1} + \ldots + v_{i_{2n}} = 0$.

28. Let $f : A^3 \to A$ where A is a nonempty set and f satisfies:

(a) for all $x,\ y \in A$, $f(x, y, y) = x = f(y, y, x)$;

(b) for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$,

$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) = f(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3)).$$

Prove that for an arbitrary fixed $a \in A$, the operation $x + y = f(x, a, y)$ is an abelian group addition.

29. Find all reals λ for which there is a nonzero polynomial P with real coefficients such that

$$\frac{P(1) + P(3) + P(5) + \ldots + P(2n-1)}{n} = \lambda P(n)$$

for all positive integers n, and find all such polynomials for $n = 2$.

30. Let R be a finite ring with the following property: for any $a,\ b \in R$ there exists an element $c \in R$ (depending on a and b) such that $a^2 + b^2 = c^2$. Prove that for any $a,\ b,\ d \in R$ there exists $f \in R$ such that $2abd = f^2$. (Note: Here $2abd$ denotes $abd + abd$. The ring R is assumed to be associative, but not necessarily commutative and not necessarily containing a unit.)

31. Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a matrix with nonnegative entries such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n.$$

(a) Prove that $|\det A| \leq 1$.

(b) If $|\det A| = 1$ and $\lambda \in \mathbb{C}$ is an arbitrary eigenvalue of A, show that $|\lambda| = 1$.

(Note: We call $\lambda \in \mathbb{C}$ an eigenvalue of A if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.)

32. a) Let u and v be two nilpotent elements in a commutative ring (with or without unity). Prove that $u + v$ is also nilpotent. (Note: An element u is called nilpotent if there exists a positive integer n for which $u^n = 0$.)

(b) Show an example of a (non-commutative) ring R and nilpotent elements $u,\ v \in R$ such that $u + v$ is not nilpotent.

33. Let $(G, \cdot)$ be a finite group of order n. Show that each element of G is a square if and only if n is odd.

34. Let A be a real $n \times n$ matrix satisfying $A + A^t = I$, where A^t denotes the transpose of A and I the $n \times n$ identity matrix. Show that $\det A > 0$.

35. Find all pairs of positive integers (n, m), with $1 < n < m$, such that the numbers 1, $\sqrt[n]{n}$, and $\sqrt[m]{m}$ are linearly dependent over the field of rational numbers $\mathbb{Q}$.

36. Let A be an $n \times n$ square matrix with integer entries. Suppose that

$$p^2 A^{p^2} = q^2 A^{q^2} + r^2 I_n$$

for some positive integers p, q, r where r is odd and $p^2 = q^2 + r^2$. Prove that $|\det A| = 1$. (Note: Here I_n means the $n \times n$ identity matrix.)

37. Let A and B be two $n \times n$ matrices with integer entries such that all of the matrices

$$A, A + B, A + 2B, A + 3B, \ldots, A + (2n)B$$

are invertible and their inverses have integer entries, too. Show that $A + (2n + 1)B$ is also invertible and that its inverse has integer entries.

38. Let a, b, c be elements of finite order in some group. Prove that if $a^{-1}ba = b^2$, $b^{-2}cb^2 = c^2$ and $c^{-3}ac^3 = a^2$, then $a = b = c = e$, where e is the unit element.

39. Let $n > k$ and let $A_1, \ldots, A_k$ be real $n \times n$ matrices of rank $n - 1$. Prove that

$$A_1 \cdot \ldots \cdot A_k \neq 0.$$

40. Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in one variable x. Find all $\mathbb{Q}$-linear maps $\Phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ such that for any irreducible polynomial $p \in \mathbb{Q}[x]$ the polynomial $\Phi(p)$ is also irreducible. (Note: A polynomial $p \in \mathbb{Q}[x]$ is called irreducible if it is non-constant and the equality $p = q_1 q_2$ is impossible for non-constant polynomials q_1, $q_2 \in \mathbb{Q}[x]$.)

41. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

42. Let a be a rational number and let n be a positive integer. Prove that the polynomial $X^{2^n}(X + a)^{2^n} + 1$ is irreducible in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients.

43. Let $n \geq 2$ be an integer. Find all real numbers a such that there exist real numbers x_1, …, x_n satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \ldots = x_{n-1}(1 - x_n) = x_n(1 - x_1) = a.$$

44. Let $c \geq 1$be a real number. Let G be an abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y = \{x + y | \, x \in X, y \in Y\}$ and $|Z|$ denotes the

cardinality of Z. Prove that

$$|\underbrace{A + A + \ldots + A}_{k}| \leq c^k |A|$$

for every positive integer k.

45. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

46. Determine all 2×2 integer matrices A having the following properties:

i) the entries of A are (positive) prime numbers;

ii) there exists a 2×2 integer matrix B such that $A = B^2$ and the determinant of B is the square of a prime number.

47. Let $(A, +, \cdot)$ be a ring with unity, having the following property: for all $x \in A$ either $x^2 = 1$ or $x^n = 0$ for some positive integer n. Show that A is a commutative ring.

48. Let M be the (tridiagonal) 10×10 matrix

$$M = \begin{pmatrix} -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & -1 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Show that M has exactly nine positive real eigenvalues (counted with multiplicities).

49. Let $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ be two real 10×10 matrices such that $a_{ij} = b_{ij} + 1$ for all i, j, and $A^3 = 0$. Prove that $\det B = 0$.

50. Let p be a prime number and let A be a subgroup of the multiplicative group $\mathbb{F}_p^*$ of the finite field $\mathbb{F}_p$ with p elements. Prove that if the order of A is a multiple of 6, then there exist x, y, $z \in A$ satisfying $x + y = z$.

51. Let $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C} \backslash \{0\}$ such that $A - A^* = 2a\, I_n$, where $A^* = (\bar{A})^t$ and $\bar{A}$ is the conjugate of the matrix A.

i) Show that $|\det A| \geq |a|^n$.

ii) Show that if $|\det A| = |a|^n$ then $A = a\, I_n$.

52. Let $A \in M_2(\mathbb{Q})$ such that there exists a positive integer n with $A^n = -I_2$. Prove that either $A^2 = -I_2$ or $A^3 = -I_2$.

53. Let $M,\, N \in M_2(\mathbb{C})$ be two nonzero matrices such that

$$M^2 = N^2 = O_2 \qquad \text{and} \qquad MN + NM = I_2,$$

where O_2 is the 2×2 zero matrix and I_2 is the 2×2 unit matrix. Prove that there exists an invertible matrix $A \in M_2(\mathbb{C})$ such that

$$M = A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A^{-1} \qquad \text{and} \qquad N = A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}A^{-1}.$$

54. Let $A = (a_{ij})_{i,j}$ be the $n\times n$ matrix, where a_{ij} is the remainder of the division of $i^j + j^i$ by 3 for $i,\, j = 1,\, 2,\, \ldots,\, n$. Find the greatest n for which $\det A \neq 0$.

55. Prove that if k is an even positive integer and A is a real symmetric $n\times n$ matrix such that $(\mathrm{Tr}(A^k))^{k+1} = (\mathrm{Tr}(A^{k+1}))^k$, then

$$A^n \;=\; \mathrm{Tr}(A)A^{n-1}.$$

Does the previous assertion also hold for odd positive integers k?

56. Let $A = (a_{ij})_{i,j}$ be a real $n\times n$ matrix such that $A^n \neq 0$ and $a_{ij}a_{ji} \leq 0$ for all $i,\, j$. Prove that there exist two non-real numbers among the eigenvalues of A.

57. Prove that:

a) for every $A \in M_2(\mathbb{R})$ there exist $B,\, C \in M_2(\mathbb{R})$ such that $A = B^2 + C^2$.

b) there do not exist $B,\, C \in M_2(\mathbb{R})$ such that $BC = CB$ and $B^2 + C^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

58. Suppose that A and B are $n\times n$ matrices with integer entries, and $\det B \neq 0$. Prove that there exists m positive integer such that

$$AB^{-1} \;=\; \sum_{k=1}^{m} N_k^{-1},$$

where N_k are $n\times n$ matrices with integer entries for all $k = 1,\, \ldots,\, m$, and $N_i \neq N_j$ for $i \neq j$.

59. Let P be a real polynomial of degree five. Assume that the graph of P has three inflection points lying on a straight line. Calculate the ratios of the areas of the bounded regions between this line and the graph of the polynomial P.

60. Let $SL_2(\mathbb{Z}) = \{A|\, A \text{ is a } 2\times 2 \text{ matrix with integer entries and } \det A = 1\}$.

a) Find an example of matrices $A,\, B,\, C \in SL_2(\mathbb{Z})$ such that $A^2 + B^2 = C^2$.

b) Show that there do not exist matrices $A,\, B,\, C \in SL_2(\mathbb{Z})$ such that $A^4 + B^4 = C^4$.

61. Given the real numbers $a_1,\, a_2,\, \ldots,\, a_n$ and $b_1,\, b_2,\, \ldots,\, b_n$, we define the $n\times n$ matrices $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ by

$$a_{ij} \;=\; a_i - b_j, \qquad b_{ij} \;=\; \begin{cases} 1, & \text{if} \quad a_{ij} \geq 0, \\ 0, & \text{if} \quad a_{ij} < 0, \end{cases}$$

for all i, $j \in \{1, 2, \ldots, n\}$. Consider also the $n \times n$ matrix $C = (c_{ij})_{i,j}$ having only 0s and 1s as its entries such that

$$\sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} c_{ij}, \qquad (\forall)\, 1 \leq j \leq n,$$
$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} c_{ij}, \qquad (\forall)\, 1 \leq i \leq n.$$

Prove that:

i) $\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = 0$ and $B = C$.

ii) B is invertible if and only if there exists two permutations σ and τ of the set $\{1, 2, \ldots, n\}$ such that

$$b_{\tau(1)} \leq a_{\sigma(1)} < b_{\tau(2)} \leq a_{\sigma(2)} < \cdots \leq a_{\sigma(n-1)} < b_{\tau(n)} \leq a_{\sigma(n)}.$$

62. Let $M_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : M_n(\mathbb{R}) \to \{0, 1, \ldots, n\}$ which satisfy

$$f(XY) \leq \min\{f(X), f(Y)\},$$

for all X, $Y \in M_n(\mathbb{R})$.

63. Let $f(x) = \max_{1 \leq i \leq n} |x_i|$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let $A \in M_n(\mathbb{R})$ such that $f(Ax) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that there exists a positive integer m such that $A^m = I_n$.

64. Let F be a field and let $P : F \times F \to F$ be a function such that for every $x_0 \in F$ the function $y \mapsto P(x_0, y)$ is a polynomial in y and for every $y_0 \in F$ the function $x \mapsto P(x, y_0)$ is a polynomial in x. Is it true that P is necessarily a polynomial in x and y, when

a) $F = \mathbb{Q}$, the field of rational numbers?

b) F is a finite field?

65. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\mathrm{tr}(M) = a$ and $\det M = b$.

66. Let n be a positive integer. Show that there are positive real numbers a_0, a_1, …, a_n such that, for each choice of signs, the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \ldots \pm a_1 x \pm a_0$$

has n distinct real roots.

67. Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let λ_1, λ_2, …, λ_n denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.