MATH 165

Midterm Solutions

Tuesday, March 20, 2007

Show all work (each step/computation) to receive full credit. No calculators. The exam contains 8 problems. Make sure it is complete.

No.	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	15	
6	15	
7	15	
8	15	
TOTAL	100	

NAME : ______________________________

SECTION : ______________________________

1. Solve the equation $y' + 2y = 1$, with $y(0) = 0$.

Solution: This is a first order linear non-homogeneous DE, so we will use the integrating factor method to solve this. The integrating factor $u(t) = \exp(\int 2dt) = \exp(2t)$, and $b(t) = 1$. By the general theory of integrating factors, the solution to the DE will be of the form

$$y(t) = (1/u(t)) \int u(t)b(t)dt + C/(u(t))$$

Thus

$$y(t) = \exp(-2t) \int (\exp(2t))dt + C \exp(-2t) = (1/2) + C \exp(-2t)$$

Now using the initial condition $y(0) = 0$ to solve for C, we see that the final answer is

$$y(t) = (1/2) + (-1/2) \exp(-2t) = (1/2)(1 - \exp(-2t))$$

2. Solve the equation $4ty' = y^2 + ty^2$, with $y(1) = 1$.

Solution: Notice that we can factor out y^2 on the right hand side of the equation, and that when we do that it is easy to see that the equation is separable. (NOTE: Although it is also possible to treat this problem as a Bernoulli Equation, that makes the computations in the problem more complicated). Thus

$$4t(dy/dt) = y^2(1+t) \Rightarrow dy/(y^2) = (1+t)/(4t).$$

Now integrating both sides of the above equation, we see that

$$-y^{-1} = (1/4)(t + \ln|t|) + C_0, \text{ and solving for } y, \text{ we obtain}$$

$$y = 1/[(-1/4)(t + \ln|t|) + C_0] = 4/(-t - \ln|t|) + C_1).$$

(where $C_1 = 4C_0$) Using the initial condition $y(1) = 1$ and solving for C_1, we see that $C_1 = 5$, so the answer is

$$y(t) = 4/(5 - t - \ln|t|)$$

3. Solve the initial value problem $y'' + y = 0$, where $y(\pi/2) = 2$ and $y'(\pi/2) = -4$.

Solution: This is a second order homogeneous DE with constant coefficients, so the key to solving this DE is to examine the roots of the corresponding characteristic polynomial $p(r) = r^2 + 1 = 0$. So $r^2 = -1$, thus the roots of $p(r)$ are $0 + 1i$ and $0 - 1i$. So in this case the solution must be of the form

$$y(t) = \exp(at)(c_1 \cos(bt) + c_2 \sin(bt))$$

where a is the real part of the complex roots, and b is the absolute value of the imaginary part of the roots. Thus

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

Notice that

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t)$$

Using the initial conditions $y(\pi/2) = 2$ and $y'(\pi/2) = -4$, we obtain

$$2 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2)$$

and

$$-4 = -c_1 \sin(\pi/2) + c_2 \cos(\pi/2)$$

So $c_1 = 4$ and $c_2 = 2$, hence the final answer is

$$y(t) = 4 \cos(t) + 2 \sin(t)$$

4. A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution: Let $y(t)$ denote the amount of salt in the tank at time t minutes. The strategy here is to come up with a DE that $y(t)$ satisfies, then solve the DE. Notice that

$$y'(t) = 2\gamma - (y(t)/120)(2) = 2\gamma - y/60$$

So the integrating factor $u(t) = \exp(\int 1/60dt) = \exp(t/60)$, and $b(t) = 2\gamma$, and by the theory for integrating factors, the final answer will be of the form

$$y(t) = (1/u(t)) \int u(t)b(t)dt + C/(u(t))$$

Thus

$$y(t) = \exp(-t/60) \int 2\gamma \exp(t/60)dt + C\exp(-t/60) = 120\gamma \exp(-t/60)\exp(t/60) + C\exp(-t/60)$$

Simplifying the above expression, we get

$$y(t) = 120\gamma + C\exp(-t/60)$$

Notice that at time $t = 0$, the water tank does NOT contain any salt, so from this fact we obtain the initial condition $y(0) = 0$. Now we use this initial condition to solve for C. So $C = -120\gamma$, hence the amount of water in the tank at time t is

$$y(t) = 120\gamma - 120\gamma \exp(-t/60)$$

And letting $t \to \infty$, we see that the limiting amount of salt in the tank is 120γ

5. Solve the following system of equations:

$$\begin{cases} x + z = 2 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 4 \end{cases}$$

Solution: In order to solve this system, write down the augmented matrix A' for the corresponding system, and put the augmented matrix into RREF. Thus

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -3 & 3 & 0 \\ 0 & 2 & -4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & -4 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -6 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

So reading off the solution from the RREF of A', the solution set S for the system is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

6. Find out whether the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ are linearly dependent or independent. Justify your answer.

Solution: By definition, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent iff the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has ONLY the trivial solution $x_1 = x_2 = x_3 = 0$. Otherwise, we say that the vectors are linearly dependent. Equivalently (rewriting the above vector equation as a matrix equation), the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent iff the matrix equation $A\mathbf{x} = \mathbf{0}$ has ONLY the trivial solution (where column i of the matrix A is the vector $\mathbf{v}_i$). Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. Equivalently, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent iff EVERY column of A is a pivot column. Otherwise, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. So let's put A into REF:

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that the third column of A is NOT a pivot column. Hence by the arguments above, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly dependent*.

7. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x-y \end{bmatrix}$$

Prove that T is a linear transformation and compute its associated matrix. Is T one-to-one? Is T onto? Justify your answer.

Solution: Part 1: Verify that T is linear. Let

$$\mathbf{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

be arbitrary vectors in $\mathbf{R}^2$, and let c be an arbitrary scalar. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T\left(\begin{bmatrix} x+x' \\ y+y' \end{bmatrix}\right) = \begin{bmatrix} x+x' \\ x+x'+y+y' \\ x+x'-(y+y') \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ x-y \end{bmatrix} + \begin{bmatrix} x' \\ x'+y' \\ x'-y' \end{bmatrix}$$
$$= T(\mathbf{v}_1) + T(\mathbf{v}_2)$$

Also,

$$T(c\mathbf{v}_1) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx \\ cx+cy \\ cx-cy \end{bmatrix} = \begin{bmatrix} cx \\ c(x+y) \\ c(x-y) \end{bmatrix} = c\begin{bmatrix} x \\ x+y \\ x-y \end{bmatrix} = cT(\mathbf{v}_1)$$

So we have verified that T is indeed linear.
Part 2: Come up with the associated matrix A for T. Recall that the i'th column of A is $T(\mathbf{e}_i)$, where $\mathbf{e}_i$ is the i'th column vector of the 2×2 identity matrix $\mathbf{I}$. So

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Part 3: Check to see if T is one-to-one and/or onto. Let's put the associated matrix A into REF in order to answer this question:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Now notice that every column of A is a pivot column, hence $A\mathbf{x} = \mathbf{0}$ has ONLY the trivial solution, hence the $\text{Ker}(T) = \{\mathbf{0}\}$, so T is one-to-one. However, notice that the third row of A does not contain a pivot, hence T is NOT onto.

8. Prove that the matrix

$$\begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$$

is invertible and find its inverse.

Solution: The strategy for this problem is to put $[A|\mathbf{I}]$ into RREF. If A is invertible, then the general theory says that after performing the appropriate row operations, we'll get $[\mathbf{I}|A^{-1}]$. So doing the row reduction:

$$\begin{pmatrix} 1 & 4 & -3 & 1 & 0 & 0 \\ -2 & -7 & 6 & 0 & 1 & 0 \\ 1 & 7 & -2 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -7 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & -20 & -9 & 3 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -7 & -3 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -28 & -13 & 3 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -7 & -3 & 1 \end{pmatrix}$$

So notice that A has row reduced to the 3×3 identity matrix $\mathbf{I}$. Thus, by the Invertible Matrix Theorem, A is invertible, and

$$A^{-1} = \begin{pmatrix} -28 & -13 & 3 \\ 2 & 1 & 0 \\ -7 & -3 & 1 \end{pmatrix}$$