3rd University of Rochester Mathematical Olympiad

Saturday, March 28, 2009

Show all work (each step/computation) to receive full credit. You may use back pages if necessary. No calculators. This test contains 4 problems. Make sure it is complete.

No.	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

REGISTRATION NO. : ____

1. Let n and k be positive integers. An n-digit whole number

$$X = \overline{A_1 A_2 \ldots A_n}$$

is called k-transposable if

$$k \cdot X = \overline{A_2 \ldots A_n A_1}$$

Prove that there exists only two 6-digit 3-transposable (i.e., $n = 6$ and $k = 3$) numbers and find them.

2. Let $\triangle ABC$ be an equilateral triangle and consider M a point on the segment BC, which is not its midpoint. The line perpendicular in M on BC intersects the parallel drawn through A to BC in the point O. The circle centered at O, having radius OM, intersects the sides AB and AC in the points P, respectively Q. Prove that A, O, P, and Q lie on the same circle.

3. Prove that for n distinct positive integers, $x_1, \ldots, x_n$, the following inequality holds:

$$(x_1 + x_2 + \ldots + x_n)^2 \ \leq \ x_1^3 + x_2^3 + \ldots + x_n^3$$

4. $\lfloor x \rfloor$, also called the floor function, is defined to be the largest integer not greater than x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor -1.4 \rfloor = -2$, $\lfloor 2 \rfloor = 2$, while $\lfloor -5 \rfloor = -5$. Find, with proof,

$$\left\lfloor \sum_{k=1}^{2^{2009}} k^{\frac{1}{2009}-1} \right\rfloor$$