

8th University of Rochester Math Olympiad

Saturday, February 8, 2014
9:30 AM - 12:30 PM

Show all work (each step and computation) to receive full credit. You may use back pages if necessary. Calculators are not permitted. The olympiad consists of four problems. Please submit your work for all the problems, even if some pages are blank.

No.	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

REGISTRATION NO. : _____

1. Let $\triangle ABC$ be an equilateral triangle whose sides all have length 1. Suppose that D , E , and F are arbitrarily chosen points on sides \overline{AB} , \overline{BC} , and \overline{AC} , respectively. Show that

$$(DE)^2 + (EF)^2 + (FD)^2 \geq \frac{3}{4}.$$

2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 4x^3 - 3x$ for all $x \in \mathbb{R}$. We define the sequence of functions $(f_n)_{n \geq 1}$ as

$$f_1 = f, \quad f_n = f \circ f_{n-1}, \quad (\forall) n \geq 2,$$

where \circ denotes the symbol for composition of functions. Prove that for all $\varepsilon > 0$ there exist $a, b \in (0, 1)$ such that all of the following conditions are satisfied:

- i) $|a - b| < \varepsilon$;
- ii) the sequence $(f_n(a))_n$ is convergent;
- iii) the sequence $(f_n(b))_n$ is divergent.

3. Without using Catalan's conjecture, find with proof all integers n for which

$$n(n+1)(n+2)(n+3)$$

is a perfect square.

4. Let \mathcal{C} be a smooth closed curve in the plane that intersects itself in only finitely many points. (The curve may pass through a single point several times, but is not tangent to itself.) Orient \mathcal{C} by tracing out the entire curve in a particular direction, indicated by the arrows in the diagram. Prove there exists an enclosed region, such as the shaded one, whose boundary is consistently oriented by \mathcal{C} , meaning that every portion of the boundary is traversed in the same direction by \mathcal{C} , either clockwise or counterclockwise.

