Second-Order Linear, Homogeneous Differential Equations with Constant Coefficient

The general form of a second-order linear, homogeneous differential equations with constant coefficients is

$$ay'' + by' + cy = 0.$$

This says that y, y', and y'' must all have a similar form, just differing by a constant multiple. We know that $y = e^{rx}$ is a function that has this property.

Is $y = e^{rx}$ a solution? We know that, if $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Plugging these into the differential equation $ay'' + by' + cy = 0$ gives

$$\begin{aligned} & ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0 \\ \implies \quad & e^{rx}(ar^2 + br + c) = 0 \end{aligned}$$

Of course since $e^{rx} \neq 0$, we know that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

makes $y = e^{rx}$ a solution.

Definition 1. *This polynomial $p(r) = ar^2 + br + c$ is called the **characteristic polynomial** for $ay'' + by' + cy = 0$.*

Example: Solve the initial value problem $y'' - 11y' + 28y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

The characteristic polynomial is $p(r) = r^2 - 11r + 28$. Setting this equal to 0 and factoring to find the roots gives

$$\begin{aligned} & r^2 - 5r + 6 = 0 \\ \Longrightarrow \quad & (r-4)(r-7) = 0 \\ \Longrightarrow \quad & r = 4 \text{ and } r = 7 \end{aligned}$$

So right away we know two different solutions, $y = e^{4x}$ and $y = e^{7x}$. Then the general solution is any *linear combination* of these

$$y = C_1 e^{4x} + C_2 e^{7x}.$$

Using the initial conditions, we can find the coefficients C_1 and C_2.

$$\begin{aligned} y = C_1 e^{4x} + C_2 e^{7x}, \quad y(0) = 0 \quad &\Longrightarrow \quad C_1 + C_2 = 0 \\ y' = 4C_1 e^{4x} + 7C_2 e^{7x}, \quad y'(0) = 1 \quad &\Longrightarrow \quad 4C_1 + 7C_2 = 1 \end{aligned}$$

To solve this system we row reduce the augmented matrix.

$$\begin{pmatrix} 1 & 1 & \vdots & 0 \\ 4 & 7 & \vdots & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \vdots & -\frac{1}{3} \\ 0 & 1 & \vdots & \frac{1}{3} \end{pmatrix} \Rightarrow C_1 = -\frac{1}{3}, C_2 = \frac{1}{3}$$

Therefore, the particular solution is $y = -\frac{1}{3}e^{4x} + \frac{1}{3}e^{7x}$.

Three Possibilities: There are three possibilities for the roots r_1 and r_2 of the characteristic polynomial $p(r) = ar^2 + br + c$:

(1) distinct real roots

(2) repeated real roots

(3) complex roots.

For case (1), $r_1 \neq r_2$ are two different real numbers, and we handle this just like in the previous example.

For case (2), $r_1 = r_2 = r$ is a real number that occurs as a root twice. In this case the general solution is given by

$$y = C_1 e^{rt} + C_2 t e^{rt}.$$

We proceed with an example to illustrate this fact.

Example: Solve the differential equation $y'' + 6y' + 9y = 0$.

The characteristic polynomial is $p(r) = r^2 + 6r + 9$. We set it equal to 0 and solve to find the roots.

$$r^2 + 6r + 9 = 0$$
$$(r+3)(r+3) = 0$$
$$r = -3,\ r = -3$$

Then the general solution is $y = C_1e^{-3t} + C_2te^{-3t}$.

Now let us take a look at another example. Suppose we want to solve the equation $y'' + y = 0$. We know $y = \sin(t)$ and $y = \cos(t)$ are solutions, but up until now we said solutions have the form $y = e^{rt}$. How can we reconcile these two? The answer is given to us by **Euler's Formula**. This formula relates exponential functions to sin and cos.

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

We use this in case (3) from above. In this case we have two complex roots to the characteristic polynomial

$$r_1 = a + bi \text{ and } r_2 = a - bi.$$

We call a the **real part** of r_1 and b the **imaginary part** of r_1.

One solution: Following all that we have done up to now, we know that if $r_1 = a + bi$ is a root of the characteristic polynomial then one solution to the differential equation is

$$\begin{aligned} y = e^{r_1t} &= e^{(a+bi)t} = e^{at}e^{bti} \\ &= e^{at}(\cos(bt) + i\sin(bt)) \\ &= [e^{at}\cos(bt)] + i[e^{at}\sin(bt)] \end{aligned}$$

This is a *complex* solution to the differential equation, but we want a *real* solution. We can get two real solutions by taking the real and imaginary parts of the complex solution.

$$\begin{aligned} \text{real part } &= e^{at}\cos(bt) \\ \text{imaginary part } &= e^{at}\sin(bt) \end{aligned}$$

We can then get the general solution by taking a linear combination of these two parts $y = C_1e^{at}\cos(bt) + C_2e^{at}\sin(bt)$.

Note: We could have used $r_2 = a - bi$, but this would lead to the same answer since $\sin(-t) = -\sin t$ and $\cos(-t) = \cos t$.

Now let us use this information to solve the equation mentioned above.

Example: Solve the differential equation $y'' + y = 0$.

The characteristic polynomial for $y'' + y = 0$ is $p(r) = r^2 + 1$. Solving for the roots, we get $r = \pm i = 0 \pm 1i$. Therefore, the general solution is

$$y = C_1 e^{0t} \cos(1t) + C_2 e^{0t} \sin(1t) = C_1 \cos t + C_2 \sin t.$$

Example: Solve the equation $y'' + 6y' + 13y = 0$.

The characteristic polynomial is

$$\begin{aligned} p(r) &= r^2 + 6r + 13 = 0 \\ r &= \frac{-6 \pm \sqrt{(6)^2 - 4(13)}}{2} \\ &= -3 \pm 2i \; . \end{aligned}$$

So the general solution is

$$y = C_1 e^{-3t} \cos(2t) + C_2 e^{-3t} \sin(2t) \; .$$