A phase diagram for a stochastic reaction diffusion system.

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January 24, 2009

Abstract

In this paper a stochastic reaction diffusion system is considered, which models the spread of a finite population reacting with a non-renewable resource in the presence of individual based noise. A two-parameter phase diagram is established to describe the large time evolution, distinguishing between certain death or possible life of the population.

KEYWORDS: Stochastic PDE; Dawson-Watanabe process; exit measures; phase diagram; oriented percolation.

1 Introduction

1.1 Statement of results

In this paper we study the stochastic reaction diffusion system on R^d, for $d \leq 3$,

$$\begin{cases} \partial_t u = \Delta u + \beta uv - \gamma u + \sqrt{u}\,\dot{W}, \\ \partial_t v = -uv. \end{cases} \tag{1.1}$$

One interpretation for this equation is that the solution $u \geq 0$ describes the distribution of a population on R^d at times $t \geq 0$ and that $v \geq 0$ is the density of a nutrient gradually used up by the population. The noise W is a space-time white noise, and the multiplicative factor $\sqrt{u}$ models 'individual based' noise. This noise arises in situations where each individual in the population contributes independently to the noise, so that the variance of the noise is proportional to the population density. In dimensions $d \leq 3$, linear scaling reduces the number of possible parameters to two. We have chosen to take the diffusion coefficient, the rate at which the nutrient is used, and the noise coefficient all to be one, leaving two parameters $\beta, \gamma \geq 0$, which we think of as a reaction rate and a death rate.

The main results concern the long time behaviour of solutions. Without noise there are travelling wave solutions and it is expected that sufficient noise can destroy these waves. For our description of the long time behaviour, we take an initial condition $u_0 = \mu$ in $\mathcal{M}(R^d)$, the space of finite measures in R^d, and an initial nutrient level $v_0(x)$ taking a constant value which, by scaling again, we have taken to be 1. We say that *certain death* occurs for the parameters values β, γ if

$$P\left[u_t = 0 \text{ for large } t\right] = 1 \text{ for any initial condition } \mu \in \mathcal{M}(R^d) \text{ and } v_0 = 1.$$

We say that *possible life* occurs for the parameter values β, γ if

$$P\left[u_t \neq 0 \text{ for all } t\right] > 0 \text{ for any non-zero initial condition } \mu \in \mathcal{M}(R^d) \text{ and } v_0 = 1.$$

The main results of the paper describe a phase diagram for the parameter values β, γ at which certain death or possible life occur.

Theorem 1. *Consider solutions to (1.1) with initial condition $v_0 = 1$. Then for any values of the parameters $\beta, \gamma \geq 0$ either possible life or certain death occurs.*

- *When **d=3**, there exists a non-decreasing function $\beta \to \Psi(\beta) \in (0, \beta)$ so that when $0 \leq \gamma < \Psi(\beta)$ possible life occurs and when $\gamma > \Psi(\beta)$ certain death occurs. Moreover*

$$0 < \liminf_{\beta \to 0} \beta^{-2} \Psi(\beta) \leq \limsup_{\beta \to 0} \beta^{-2} \Psi(\beta) < \infty,$$

- *When **d=2** there exists a critical curve $\beta \to \Psi(\beta) \in [0, \beta)$ as in dimension $d = 3$ (but allowing the possibility that $\Psi(\beta) = 0$ for small β).*
- *When **d=1** certain death occurs for any $\gamma, \beta \geq 0$.*

Remarks

1. The estimates in the paper imply certain bounds on the critical curve. In section 6.3 we show that $\Psi(\beta)/\beta \to 1$ as $\beta \to \infty$. This asymptotic is not sharp, and the correct large β asymptotic is, we believe, $\beta - \Psi(\beta) \sim \beta^{-2/(6-d)}$. The remarks in section 6.3 indicate that a certain improved spatial Markov property is required to establish these asymptotics, and the details are left to a subsequent paper. Many other questions remain, such as regularity of Ψ and the behaviour on the critical curve. Most importantly perhaps, we do not know the small β behaviour in $d = 2$, though we conjecture (see section 7.3) that there is certain death for $\gamma = 0$ and small enough β.

2. The possible life behaviour is not an example of global coexistence of two species, for locally either the population u_t becomes extinct or the nutrient levels converge to zero (see Lemma 9). Rather one expects the process to live on a moving front travelling through space, and behind this moving front the nutrient is used up and the process dies away. It is therefore more analogous to the weak survival results for the contact process on a homogeneous tree (see [8] section I.4).

3. A key difficulty in dealing with many reaction diffusion systems is the lack of pathwise comparison results (which were crucial to the arguments for the scalar equation studied in [9]). For example if two solutions $u, \tilde{u}$ satisfy $u_0 \leq \tilde{u}_0$ one should not expect that $u_t \leq \tilde{u}_t$ for $t > 0$. To replace these arguments, the key new method in this paper is the use of comparison results for the total occupation measure $\int_0^\infty u_t dt$ and total exit measure $u_\infty^{\partial D}$ of solutions on a domain D. Some intuition behind this is given in section 1.2.

4. Using scaling (as in Lemma 3) one can investigate the dependence on other parameters. For example the equation

$$\begin{cases} \partial_t u = \Delta u + uv - \gamma_0 u + \sqrt{\sigma u}\, \dot{W}, \\ \partial_t v = -uv, \quad v_0 = 1. \end{cases}$$

can be reduced to the standard form (1.1), by a linear change of variables, with the parameters becoming $\gamma = \gamma_0 \sigma^{-2/(4-d)}$ and $\beta = \sigma^{-2/(4-d)}$. The main theorem shows that possible life occurs, for example when $d = 3$, if $0 \leq \gamma_0 < \Phi(\sigma)$ for some $\Phi : (0, \infty) \to (0, 1)$ and satisfying $\lim_{\sigma \to 0} \Phi(\sigma) = 1$ and $\lim_{\sigma \to \infty} \Phi(\sigma) = 0$ at certain rates. One reason for choosing β, γ as our parameters is that we have natural monotonicity in these parameters which is unclear for other choices - for example we do not know if Φ is decreasing.

5. One can investigate, via phase plane methods, the possibility of travelling waves for the corresponding deterministic system $\partial_t u = \Delta u + uv - \gamma u,\ \partial_t v = -uv$ with $v_0 = 1$. (With no noise term and $\beta > 0$ we can remove one more parameter by linear scaling, leaving only $\gamma \geq 0$.) We require $\gamma \leq 1$, else the death term exceeds the reaction term resulting in death. Fix $k \in R^d$ with $|k| = 1$. Looking for a travelling wave solution of the form $u_t(x) = U((x \cdot k) - ct)$, $v_t(x) = V((x \cdot k) - ct)$ with $c > 0$, we set $W = U'$ to obtain the first order system

$$U' = W, \quad V' = \frac{1}{c} UV, \quad W' = \gamma U - cW - UV.$$

The steady states are $(U, V, W) = (0, a, 0)$ for any a. By scaling we assume the untouched nutrient level is 1 and seek a path in phase space connecting $(0,0,0)$ to $(0,1,0)$, so that u is a travelling 'hump', behind the wave the nutrient is used up and ahead it is untouched. Linearizing around $(0,0,0)$ we find the eigenvalues satisfy $\lambda = 0$ or $\lambda^2 + c\lambda - \gamma = 0$. So for any $\gamma > 0$ we have real eigenvalues $\lambda_1 < 0 = \lambda_2 < \lambda_3$. Linearizing around $(0,1,0)$ we obtain $\lambda = 0$ or $\lambda^2 + c\lambda + (1 - \gamma) = 0$. For $\gamma < 1$ we get $\lambda_1 < \lambda_2 < 0 = \lambda_3$ when $c^2 \geq 4(1 - \gamma)$. Complex roots, which imply the impossibility of a (non-negative) travelling wave, occur when $c^2 < 4(1 - \gamma)$. The equations are three dimensional and degenerate and we have not done a rigorous analysis. Based on the above local picture, the following seems the reasonable best guess: there is a unique path connecting $(0,0,0)$ to $(0,1,0)$ which stays in the good region $U, V > 0$ precisely if $c^2 \geq 4(1 - \gamma)$ and hence a travelling wave for all $\gamma < 1$ and with speeds greater than or equal to $2\sqrt{1 - \gamma}$. The main result supports the conjecture that the deterministic travelling waves are stable to small enough individual based noise noise in dimensions $d = 2, 3$.

1.2 A description of key techniques and the layout of the paper

In this section we explain some of the intuition and tools used in the proofs, emphasising the new ideas.

When $\beta = 0$, solutions u to (1.1) are Dawson-Watanabe processes with underlying Brownian motion and mass annihilation at rate γ. In dimensions $d \geq 2$, Dawson-Watanabe processes are singular measures $(u_t(dx) : t \geq 0)$. However, as shown by Sugitani [13], when $d \leq 3$ the occupation measures $\int_s^t u_r dr$ are absolutely continuous with repsect to Lebesgue measure, with densities $u(s, t, x)$. This will remain true for for solutions to (1.1) and hence the the formal solution $v_t(x) = v_0(x) \exp(-u(0, t, x))$ will make sense as a function. Using this idea, we construct solutions to (1.1) in $d \leq 3$ by a change of measure starting from a Dawson-Watanabe process; a reverse argument yields uniqueness in law of solutions.

The results on possible life use a construction of a supercritical oriented percolation process, coupled to a solution of (1.1), so that if the oriented percolation survives so too does the solution. The use of an embedded oriented percolation to show survival was first developed for interacting particle systems (see Durrett [4] for many examples). Similar methods have been used before by the authors to establish a one-parameter phase transition for a noisy version of the KPP equation [9]. The results on certain death in dimensions $d \geq 2$ are simpler, and the background idea is a comparison with a subcritical branching process (for which simple moment estimates show death).

When possible life occurs, one expects that the process lives on a front propagating with asymptotically linear speed. Noisy oscillations around this linear speed mean that it is not possible to predict exactly where in space, at a given time t, to look for the living part of the solution (that is where it is in the process of exploiting the nutrient). This leads to the wish to apply the percolation comparison in space at random times. To overcome this technical problem, we freeze the process at random stopping times, by means of exit measures. The exit measures can be constructed by solving (1.1) on a domain D with Dirichlet boundary conditions and measuring the total flux of mass out of the boundary of the domain. Suitable comparison results do hold for this model, not for $u_t(dx)$ but for the total exit measures $u_\infty^{\partial D}(dx)$. For example if $u_0(dx)$ and $\tilde{u}_0(dx)$ are supported inside D and $u_0(dx) \leq \tilde{u}_0(dx)$, then one can construct solutions for which $u_\infty^{\partial D}(dx) \leq \tilde{u}_\infty^{\partial D}(dx)$. It turns out that the total exit measures are stochastically monotone both in the initial conditions and in the parameters β, γ. A spatial Markov property, analogous to that found for Dawson-Watanabe processes by Dynkin in [5], shows that exit measures can be constructed iteratively on larger and larger domains, and is used to set up the percolation comparison. It is this spatial Markov property especially, that means the use of the special form of noise $\sqrt{u}\dot{W}$ is needed for our methods.

Intuition as to why comparison results hold for the total exit measures is easiest for particle approximations to (1.1), where population particles may react with nutrient particles leading to death of a nutrient particle and the creation of extra population particles. The total exit measures do not depend on the time at which particles hit the exit of the domain D. It does not affect the final level of the nutrient $v_\infty(x)$ if we change the order that particles pass through position x. For example, we may think of the initial mass $\tilde{u}_0(dx)$ as being composed of blue particles, identical to those from $u_0(dx)$ and some extra red particles represented by $\tilde{u}_0(dx) - u_0(dx)$. We freeze the extra red particles until all the blue particles have died or are frozen on ∂D. Then we may run the red particles through whatever is left of the nutrient, but this can only lead to a larger exit measure. The reader may wish to draw a diagram illustrating the possible interactions for the simplest non-trivial case of two population particles and a single nutrient particle - it was such a diagram that initiated this work.

Layout. In section 2 we state the existence, uniqueness and comparison results for occupation and exit measures. Section 3 contains the change of measure arguments needed for the proof of existence and uniqueness, and the proof of the existence of a critical curve $\Psi(\beta)$. Approximations to solutions, using a discretized nutrient, are given in section 4, together with tightness and convergence results. Comparison results for the approximations are established in section 5 together with the weak convergence arguments allowing us to pass to the continuous limit. However, the arguments for life (in section 6) and death (in section 7) can be read after the statement of the comparison results in section 2.

2 Statements of existence, uniqueness and comparison results

2.1 Definition of solutions

We need to consider solutions to (1.1) on domains D. Throughout the paper our domains are taken to be either (i) connected open sets with smooth boundary; or (ii) open boxes, that is of the form $D = \prod_i (a_i, b_i)$ for $-\infty \leq a_i < b_i \leq \infty$. This restriction allows us to quote estimates on the Green's functions for the domain. Moreover, boxes will be all that we need for the blocking arguments in sections 6 and 7. We will look for solutions with zero Dirichlet boundary conditions, that is where the population is killed on reaching the boundary. This will be encoded by the test functions for the martingale problem below. We let $C_b^k(\overline{D})$ be the space of k-times continuously differentiable functions $\phi : D \to R$ for which ϕ and its derivatives can be extended to be continuous bounded functions on the closure $\overline{D}$. Let $C_0^2(\overline{D})$ be the subset of those $\phi \in C_b^2(\overline{D})$ for which $\phi = 0$ on ∂D. Note that solutions on $D = R^d$ are a special case of the definition below, and in this case $C_0^2(\overline{D})$ reduces to the space of bounded functions with two bounded continuous derivatives.

Notation. We write $\mathcal{M}(D)$ for the space of finite measures on D with the topology of weak convergence. For $\mu \in \mathcal{M}$ and (suitable) measurable functions $f, g : D \to R$ we will write $\mu(f)$ for $\int_D f(x)\mu(dx)$ and $\langle f, g \rangle$ for $\int_D f(x)g(x)dx$.

We look for solutions $(u, v) = (u_t(dx), v_t(x) : t \geq 0, x \in D)$ to (1.1) defined on some filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$. The solutions will be in the sense of martingale problems. The adapted process u will have continuous paths in $\mathcal{M}(D)$. Let $\mathcal{P}$ be the $(\mathcal{F}_t)$ predictable sets in $\Omega \times [0, \infty)$ and $\mathcal{B}(D)$ be the Borel subsets of D. The process v will be predictable and non-increasing, that is the map $(\omega, t, x) \to v_t(x)(\omega)$ will by $\mathcal{P} \times \mathcal{B}(D)$ measurable, and such that the paths $t \to v_t(x) \in [0, 1]$ are non-increasing for all x, almost surely. We say that such a pair (u, v) is a solution to (1.1) on D if for $\phi \in C_0^2(\overline{D})$

$$u_t(\phi) \;=\; u_0(\phi) + \int_0^t u_s\left(\Delta\phi + \beta v_s \phi - \gamma\phi\right) ds + m_t(\phi), \quad P \text{ a.s.} \tag{2.1}$$

$$\langle v_t, \phi \rangle \;=\; \langle v_0, \phi \rangle - \int_0^t u_s(v_s \phi) ds, \quad P \text{ a.s.} \tag{2.2}$$

where $m_t(\phi)$ is a continuous $(\mathcal{F}_t)$ local martingale, with $m_0(\phi) = 0$ P a.s., and with quadratic variation

$$[m(\phi)]_t = \int_0^t u_s(\phi^2) ds, \quad P \text{ a.s.} \tag{2.3}$$

If, in addition, $P[u_0 = \mu, v_0 = f] = 1$, for some measurable $f : D \to [0,1]$ and $\mu \in \mathcal{M}(D)$, we say the solution has initial condition (μ, f).

We will wish to keep track of the flux of mass that exits to the boundary ∂D by time t, which creates the so called exit measure $u_t^{\partial D}(dx)$ on ∂D. For a deterministic equation, this flux would be given by a surface integral $\int_0^t \int_{\partial D} \nabla u \cdot \hat{N} dS \, ds$ for an outward unit normal $\hat{N}$. A simple way to treat this process in the noisy setting, where these derivatives do not exist, is to extend the martingale problem formulation to test functions $\phi \in C_b^2(\overline{D})$. The exit measure will take values in $\mathcal{M}(\partial D)$, the space of finite measures on ∂D with the topology of weak convergence. We look for an adapted $\mathcal{M}(\partial D)$ valued process $t \to u_t^{\partial D}$, with non-decreasing continuous paths, so that $u_0^{\partial D} = 0$ P a.s. and for $\phi \in C_b^2(\overline{D})$

$$u_t(\phi) = u_0(\phi) + \int_0^t u_s \left(\Delta \phi + \beta v_s \phi - \gamma \phi\right) ds - u_t^{\partial D}(\phi) + m_t(\phi), \quad P \text{ a.s.} \tag{2.4}$$

where $m_t(\phi)$ is a continuous $(\mathcal{F}_t)$ local martingale, with $m_0(\phi) = 0$ P a.s., and with quadratic variation as in (2.3).

The following existence and uniqueness result is shown in section 3.2, using a change of measure argument starting with a Dawson-Watanabe process on $\overline{D}$.

Notation. We let $C([0,\infty), E)$ be the space of continuous functions with values in a metric space E, with the topology of uniform convergence on compacts. Both $\mathcal{M}(D)$ and $\mathcal{M}(\partial D)$ are metrizable as Polish spaces. Let Ω_D be the space $C([0,\infty), \mathcal{M}(D) \times \mathcal{M}(\partial D))$. We write $(U_t, U_t^{\partial D})$ for the canonical random variables on Ω_D, $\mathcal{U}$ for the Borel subsets of Ω_D and $(\mathcal{U}_t)$ for the canonical filtration. Note that in the case $D = R^d$ where $\partial D = \emptyset$ the second component of the path space becomes trivial.

Notation. Let $\mathcal{B}(D, [0,1])$ be the space of Borel measurable $f : D \to [0,1]$. We give $\mathcal{B}(D, [0,1])$ the sigma field generated by $\langle f, I(A) \rangle$ for all bounded Borel $A \subseteq D$.

Theorem 2. *Suppose $d \le 3$ and D is a domain as described above. Fix $\beta, \gamma \ge 0$.*

(i) For any $\mu \in \mathcal{M}(D)$ and $f \in \mathcal{B}(D, [0,1])$, there exists a solution (u, v) to (1.1) on D with initial conditions (μ, f).

(ii) For any solution, the law of $(u_t : t \ge 0)$ on $C([0,\infty), \mathcal{M}(D))$ is uniquely determined by the law of (u_0, v_0).

(iii) For any solution and $0 \le s \le t$, the occupation measure $\int_s^t u_r dr$ is absolutely continuous with respect to Lebesgue measure, and there is a continuous version of the density $(s,t,x) \to u(s,t,x)$ on $0 < s \le t, x \in D$, almost surely. Moreover, setting $u(0,t,x) = \lim_{s \downarrow 0} u(s,t,x)$ we have,

$$v_t(x) = v_0(x) e^{-u(0,t,x)} \quad \textit{for all } t > 0, \textit{ for almost all } x \in D, \ P \ \textit{a.s.}$$

(iv) For any solution (u, v) there exists an adapted exit measure process $u^{\partial D} = (u_t^{\partial D} : t \ge 0)$, with non-decreasing continuous paths in $\mathcal{M}(\partial D)$, satisfying the extended martingale problem (2.4). Moreover, for all $t \ge 0$, there exists a measurable map $R_t : C([0,t], \mathcal{M}(D)) \to \mathcal{M}(\partial D)$ so that $u_t^{\partial D} = R_t((u_s : s \le t))$ almost surely.

(v) Let $Q_{\mu,f}^{D,\beta,\gamma}$ be the law of $(u, u^{\partial D})$ on Ω_D for a solution with initial condition (μ, f). Then $(\mu, f) \to Q_{\mu,f}^{D,\beta,\gamma}(B)$ is measurable for any Borel $B \subseteq \Omega_D$. Moreover, the family of laws $(Q_{\mu,f}^{D,\beta,\gamma} : \mu \in \mathcal{M}(D), f \in \mathcal{B}(D,[0,1]))$ satisfies the following strong Markov property: for any solution (u, v) defined on $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$, and any finite $(\mathcal{F}_t)$ stopping time τ,

$$E\left[h(u_{\tau+\cdot}) | \mathcal{F}_\tau\right] = Q^{D,\beta,\gamma}_{u_\tau, v_0 \exp(-u(0,\tau))}\left[h(U_\cdot)\right] \qquad P \ \textit{a.s},$$

for all bounded measurable $h : C([0,\infty), \mathcal{M}(D)) \to R$.

We note one consequence of these results: using part (iii), whenever (u,v) is a solution we may replace v by the process $v_0(x)\exp(-u(0,t,x))$ and it will still be a solution.

The following scaling lemma will be very useful in allowing us to transfer large or small parameters to terms in the equations that are convenient. We find our intuition stronger when working in regimes with small or large values of parameters, rather than small or large space and time scales. This lemma allows us to transfer between these two regimes.

Lemma 3. *Suppose that (u,v) is a solution to (1.1) on D. For $a,b,c,e>0$ define*

$$\tilde u_t(A) = \frac{a}{c^d} u_{bt}(cA), \quad \tilde v_t(x) = e v_{bt}(cx)$$

where $cA = \{cx : x \in A\}$. Then $(\tilde u, \tilde v)$ is a solution on $c^{-1}D$, in the sense described above, to the equation

$$\begin{cases} \partial_t \tilde u = \frac{b}{c^2}\Delta \tilde u + \frac{b\beta}{e}\tilde u\,\tilde v - b\gamma \tilde u + \sqrt{\frac{ab}{c^d}\tilde u}\,\dot W, \\ \partial_t \tilde v = -\frac{b}{a}\tilde u\,\tilde v. \end{cases}$$

Furthermore, for $A \subseteq \partial(c^{-1}D)$, we have $\tilde u_t^{\partial(c^{-1}D)}(A) = (a/c^d) u_{bt}^{\partial D}(cA)$.

The equation for $(\tilde u, \tilde v)$ follows immediately by scaling the martingale problem (2.1,2.2). The scaling of the exit measures follows by matching the finite variation parts of the semimartingale decomposition of $\tilde u_t(\phi)$ and of $(a/c^d)u_{bt}(\phi(\cdot/c))$ for $\phi \in C_b^2(c^{-1}\overline{D})$ given in the corresponding extended martingale problems.

2.2 Decomposition results for the total occupation and exit measures

Notation. For a solution (u,v) to (1.1) we write $u_{[s,t]}$ for the occupation measure $\int_s^t u_r dr$ (so that $u_{[s,t]}(dx) = u(s,t,x)dx$ P a.s.) and write

$$u_{[0,\infty)} = \lim_{t\to\infty} u_{[0,t]}, \quad u(0,\infty,x) = \lim_{t\to\infty} u(0,t,x), \quad \text{and} \quad u_\infty^{\partial D} = \lim_{t\to\infty} u_t^{\partial D}$$

for the total occupation measure on D and its density, and for the total exit measure on ∂D.

The lemmas below are stated under the restriction that D is bounded, which implies (see section 3.3) that the total occupation and exit measures are almost surely finite. The first lemma describes the monotonicity of the total occupation and exit measures with respect to the initial conditions (μ, f). We write $\stackrel{\mathcal{D}}{=}$ for equality in distribution.

Lemma 4. *Fix a bounded domain D and initial conditions $\mu = \mu^- + \mu^+$ and $f = f^- + f^+$. The law of the total occupation and exit measures of a solution (u,v) to (1.1) on D with initial conditions (μ, f) can be decomposed as*

$$\left(u_{[0,\infty)}, u_\infty^{\partial D}\right) \stackrel{\mathcal{D}}{=} \left(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u_\infty^{\partial D,-} + u_\infty^{\partial D,+}\right) \tag{2.5}$$

in either of the following two ways:

(i) *(u^-, v^-) is a solution with initial conditions (μ^-, f) and, conditional on $\sigma\{u^-\}$, the process (u^+, v^+) is a solution with initial conditions $u_0^+ = \mu^+$ and $v_0^+ = v_\infty^-$ (where $v_\infty^- = \lim_{t\to\infty} v_t^-$);*

(ii) *(u^-, v^-) is a solution with initial conditions (μ, f^-) and, conditional on $\sigma\{u^-\}$, the process (u^+, v^+) is a solution with initial conditions $u_0^+ = \beta f^+(1-\exp(-u^-(0,\infty)))\,dx$ and $v_0^+ = f\exp(-u^-(0,\infty))$.*

The next lemma shows monotonicity in the parameters β, γ.

Lemma 5. *Fix a bounded domain D, and parameter values $\beta = \beta^- + \beta^+$ and $\gamma^- = \gamma + \gamma^+$. The law of the total occupation and exit measures of a solution (u, v) to (1.1) on a bounded domain D with initial conditions (μ, f) and parameter values (β, γ) can be decomposed as in (2.5) in either of the following two ways:*

(i) *(u^-, v^-) is a solution with initial conditions (μ, f) and with parameter values β^-, γ and, conditional on $\sigma\{u^-\}$, the process (u^+, v^+) is a solution with initial conditions $u_0^+ = \beta^+(f - v_\infty^-)dx$ and $v_0^+ = v_\infty^-$ and with parameter values β, γ;*

(ii) *(u^-, v^-) is a solution to (1.1) with initial conditions (μ, f) and parameter values β, γ^- and, conditional on $\sigma\{u^-, v^-\}$, the process (u^+, v^+) is a solution with initial conditions $u_0^+ = \gamma^+ u^-_{[0,\infty)}$ and $v_0^+ = v_\infty^-$ and with parameter values β, γ.*

A special case of the Lemma 5 (i) will be particularly useful. The total exit measure can be built up in two stages: first run a process with $\beta = 0$ (which is the well understood Dawson-Watanabe process) and then run a second solution started with the mass that the nutrient would have produced from the first process.

The following simpler comparison will often be useful. It states, roughly, that if we convert some of the nutrient available into the equivalent amount of initial mass at time zero then we obtain a healthier process, in the sense of stochastic ordering.

Lemma 6. *Fix a bounded domain D and initial conditions μ and f, g. Let (u, v) and $(\tilde{u}, \tilde{v})$ be solutions to (1.1) on D with initial conditions $(\mu, f + g)$ and $(\mu + \beta g\, dx, f)$ respectively. Then for any bounded measurable $F : \mathcal{M}(D) \times \mathcal{M}(\partial D) \to R$ that is non-decreasing in both variables*

$$E[F(u_{[0,\infty)}, u_\infty^{\partial D})] \leq E[F(\tilde{u}_{[0,\infty)}, \tilde{u}_\infty^{\partial D})].$$

The final lemma is a spatial Markov property, analogous to that known for Dawson-Watanabe processes. We choose two domains $D^- \subseteq D^+$. We allow the boundaries ∂D^- and ∂D^+ to intersect, but we need this intersection not to be too complicated, and for this we ask that

$$(\partial D^- \cap \partial D^+) \cap \overline{(\partial D^- \setminus \partial D^+)} \text{ has surface measure zero in } \partial D^-. \tag{2.6}$$

This is satisfied for example if both domains are boxes. We write $\mu|_A$ for the restriction of a measure μ to a set A.

Lemma 7. *Fix bounded domains $D^- \subseteq D^+$ satisfying (2.6). Suppose (u, v) is a solution to (1.1) on D^+ started at (μ, f). Then the law of the total occupation and exit measures satisfies*

$$\left(u_{[0,\infty)}, u_\infty^{\partial D^+}\right) \stackrel{\mathcal{D}}{=} \left(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u_\infty^{\partial D^-,-}|_{\partial D^+} + u_\infty^{\partial D^+,+}\right)$$

where (u^-, v^-) is a solution to (1.1) on D^- with initial conditions $(\mu I(D^-), f I(D^-))$ and, conditional on $\sigma\{u^-\}$, the process (u^+, v^+) is a solution on D^+ with initial conditions

$$u_0^+ = u_\infty^{\partial D^-,-}|_{D^+} + \mu I(D^+ \setminus D^-) \quad \text{and} \quad v_0^+ = v_\infty^- I(D^-) + f I(D^+ \setminus D^-).$$

Remark. As indicated in the introduction, these lemmas will be proved for discretized approximations to solutions, where the nutrient is used up in finitely many steps, and then via a passage to the limit. We spent considerable energy trying to find a proof using the continuum equation (1.1) directly. Approaches using a change of measure (since the lemmas are easy to prove for a Dawson-Watanabe process) typically failed because the decomposition (2.5) holds only for the total occupation and exit measures and fails for these measures at times $t < \infty$. Another approach, using Laplace functionals, is sketched in remark (b) of section 3.1. On the other hand, we found the discretized approximations, in particular particle pictures, were the easiest testing ground for exploring possible extensions of these results to other models.

3 Change of measure arguments

In section 3.1 we review some results on Dawson-Watanabe processes and indicate when these imply estimates for solutions of (1.1). Section 3.2 gives the proof of Theorem 2, and section 3.3 contains the proof of the existence of the critical curve $\Psi(\beta)$.

3.1 Results from Dawson-Watanabe processes

A Dawson-Watanabe process, with underlying Brownian spatial motion, killed at the exit of a domain D, and with a (deterministic) bounded, measurable mass annihilation/creation rate $\eta : D \to R$, is the solution on D, in the same sense as in section 2.1, to the equation

$$\partial_t u = \Delta u - \eta u + \sqrt{u}\,\dot{W}. \tag{3.1}$$

We shall abbreviate Dawson-Watanabe processes as DW processes. In the case $\eta(x) = \gamma \in R$ is constant we will call the process a DW(D,γ) process. We describe below some results for DW processes that we shall need, and we mostly indicate where to find the proofs, or how to adapt them, from the literature. The surveys by Perkins [10] and Dawson [2] are our main sources. However, we give proofs for some of the results on exit measures, that are perhaps not in the literature, at the end of this subsection.

A number of the estimates (for example moment bounds, estimates on the speed of the support, estimates on the extinction probability) also hold for solutions to (1.1). One way to see this is to set up a pathwise coupling for (u, v) a solution to (1.1) on D so that $u^{(1)}_t \le u_t \le u^{(2)}_t$ where $u^{(1)}$ is a DW(D,γ) process and $u^{(2)}$ is a DW$(D,\gamma-\beta)$ process. We do not go through the steps to construct this coupling since, as we indicate below, the proofs of the desired estimates are typically based on stochastic calculus and are easily adapted to follow from the martingale problem for solutions to (1.1).

1. Martingale problems. We quickly repeat here the martingale problem definition for solutions. A continuous, adapted $\mathcal{M}(D)$ valued process u, on some filtered space $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$, is a solution to (3.1) if for $\phi \in C^2_0(\overline{D})$

$$u_t(\phi) = u_0(\phi) + \int_0^t u_s\left(\Delta\phi - \eta\phi\right) ds + m_t(\phi), \quad P \text{ a.s.} \tag{3.2}$$

where $m_t(\phi)$ is a continuous $(\mathcal{F}_t)$ local martingale with quadratic variation $[m(\phi)]_t = \int_0^t u_s(\phi^2) ds$ almost surely. Solutions, starting from $\mu \in \mathcal{M}(D)$, exist and are unique in law on $C([0,\infty), \mathcal{M}(D))$. Furthermore, for any solution there will exist an adapted process $(u^{\partial D}_t)$ with non-decreasing continuous paths in $\mathcal{M}(\partial D)$, called the exit measure process, so that for $\phi \in C^2_b(\overline{D})$

$$u_t(\phi) = u_0(\phi) + \int_0^t u_s\left(\Delta\phi - \eta\phi\right) ds - u^{\partial D}_t(\phi) + m_t(\phi), \quad P \text{ a.s.} \tag{3.3}$$

where $m_t(\phi)$ is a continuous local martingale with variation as before. The almost sure linearity $m_t(\phi + \psi) = m_t(\phi) + m_t(\psi)$, implies that if $\phi_n \in C^2_0(\overline{D})$ converge bounded pointwise to $\phi \in C^2_b(\overline{D})$ then $m_t(\phi_n) \to m_t(\phi)$ in probability (since $[m(\phi - \phi_n)]_t \to 0$). Now (3.2) and (3.3) imply that $u^{\partial D}_t(\phi)$ is measurable function of the path $(u_s : s \le t)$. Choose is a countable family $(\phi_k)_{k=1,\ldots}$ in $C^2_b(\overline{D})$ so that the map $\mu \to (\mu(\phi_k))_{k=1,\ldots}$ is a continuous injection from $\mathcal{M}(\partial D)$ into R^∞ (with the topology of uniform convergence on compacts). Using the measurable inverse of this map, one one can construct a measurable map $R_t : C([0,t], \mathcal{M}(D)) \to \mathcal{M}(\partial D)$ so that

$$u^{\partial D}_t = R_t(\{u_s : s \le t\}) \quad P \text{ a.s.} \tag{3.4}$$

These maps show that the law of $(u, u^{\partial D})$ on Ω_D is determined by the law of u on $C([0,\infty), \mathcal{M}(D))$. Also, for any DW process on D, the exit measure process can be constructed by using the maps R_t and then regularizing the path over the rationals.

Notation. We denote by $Q_\mu^{D,\eta}$ the law on Ω_D of a DW process and its exit measure $(u, u^{\partial D})$ on D, with mass annihilation/creation rate η as above, and starting at $\mu \in \mathcal{M}(D)$. We write $Q_\mu^{D,\gamma}$ in the case that $\eta(x) = \gamma$.

The map $\mu \to Q_\mu^{D,\eta}(B)$ is measurable for any Borel $B \subseteq \Omega_D$; furthermore the family of laws $(Q_\mu^{D,\eta} : \mu \in \mathcal{M}(D))$ form a strong Markov family in that for any solution u defined on $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$, and any finite $(\mathcal{F}_t)$ stopping time τ,

$$E\left[h(u_{\tau+\cdot})|\mathcal{F}_\tau\right] = Q_{u_\tau}^{D,\eta}\left[h(U_\cdot)\right] \qquad P \text{ a.s,}$$

for all bounded measurable $h : C([0,\infty), \mathcal{M}(D)) \to R$. The measurability and strong Markov property follow from the Laplace functional (3.6) below by, for example, the arguments used in the proof of Theorem II.5.1 of [10].

For a DW process u, one can extend the local martingales to a local martingale measure, which we write as $m_t(\phi) = \int_0^t \int_D \phi_s m(dx, ds)$, and is defined for $\phi : [0,\infty) \times \Omega \times D \to R$ that are $\mathcal{P} \times \mathcal{B}(D)$ measurable and satisfy $\int_0^t u_s(\phi_s^2) ds < \infty$ for all $t \geq 0$. The integral produces a continuous $(\mathcal{F}_t)$ local martingale with quadratic variation $[m(\phi)]_t = \int_0^t u_s(\phi_s^2) ds$. This follows as in [10] Proposition II.5.4. The same construction applies to give a local martingale measure for solutions to (1.1), and this will be needed in our change of measure arguments. One simple use of these extensions is to allow time dependent test functions in the martingale problem. Since the exit measures $u_t^{\partial D}$ are non-decreasing, they induce a unique measure $du_t^{\partial D}(dx)$ on $[0,\infty) \times \partial D$ characterized by the measure of the rectangles $(s,t] \times A$ being $u_t^{\partial D}(A) - u_s^{\partial D}(A)$. We will write the integral of $\psi : [0,t] \times \partial D \to R$ with respect to this measure as $\int_0^t du_s^{\partial D}(\psi_s)$. Then, if $\phi : [0,T] \times \overline{D} \to R$ is in the class $C_b^{1,2}([0,T] \times \overline{D})$, where ϕ and one partial derivative in time (denoted $\dot{\phi}$) and two partial derivatives in space exist and have continuous bounded extensions to $[0,T] \times \overline{D}$, the following decomposition holds:

$$u_t(\phi_t) = u_0(\phi_0) + \int_0^t u_s\left(\Delta\phi_s + \dot{\phi}_s - \eta\phi_s\right) ds - \int_0^t du_s^{\partial D}(\phi_s) + m_t(\phi) \quad P \text{ a.s.} \tag{3.5}$$

This follows using easier versions of the arguments in [10] Proposition II.5.7. Moreover, the analogous extension works for solutions to (1.1).

2. Laplace functionals. Suppose first that D has a smooth boundary and that $\eta : \overline{D} \to R$ is smooth and bounded. The Laplace functional of the solution, the occupation measure and the exit measure is given, for smooth bounded $h^{(1)} : \overline{D} \to R$, $h^{(2)} : [0,t] \times \overline{D} \to R$ and $h^{(3)} : [0,t] \times \partial D \to R$ by

$$Q_\mu^{D,\eta}\left[e^{-U_t(h^{(1)}) - \int_0^t U_s(h_s^{(2)}) - \int_0^t dU_s^{\partial D}(h_s^{(3)})}\right] = e^{-\mu(\phi_t)}, \tag{3.6}$$

where $(\phi_s(x) : s \in [0,t],\, x \in D)$ is the unique solution, smooth on $[0,t] \times \overline{D}$, to

$$\begin{cases} \partial_s\phi_s = \Delta\phi_s - \eta\phi_s - \frac{\phi_s^2}{2} + h_{t-s}^{(2)} \quad \text{on } D, \\ \quad \phi_s = h_{t-s}^{(3)} \text{ on } \partial D \text{ for } s \in [0,t], \quad \text{ and } \phi_0 = h^{(1)} \text{ on } D. \end{cases} \tag{3.7}$$

This follows by checking that $s \to \exp(-U_s(\phi_{t-s}) - \int_0^t U_r(h_s^{(2)}) dr - \int_0^t dU_r^{\partial D}(h_r^{(3)}))$ is a martingale for $s \in [0,t]$. If in addition $\eta \geq 0$ and D is bounded, the total occupation $U_{[0,\infty)} = \lim_{t\uparrow\infty} U_{[0,t]}$ and total exit measures $U_\infty^{\partial D} = \lim_{t\uparrow\infty} U_t^{\partial D}$ are finite almost surely and have Laplace functional

$$Q_\mu^{D,\eta}\left[e^{-U_{[0,\infty)}(h_1) - U_\infty^{\partial D}(h_2)}\right] = e^{-\mu(\phi)}, \tag{3.8}$$

for smooth bounded $h_1 : \overline{D} \to [0,\infty)$ and $h_2 : \partial D \to [0,\infty)$, where $(\phi(x) : x \in D)$ is the unique solution, smooth on $\overline{D}$, to

$$\Delta\phi = \frac{\phi^2}{2} + \eta\phi - h_1 \quad \text{on } D, \quad \phi = h_2 \text{ on } \partial D. \tag{3.9}$$

This follows by examining the martingale $t \to \exp(-U_t(\phi) - \int_0^t U_s(h_1) - U_t^{\partial D}(h_2))$. On bounded domains, and when $\eta \geq 0$, there exist finite solutions to (3.9) when h_1, h_2 are negative and sufficiently small. In particular, the exponential moment $Q_\mu^{D,\eta}[\exp(+\theta U_\infty^{\partial D}(1) + \theta U_{[0,\infty)}(1))]$ is finite for sufficiently small $\theta > 0$.

When the domain D is a box the associated p.d.e.'s have unique solutions that are smooth inside D and continuous on $\overline{D}$, and the formulae for the Laplace functionals still hold, and can be established by an approximation argument. Indeed, one concrete way to do this is to suppose, without loss, that D contains the origin and to replace the test functions above by $\phi_t(x/(1+\epsilon))$ and $\phi(x/(1+\epsilon)$, which are smooth in $\overline{D}$.

Remark (a). From the pde viewpoint, it is natural to use (3.3) as a means of characterizing the exit measures $t \to u_t^{\partial D}(dx)$, that is purely as exit fluxes (as is suggested in [11]). Although we do not need such a characterization, since we have other constructions of the exit measure, we briefly sketch such an approach, and for convenience we consider deterministic initial condition μ and smooth bounded domain D. Construct the martingale measure $m(dx, ds)$ from a DW(D, γ) process. Then for smooth $h : \partial D \to R$ we may uniquely find $\phi^h \in C_b^2(\overline{D})$ solving the Dirichlet problem $\Delta \phi^h = 0$ on D and $\phi^h = h$ on ∂D. Use the formula (3.3) with the test function ϕ^h to define a continuous path $t \to u_t^{\partial D}(h)$ (although we have yet to establish that the values $u_t^{\partial D}(h)$ arise from a measure $u_t^{\partial D}$ on ∂D). By adding the decompositions (3.3) for ϕ^g and ϕ^h we find that $u_t^{\partial D}(g+h) = u_t^{\partial D}(g) + u_t^{\partial D}(h)$ almost surely. Since $\|\phi^h\|_\infty \leq \|h\|_\infty$, we can deduce a first moment bound $\sup_{s \leq t} E[|u_s^{\partial D}(h)|] \leq C(\mu, t, \gamma)\|h\|_\infty$ from the corresponding bounds for u. For $\phi \in C_b^2(\overline{D})$ with $\phi = h$ on ∂D, the decomposition (3.3) holds by adding the decompositions for ϕ^h and $\phi - \phi^h$. Then the first moment bounds allow the decompositions to be extended to time dependent test functions $\phi(t, x)$ as above. In particular we find, for $h \geq 0$, that $E[\exp(-\lambda u_t^{\partial D}(h))] \leq \exp(-\mu(\phi_t))$ where $\partial_t \phi = \Delta \phi - \gamma \phi - \frac{\phi^2}{2}$ on D, and $\phi_t = \lambda h$ on ∂D, and $\phi_0 = 0$. The inequality here arises since we do not yet know that $u_t^{\partial D}(h) \geq 0$ and so we have to localize via the stopping times $\tau_n = \inf\{t : u_t(1) \geq n\}$, and we obtain an upper bound by passing to the limit using Fatou's Lemma. However letting $\lambda \to \infty$ now implies that $u_t^{\partial D}(h) \geq 0$ almost surely. It is then not hard to establish the existence of a measure $u_t^{\partial D}$ so that $u_t^{\partial D}(h) = \int_{\partial D} h \, du_t^{\partial D}$. Finally the Markov property implies that $u_t^{\partial D} - u_s^{\partial D}$ is non-negative and hence the paths of $u_t^{\partial D}$ are non-decreasing.

Remark (b). Mimicking the calculus that leads to the Laplace functional above, one can show that for solutions to (1.1) that

$$\exp\left(-u_t(\phi) - u_{[0,t]}(h_1) - u_t^{\partial D}(h_2) + \beta \int_0^t u_s(v_s \phi) ds\right)$$

is a martingale when ϕ solves (3.9) with $\eta = \gamma$. One deduces, by letting $t \to \infty$, that

$$E\left[e^{-u_{[0,\infty)}(h_2) - u_\infty^{\partial D}(h_3) + \beta \langle \phi f, 1 - \exp(-u(0,\infty)) \rangle}\right] = e^{-\mu(\phi)}.$$

We do not know if this formula, as one varies h_1, h_2, characterizes the law of $(u_{[0,\infty)}, u_\infty^{\partial D})$. If it did characterize the law, a simple proof of all the decomposition lemmas from section 2.2 would result, since it is straightforward to use calculus for both parts of the decompositions to yield the above formula for the sum.

3. Some moments. Estimates on moments for the total mass process $u_t(1)$, starting from a deterministic initial condition, follow by choosing $\phi = 1$ in (3.3), using $u_t^{\partial D}(1) \geq 0$, by the usual arguments (localizing by the stopping times $\tau_n = \inf\{t : u_t(1) \geq n\}$ and applying Gronwall's inequality). By rewriting (3.3), again with $\phi = 1$, as an equation for $u_t^{\partial D}(1)$, one can then deduce moments for the exit measures. These arguments show, for instance, that for for $p \geq 1$

$$Q_\mu^{D,\gamma}\left[\sup_{t \leq T}(U_t(1))^p + |U_T^{\partial D}(1)|^p\right] \leq C(p, T, \gamma)\left((\mu(1) + (\mu(1))^p\right). \tag{3.10}$$

In particular, starting from a deterministic initial condition the processes $m_t(\phi)$ are true martingales. The same arguments apply to solutions to (1.1), and lead to

$$Q_{\mu,f}^{D,\beta,\gamma}\left[\sup_{t \leq T}(U_t(1))^p + |U_T^{\partial D}(1)|^p\right] \leq Q_\mu^{D,\gamma-\beta}\left[\sup_{t \leq T}(U_t(1))^p + |U_T^{\partial D}(1)|^p\right]. \tag{3.11}$$

(We abuse logical order here, and below, by writing $Q_{\mu,f}^{D,\beta,\gamma}$ before checking that these laws are uniquely defined. In such cases we mean that the arguments apply to all solutions to (1.1).)

We will need the first and second moment formulae, for measurable $g, h \geq 0$,

$$\begin{aligned} Q_\mu^{D,\gamma}\left[U_t(g)\right] &= \mu(G_t^{D,\gamma} g), \\ Q_\mu^{D,\gamma}\left[U_t(g)U_s(h)\right] &= \mu(G_t^{D,\gamma} g)\mu(G_s^{D,\gamma} h) + \int_0^{t\wedge s} \mu(G_r^{D,\gamma}(G_{t-r}^{D,\gamma} g\, G_{s-r}^{D,\gamma} h))dr, \end{aligned} \tag{3.12}$$

where $G_t^{D,\gamma} g(x) = \int_D G_t^{D,\gamma,x}(y)g(y)dy$ and $G_t^{D,\gamma,x}(y)$ is the killed Green's function $G_t^{D,\gamma,x}(y) = e^{-\gamma t} G_t^{D,x}(y)$ for the domain D (that is $G_t^{D,x}(y)$ is the transition density of a Brownian motion killed on its exit from D). These can be proved by using the time dependent test functions $\phi^{(g)}$ and $\phi^{(h)}$, where $\phi_s^{(g)} = G_{t-s}^{D,\gamma} g$ for $s \in [0,t]$, in (3.5). For smooth g, h compactly supported in D, this test function has the required regularity, and the moment formulae can then be extended to general g, h by monotone class methods. In particular the same methods apply to the martingale problem for solutions to (1.1) and give the bounds, for $g \geq 0$ and $k = 1, 2$,

$$Q_\mu^{D,\gamma}\left[(U_t(g))^k\right] \leq Q_{\mu,f}^{D,\beta,\gamma}\left[(U_t(g))^k\right] \leq Q_\mu^{D,\gamma-\beta}\left[(U_t(g))^k\right], \tag{3.13}$$

illustrating the natural intuition that less killing leads to larger solutions.

4. Extinction probabilities and speed of propagation. The probability of extinction for DW(R^d, γ) processes is given by $Q_\mu^{R^d,\gamma}[U_t = 0] = \exp(-\lambda_t^{(\gamma)} \mu(1))$ where $\lambda_s^{(\gamma)}$ is determined for $s \in (0,t]$ by

$$\dot{\lambda}^{(\gamma)} = -\frac{1}{2}(\lambda^{(\gamma)})^2 - \gamma \lambda^{(\gamma)}, \qquad \text{and} \qquad \lambda_s^{(\gamma)} \to \infty \text{ as } s \downarrow 0.$$

This follows by examining the martingale $s \to \exp(-\lambda_{t-s}^{(\gamma)} U_s(1))$ for $s \in (0,t]$ (see the derivation of (3.15) below). This same argument applies to solutions of (1.1) to give the bounds

$$Q_\mu^{R^d,\gamma-\beta}[U_t = 0] \leq Q_{\mu,f}^{D,\beta,\gamma}[U_t = 0] \quad \text{and} \quad Q_{\mu,f}^{R^d,\beta,\gamma}[U_t = 0] \leq Q_\mu^{R^d,\gamma}[U_t = 0]. \tag{3.14}$$

DW processes, started from finite measures, are compactly supported at all times $t > 0$. The same will apply for solutions u to (1.1), by the absolute continuity of the law of u with respect to a DW process (see section 3.2). However we need a quantitative bound on the size of the support and we use two estimates for this behaviour, based on the results from Dawson, Iscoe and Perkins [3]. For $R > 0$ let $D_R = (-R,R)^d$. Choose $0 \leq \xi_R \leq 1$ smooth and satisfying $\{\xi_R > 0\} = \overline{D}_R^c$. Then for $M \geq 0$ and smooth bounded $\eta : R \to R$, let $\phi_s = \phi_s^{(M,R,\eta)}$ be the unique non-negative solution to

$$\dot{\phi} = \Delta\phi - \frac{1}{2}\phi^2 - \eta\phi, \quad \psi_0 = M\xi_R.$$

Then, for $s > 0$, $\phi_s^{(M,R,\eta)} \uparrow \phi_s^{(R,\eta)} < \infty$ as $M \uparrow \infty$ and moreover $\phi_s^{(R,\eta)}(x) \to 0$ as $R \to \infty$ (use a comparsion argument and [3] Lemma 3.6). Taking expectations of the martingale $\exp(-U_s(\phi_{t-s}^{(M,R,\eta)}))$ at times $s = 0$ and $s = t$, and then letting $M \to \infty$, one finds that $Q_\mu^{R^d,\eta}[U_t(\overline{D}_R^c) = 0] = \exp(-\mu(\phi_t^{(R,\eta)}))$. The same proof shows, when $\gamma - \beta f \geq \eta$, the bound

$$Q_{\mu,f}^{R^d,\beta,\gamma}\left[U_t(\overline{D}_R^c) = 0\right] \geq Q_\mu^{R^d,\eta}\left[U_t(\overline{D}_R^c) = 0\right]. \tag{3.15}$$

Letting $R \to \infty$ shows that solutions to (1.1) started from compactly supported $\mu \in \mathcal{M}(R^d)$ must have compact support at any time $t > 0$.

Similarly, let $\psi_s = \psi_s^{(M,R,\eta)}$ be the unique non-negative solution to

$$\dot{\psi} = \Delta\psi - \frac{1}{2}\psi^2 - \eta\psi + M\xi_R, \quad \psi_0 = 0.$$

Then, for $|x| < R$, $\psi_s^{(M,R,\eta)}(x) \uparrow \psi_s^{(R,\eta)}(x) < \infty$ as $M \uparrow \infty$ and moreover $\psi_s^{(2,R,\gamma)}(x) \to 0$ as $R \to \infty$ (argue as in [3] Theorem 3.3 (ii)). Examining the process $\exp(-U_s(\psi_{t-s}^{(M,R,\eta)}))$ as above one finds that $Q_\mu^{R^d,\eta}[U_{[0,t]}(\overline{D}_R^c) = 0] = \exp(-\mu(\psi_t^{(R,\eta)}))$ and the same proof shows, when $\gamma - \beta f \geq \eta$, the bound

$$Q_{\mu,f}^{R^d,\beta,\gamma}\left[U_{[0,t]}(\overline{D}_R^c) = 0\right] \geq Q_\mu^{R^d,\eta}\left[U_{[0,t]}(\overline{D}_R^c) = 0\right]. \tag{3.16}$$

Letting $R \to \infty$ shows that solutions to (1.1) started from compactly supported μ remain compactly supported at all times. Comparisons can be used to estimate $\psi^{(R,\eta)}$. For example suppose that $\Delta w \le (1/2)w^2 - \eta w$ on D_R and $\inf\{w(x) : x \in \partial D_{R-\epsilon}\} \uparrow \infty$ as $\epsilon \downarrow 0$. Then $\psi_t^{(R,\eta)} \le w$ on D_R for all t so that, when $\eta \ge \beta f - \gamma$, for μ supported inside D_R

$$Q_{\mu,f}^{R^d,\beta,\gamma}\left[U_{[0,\infty)}(\overline{D}_R^c) = 0\right] \ge Q_{\mu}^{R^d,\eta}\left[U_{[0,\infty)}(\overline{D}_R^c) = 0\right] \ge \exp(-\mu(w)). \quad (3.17)$$

5. Occupation density. Define a putative density $U(s,t,x)$ for $U_{[s,t]}$ on path space Ω_D by setting, for $0 < s \le t$, $x \in D$,

$$U(s,t,x) = \liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon^d} \int_s^t U_r\left([x, x+\epsilon)^d \cap D\right) dr. \quad (3.18)$$

We also set $U(0,t,x) = \lim_{s \downarrow 0} U(s,t,x)$.

Sugitani [13] studied the occupation measure under the law $Q_\mu^{R^d,0}$ of a DW process on R^d. His main result extends to our processes on more general domains, and with bounded annihilation/creation rates, as follows: $Q_\mu^{D,\eta}$ almost surely, the process $(s,t,x) \to U(s,t,x)$ is continuous over $0 < s \le t$, $x \in D$, and $x \to U(s,t,x)$ acts as a density for $U_{[s,t]}$. One way of representing the density is via a Green's function formula, as follows. Let $G_{[0,t]}^{D,x} = \int_0^t G_s^{D,x} ds$ and $G_{[0,t]}^{D,\gamma,x} = \int_0^t G_s^{D,\gamma,x} ds$. Then, for a fixed $t > \delta > 0$ and $x \in D$,

$$U(\delta,t,x) = U_\delta(G_{[0,t-\delta]}^{D,x}) + \int_\delta^t \int_D G_{[0,t-s]}^{D,x}(z)(-\eta(z)U_s(dz)ds + M(dz,ds)) \quad Q_\mu^{D,\eta} \text{ a.s.} \quad (3.19)$$

where $M(dz,ds)$ is the martingale measure associated to U. Moreover, for $k \in N$,

$$Q_\mu^{D,\eta}[U(\delta,t,x)^k] \le C(k,\delta,\mu,\|\eta\|_\infty,T) \quad \text{for all } x \in D, t \in [\delta,T]. \quad (3.20)$$

When the initial condition is more one may take $\delta = 0$ in (3.19), and indeed this is the case if μ has a bounded density. In particular, when $\mu(dx) = \beta dx$ we have, for $k \in N$,

$$Q_{\beta dx}^{D,\eta}[U(0,t,x)^k] \le C(k,\beta,\|\eta\|_\infty,T) \quad \text{for all } x \in D, t \in [0,T]. \quad (3.21)$$

It is not hard to extend Sugitani's proof of the representation formula to hold for domains D and with annihilation/creation terms, and also to bound the moments (inductively in k). Sugitani's proof of continuity is via Kolmogorov's continuity criterion and moments of increments. We omit the details since we will follow Sugitani's arguments to establish a similar increment estimates for approximations to solutions in section 4.2, and some of the key steps are illustrated there.

The second moment formula (3.12) lead to the moment

$$Q_\mu^{D,\gamma}\left[U^2(0,t,x)\right] = \left(\mu(G_{[0,t]}^{D,\gamma,x})\right)^2 + \int_0^t \mu(G_s^{D,\gamma}((G_{[0,t-s]}^{D,\gamma,x})^2))ds.$$

We use the bound $G_t^{D,\gamma,x} \le e^{-\gamma t} G_t^{R^d,x}$ and the estimate

$$q(t) := \int (G_{[0,t]}^{R^d,x}(y))^2 dx \le C(d) t^{(4-d)/2} \quad \text{for all } x, y \text{ when } d \le 3.$$

One can now deduce that

$$Q_\mu^{D,\gamma}\left[\int_D (U(0,t,x))^2 dx\right] \le C(d,\gamma)\left(\mu^2(1)q(t) + \mu(1)\int_0^t q(s)ds\right). \quad (3.22)$$

Again, one has that $Q_{\mu,f}^{D,\beta,\gamma}\left[\langle U^2(0,t),1\rangle\right] \le Q_\mu^{D,\gamma-\beta}\left[\langle U^2(0,t),1\rangle\right]$.

6. A construction of the exit measure process and the spatial Markov property. A convenient way to construct the exit measure processes was developed in Perkins [11], which will be useful in our proof of a coupling version of the spatial Markov property lemma below. This exploits a historical DW process, which is a process of random measures on the space of paths, where intuitively the paths trace the positions of the ancestors of the particles in a DW process. To explain the construction, we recall some notation from [11] for the historical Brownian DW process (which is only required in this one subsection): on the path space $C([0,\infty),R^d)$ let $\mathcal{C}$ be the Borel σ-field and $(\mathcal{C}_t)$ the canonical filtration; $\mathcal{M}(C)$ is the space of finite measures on $C([0,\infty),R^d)$ with the topology of weak convergence; on $\Omega_H = C([0,\infty),\mathcal{M}(C))$ let $\mathcal{H}$ be the Borel sets and (H_t) the canonical coordinate variables. Write $Q^{\eta}_{H,\mu}$ for the law on $(\Omega_H,\mathcal{H})$ of a historical Brownian DW process, with branching rate one, with bounded measurable annihilation/creation rate $\eta : R^d \to R$, and with initial condition $\mu \in M_F(R^d)$. Letting $\mathcal{N}$ be the $Q^{\eta}_{H,\mu}$ null sets we define a filtration by $\mathcal{H}_t = \cap_{s>t}\sigma\{H_r : r \le s\} \vee \mathcal{N}$. The canonical process $H = (H_t)$ is a historical DW process with creation/annihilation η on $(\Omega_H,\mathcal{H},\mathcal{H}_t,Q^{\eta}_{H,\mu})$. In the notation from [11], it solves the martingale problem $(MP)^{0,\mu}_{1,2I,0,\hat{\eta}}$, where $\hat{\eta}(s,y) = \eta(y_s)$.

For $(\mathcal{C}_t)$ predictable $C : [0,\infty) \times C([0,\infty),R^d) \to R$, having left continuous paths of bounded variation, Theorem 2.23 of Perkins [11] constructs the integral of C along the paths of a historical DW process as the limit (in probability, or almost surely along a subsequence)

$$\int_0^t H_s(dC_s) = \lim_{n\to\infty} \sum_{i=1}^{\infty} I(t_i^n < t) H_{t_i^n}(C_{t_i^n} - C_{t_{i-1}^n}), \quad \text{where } t_i^n = i2^n. \tag{3.23}$$

Note that the process $\int_0^t H_s(dC_s)$ is non-decreasing whenever C_s is also non-decreasing.

Fix a bounded domain D and define $\tau = \tau(y) = \inf\{t \ge 0 : y_t \notin D\}$ for $y \in C([0,\infty),R^d)$, and where $\inf\{\emptyset\} = 0$. Define a measure $u_t(dx)$ on D by

$$u_t(dx) = \int_{C([0,\infty),R^d)} I(y_t \in dx,\, t \le \tau,\, \tau > 0) H_t(dy).$$

If μ is supported inside D we may omit the indicator $I(\tau > 0)$. For bounded measurable $\phi : \partial D \to R$ define

$$u_t^{\partial D}(\phi) = \int_0^t H_s(dC_s^{\phi}), \quad \text{where } C_s^{\phi}(y) = \phi(y_\tau) I(s > \tau > 0).$$

By definition $u^{\partial D}(\phi) \ge 0$ if $\phi \ge 0$ and $u^{\partial D}(\phi+\psi) = u^{\partial D}(\phi) + u^{\partial D}(\psi)$ almost surely. Using these properties, it is not hard to show that there is an ($\mathcal{H}_t$ measurable) random measure $u_t^{\partial D}$ on ∂D so that $u_t^{\partial D}(\phi) = \int_{\partial D} \phi(x) du_t^{\partial D}(dx)$ almost surely.

Let $Z(dy,dt)$ denote the martingale measure associated to H (constructed for example in section 2 of [11]) and use the notation $Z^0(dy,ds) = Z(dy,ds) - \eta(y_s)H_s(dy)ds$, that is an integral against $Z^0(dy,ds)$ means the difference of two integrals. This notation is borrowed from [11] and will save space since these two integrals frequently appear with the same integrand. We may apply the historical Ito formula ([11] Theorem 2.14) to the process $Y_t(y) = y_{t\wedge\tau}$, since the stopped path $y_{t\wedge\tau}$. For $\phi \in C_b^{1,2}([0,\infty) \times R^d)$ this yields

$$\begin{aligned} \int \phi_{t\wedge\tau}(y_{t\wedge\tau}) H_t(dy) &= \mu(\phi_0) + \int_0^t \int (\dot{\phi} + \Delta\phi)_{s\wedge\tau}(y_{s\wedge\tau}) I(s \le \tau) H_s(dy) ds \\ &\quad + \int_0^t \int \phi_{s\wedge\tau}(y_{s\wedge\tau}) Z^0(dy,ds) \quad \text{for } t \ge 0, \text{ a.s.} \end{aligned} \tag{3.24}$$

Using Proposition 2.7 from [11] we have also, $Q^{\eta}_{H,\mu}$ a.s. for all $t \ge 0$,

$$\int \phi_0(y_0) I(\tau = 0) H_t(dy) = \mu(I(D^c)\phi_0) + \int_0^t \int \phi_0(y_0) I(\tau = 0) Z^0(dy,ds). \tag{3.25}$$

Finally, Theorem 2.23 of [11] shows, when $C_t^{\phi}(y) = \phi_\tau(y_\tau) I(t > \tau > 0)$,

$$H_t(C_t^{\phi}) = \int_0^t H_s(dC_s^{\phi}) + \int_0^t \int \phi_\tau(y_\tau) I(s > \tau > 0) Z^0(dy,ds) \quad \text{for } t \ge 0 \text{ a.s.} \tag{3.26}$$

Note that

$$u_t(\phi_t) = \int \phi_{t\wedge\tau}(y_{t\wedge\tau})H_t(dy) - \int \phi_0(y_0)I(\tau=0)H_t(dy) - H_t(C_t^\phi).$$

Combining this with (3.24,3.25,3.26) shows that

$$\begin{aligned} u_t(\phi_t) &= \mu(\phi_0 I(D)) - \int_0^t H_s(dC_s^\phi) + \int_0^t \int \phi_s(y_s)I(s\le\tau,\tau>0)Z(dy,ds) \\ &\quad + \int_0^t \int (\dot\phi + \Delta\phi - \eta\phi)_s(y_s)I(s\le\tau)H_s(dy)ds \quad \text{for } t\ge 0 \text{ a.s.} \end{aligned} \tag{3.27}$$

An approximation argument shows that (3.27) continues to hold for $\phi \in C^{1,2}([0,t]\times\overline{D})$. Taking time independent ϕ shows that the extended martingale problem (3.3) holds for $(u, u^{\partial D})$. To check the regularity of the paths, note that the path $t \to u_t(\phi)$ is continuous for $\phi \in C_0^2(\overline{D})$ by (3.2). To establish continuity when $\phi \in C_b^2(\overline{D})$, we use an estimate of the amount of mass near the boundary of D. For $\epsilon > 0$ let $\phi_{D,\epsilon}(x) = (1-\epsilon^{-1}d(x,\partial D))\vee 0$. If $\mu(\partial D) = 0$ then $U_t(\partial D) = 0$ for all $t \ge 0$, $Q_\mu^{R^d,\eta}$ almost surely. This follows since $t \to U_t(\partial D)$ is continuous (see [10] Theorem III.5.1 and use absolute continuity between $Q_\mu^{R^d,\eta}$ and $Q_\mu^{R^d,0}$) and has zero first moment. Since $\phi_{D,\epsilon} \downarrow I(\partial D)$, for such μ we have

$$\sup_{t\le T} U_t(\phi_{D,\epsilon}) \downarrow 0 \text{ as } \epsilon\downarrow 0, \quad Q_\mu^{R^d,\eta} \text{ almost surely.} \tag{3.28}$$

Note the simple coupling

$$u_t \le \tilde u_t \quad \text{for all } t \ge 0 \tag{3.29}$$

where $\tilde u$ is a DW(R^d) process with creation/annihilation η, both with the same initial condition. This is natural since $\tilde u$ does not have the killing on the boundary of D, and follows by setting $\tilde u_t(dx) = \int I(y_t \in dx)H_t(dy)$. This coupling ensures that (3.28) is also true for $\sup_{t\le T} u_t(\phi_{D,\epsilon})$. This implies the continuity of $t \to u_t(\phi)$ for $\phi \in C_b^2(\overline{D})$, and hence of $t \to u_t \in \mathcal{M}(D)$. The continuity of $t \to u_t^{\partial D}(h)$ for $h \in C_b^2(\partial D)$ follows from the continuity of all other terms in (3.3). This implies the paths $t \to u_t^{\partial D}$ are continuous, and they are non-decreasing by definition. So u is a DW process on D, with annihilation/creation rate η, with exit measure $u^{\partial D}$, and with initial condition $\mu I(D)$.

Dynkin [5] established a spatial Markov property for DW processes. We will need the following slight variant of the standard statement. The lemma is reasonably clear at the approximation level of discrete branching trees, but we choose to give a proof in the continuum setting using historical calculus.

Lemma 8. *Let $D^- \subseteq D^+$ be bounded domains. Fix $\mu \in \mathcal{M}(D^+)$. There exists a coupling of three processes: u a DW(D^+,γ) process with initial condition μ; u^- a DW(D^-,γ) process with initial condition $\mu I(D^-)$; and u^+ which, conditional on $\sigma\{u^-\}$, is a DW(D^+,γ) process with initial condition $\mu I(D^+\setminus D^-) + u_{[0,\infty)}^{\partial D^-,-}|_{D^+}$; and moreover these processes satisfy the splitting*

$$\left(u_{[0,\infty)}, u_\infty^{\partial D^+}\right) = \left(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u_\infty^{\partial D^-,-}|_{\partial D^+} + u_\infty^{\partial D^+,+}\right) \qquad \textit{almost surely.} \tag{3.30}$$

Proof. Define $\tau^\pm = \inf\{t : y_t \notin D^\pm\}$. As above, we suppose the canonical process (H_t) is a historical DW process with constant annihilation rate γ on $(\Omega_H, \mathcal{H}, \mathcal{H}_t, Q_{H,\mu}^\gamma)$. Define

$$\begin{aligned} u_t(dx) &= \int I(y_t\in dx,\, t\le\tau^+)\,H_t(dy), \quad u_t^-(dx) = \int I(y_t\in dx,\, t\le\tau^-,\, \tau^->0)\,H_t(dy) \\ u_t^{\partial D^+}(\phi) &= \int_0^t H_s(dC_s^{\phi,+}), \qquad u_t^{\partial D^-,-}(\phi) = \int_0^t H_s(dC_s^{\phi,-}), \end{aligned}$$

where $C_s^{\phi,+}(y) = \phi(y_{\tau^+})I(s>\tau^+)$ for bounded measurable $\phi : \partial D^+ \to R$ and $C_s^{\phi,-}(y) = \phi(y_{\tau^-})I(s>\tau^->0)$ for bounded measurable $\phi : \partial D^- \to R$. The earlier arguments leading to (3.27) show that this defines processes $(u, u^{\partial D^+})$ and $(u^-, u^{\partial D^-,-})$ having laws $Q_\mu^{D^+,\gamma}$ and $Q_{\mu I(D^-)}^{D^-,\gamma}$ as desired.

There exist random measures $\mathbf{I}$ on $[0,\infty)\times D^+$ and $\mathbf{I}^{\partial D^+}$ on $[0,\infty)\times\partial D^+$ satisfying, for bounded measurable $\phi:[0,\infty)\times D^+\to R$ and $\psi:[0,\infty)\times\partial D^+\to R$,

$$\begin{aligned}\mathbf{I}(\phi) &= \int_0^\infty\int \phi_{t-\tau^-}(y_t)I(t\in(\tau^-,\tau^+])H_t(dy)dt\\ \mathbf{I}^{\partial D^+}(\psi) &= \int_0^\infty H_s(d\hat{C}^\psi_s) \quad \text{where } \hat{C}^\psi_s(y)=\psi_{\tau^+-\tau^-}(y_{\tau^+})I(s>\tau^+>\tau^-).\end{aligned}$$

We will show that conditional on $\sigma\{u^-\}$, the law of $(\mathbf{I},\mathbf{I}^{\partial D^+})$ is that of $(U_t(dx)dt, dU^{\partial D}_t(dx))$ under $Q^{D^+,\gamma}_{\mu I(D^c)+u^{\partial D^-,-}_\infty|_{D^+}}$. This allows us to define $(u^+,u^{\partial D^+,+})$ with the required law so that

$$\mathbf{I}(\phi)=\int_0^\infty u^+_t(\phi_t)dt \quad \text{and} \quad \mathbf{I}^{\partial D^+}(\psi)=\int_0^\infty du^{\partial D^+,+}_s(\psi_s).$$

The splitting (3.30) for the total occupation measures holds since

$$u^-_{[0,\infty)}(A)=\int_0^\infty\int I(y_t\in A, t\le\tau^-,\tau^->0)H_t(dy)dt=\int_0^\infty\int I(y_t\in A,t\le\tau^-)H_t(dy)dt$$

while

$$u_{[0,\infty)}(A)=\int_0^\infty\int I(y_t\in A,t\le\tau^+)H_t(dy)dt, \quad \mathbf{I}([0,\infty)\times A)=\int_0^\infty\int I(y_t\in A,t\in(\tau^-,\tau^+])H_t(dy)dt.$$

Note that for $A\subseteq\partial D^+$

$$u^{\partial D^+}_\infty(A)=\int_0^\infty H_s(dC^{A,+}_s), \quad u^{\partial D^-,-}_\infty(A)=\int_0^\infty H_s(dC^{A,-}_s), \quad \mathbf{I}^{\partial D^+}([0,\infty)\times A)=\int_0^\infty H_s(d\hat{C}^A_s)$$

where

$$C^{A,+}_s(y)=I(y_{\tau^+}\in A, s>\tau^+), \quad C^{A,-}_s(y)=I(y_{\tau^-}\in A,s>\tau^->0), \quad \hat{C}^A_s(y)=I(y_{\tau^+}\in A,s>\tau^+>\tau^-).$$

Note also that $C^{A,-}_s(y)=I(y_{\tau^+}\in A,s>\tau^+=\tau^->0)$ since $A\subseteq\partial D^+$. Then the splitting for the total exit measures holds since

$$C^{A,+}_s=\hat{C}^A_s+C^{A,-}_s+I(y_{\tau^+}\in A,s>\tau^+=0)$$

and $H_t(\tau^+=0)=0$ for all $t\ge 0$, $Q^\gamma_{H,\mu}$ a.s.

To identify the conditional law of $(\mathbf{I},\mathbf{I}^{\partial D^+})$ we will use the Laplace functionals. Choose non-negative, continuous bounded $h^{(2),\pm}:[0,\infty)\times D^\pm$ and $h^{(3),\pm}:[0,\infty)\times\partial D^\pm$ which are zero for $t\ge T$. Let $\hat{\phi}^\pm\in C^{1,2}([0,T]\times D^\pm)\cap C([0,T]\times\overline{D^\pm})$ be the unique non-negative mild solution to

$$-\dot{\hat{\phi}}^\pm=\Delta\hat{\phi}^\pm-\gamma\hat{\phi}^\pm-\tfrac{1}{2}(\hat{\phi}^\pm)^2+h^{(2),\pm} \text{ on } D^\pm, \quad \hat{\phi}^\pm_T=0 \text{ and } \hat{\phi}^\pm=h^{(3),\pm} \text{ on } \partial D^\pm.$$

We extend these by letting $\hat{\phi}^\pm_t=0$ for $t\ge T$. It is enough to show

$$E\left[e^{-\mathbf{I}(h^{(2),+})-\mathbf{I}^{\partial D^+}(h^{(3),+})}\,\big|\sigma\{u^-\}\right]=e^{-u^+_0(\hat{\phi}^+_0)} \tag{3.31}$$

where u^+_0 is defined as $\mu I(D^+\setminus D^-)+u^{\partial D^-,-}_\infty|_{D^+}$. This in turn will be implied by

$$E\left[e^{+u^+_0(\hat{\phi}^+_0)}\,e^{-\mathbf{I}(h^{(2),+})-\mathbf{I}^{\partial D^+}(h^{(3),+})}\,e^{-\int_0^\infty u^-_s(h^{(2),-}_s)ds-\int_0^\infty du^{\partial D^-,-}_s(h^{(3),-}_s)}\right]=e^{-\mu(\hat{\phi}^-_0)} \tag{3.32}$$

for all $h^{(2),-},h^{(3),-}$.

A comparison of the martingale problem for time dependent test functions (3.5) with (3.27) shows that if $\phi : [0,\infty) \times \partial D^+ \to R$ and $C^{\phi,+}_s(y) = \phi_{\tau^+}(y_{\tau^+}) I(s > \tau^+)$ then $\int_0^t H_s(dC^{\phi,+}_s) = \int_0^t du^{\partial D^+}_s(\phi_s)$ (and a similar representation for a $du^{\partial D^-,-}_t$ integral). Suppose first that $\hat\phi^-$ is in $C^{1,2}_b([0,T]\times\overline{D^-})$. Then using (3.27) on the domain D^-, one finds that Ξ^-_t defined by

$$\Xi^-_t = e^{-u^-_t(\hat\phi^-_t) - \int_0^t u^-_s(h^{(2),-}_s)ds - \int_0^t du^{\partial D^-,-}_s(h^{(3),-}_s)}$$

is a martingale which satisfies $\Xi^-_\infty = e^{-\int_0^T u^-_t(h^{(2),-}_t)dt - \int_0^T du^{\partial D^-,-}_t(h^{(3),-}_t)}$ and

$$\Xi^-_t = e^{-\mu(\hat\phi^-_0)} - \int_0^t \int \Xi^-_s \, \hat\phi^-_s(y_s) I(s \le \tau^-, \tau^- > 0) Z(dy,ds) \quad \text{for } t \ge 0 \text{ a.s.} \tag{3.33}$$

An approximation argument shows (3.33) continues to hold when $\hat\phi^- \in C^{1,2}([0,T]\times D^-) \cap C([0,T]\times\overline{D^-})$. We now develop a similar representation for the Laplace functional of $(\mathbf{I}, \mathbf{I}^{\partial D^+})$. Set

$$Y_t = ((t-\tau^-)_+ - (t-\tau^+)_+), y_{t\wedge\tau^+}) = \left(\int_0^t I(s \in (\tau^-,\tau^+])ds, \int_0^t I(s \le \tau^+)dy(s)\right).$$

For $\phi \in C^{1,2}_b([0,\infty)\times R^d)$ we have, using the historical Ito formula again,

$$\begin{aligned}\int \phi(Y_t) H_t(dy) &= \mu(\phi_0) + \int_0^t \int \phi(Y_s) Z^0(dy,ds) \\ &\quad + \int_0^t \int \dot\phi(Y_s) I(s \in (\tau^-,\tau^+]) + \Delta\phi(Y_s) I(s \le \tau^+) H_s(dy)ds.\end{aligned}$$

On $\{t > \tau^+\}$ we have $Y_t = (\tau^+ - \tau^-, y_{\tau^+})$ so that

$$\begin{aligned}\int \phi(Y_t) I(t > \tau^+) H_t(dy) &= \int \phi_{\tau^+-\tau^-}(y_{\tau^+}) I(t > \tau^+) H_t(dy) \\ &= H_t(\hat C^\phi_t) + \int \phi_0(y_{\tau^-}) I(t > \tau^+ = \tau^-) H_t(dy).\end{aligned} \tag{3.34}$$

Using Theorem 2.23 of [11],

$$H_t(\hat C^\phi_t) = \int_0^t H_s(d\hat C^\phi_s) + \int_0^t \int \phi_{\tau^+-\tau^-}(y_{\tau^+}) I(s > \tau^+ > \tau^-) Z^0(dy,ds).$$

On $\{t \le \tau^-\}$ we have $Y_t = (0, y_t)$ so that

$$\begin{aligned}\int \phi(Y_t) I(t \le \tau^-) H_t(dy) &= \int \phi_0(y_{t\wedge\tau^-}) I(t \le \tau^-) H_t(dy) \\ &= \int \phi_0(y_{t\wedge\tau^-}) H_t(dy) + \int \phi_0(y_{\tau^-}) I(t > \tau^-) H_t(dy).\end{aligned} \tag{3.35}$$

By the historical Ito formula again

$$\int \phi_0(y_{t\wedge\tau^-}) H_t(dy) = \mu(\phi_0) + \int_0^t \int \phi_0(y_{s\wedge\tau^-}) Z^0(dy,ds) + \int_0^t \int \Delta\phi_0(y_s) I(s \le \tau^-) H_s(dy)ds.$$

We may combine some terms from (3.34) and (3.35) as follows, again using Theorem 2.23 of [11],

$$\begin{aligned}&\int \phi_0(y_{\tau^-}) I(t > \tau^-) H_t(dy) - \int \phi_0(y_{\tau^-}) I(t > \tau^+ = \tau^-) H_t(dy) \\ &= \int \phi_0 I(D^+)(y_{\tau^-}) I(t > \tau^-) H_t(dy) \\ &= H_t(C^{\phi_0 I(D^+),-}_t) + \int \phi_0 I(D^+)(y_0) I(t > \tau^- = 0) H_t(dy) \\ &= \int_0^t H_s(dC^{\phi_0 I(D^+),-}_s) + \mu(\phi_0 I(D^+ \setminus D^-)) + \int_0^t \int \phi_0 I(D^+)(y_{\tau^-}) I(t > \tau^-) Z^0(dy,ds).\end{aligned}$$

The last six displayed equations hold for all $t \geq 0$ almost surely, and a little book-keeping combines them to yield

$$\begin{aligned}
&\int \phi_{t-\tau^-}(y_t) I(t \in (\tau^-, \tau^+]) H_t(dy) + \int_0^t H_s(d\hat{C}_s^\phi) - \left(\mu(\phi_0 I(D^+ \setminus D^-)) + \int_0^t H_s(dC_s^{\phi_0 I(D^+),-}) \right) \\
&= \int_0^t \int \phi_{s-\tau^-}(y_s) I(s \in (\tau^-, \tau^+]) Z(dy, ds) + \int_0^t \int (\dot{\phi} + \Delta\phi - \gamma\phi)_{s-\tau^-}(y_s) I(s \in (\tau^-, \tau^+]) H_s(dy) ds.
\end{aligned}$$

If we now suppose that $\dot{\phi} + \Delta\phi - \gamma\phi - (1/2)\phi^2 + h = 0$ for some $h \in C_b([0,\infty) \times R^d)$ then Ξ_t^+ defined by

$$\begin{aligned}
\Xi_t^+ &= \exp(+\mu(\phi_0 I(D^+ \setminus D^-)) + \int_0^t H_s(dC_s^{\phi_0 I(D^+),-}) - \int \phi_{t-\tau^-}(y_t) I(t \in (\tau^-, \tau^+]) H_t(dy)) \\
&\quad \exp(- \int_0^t H_s(d\hat{C}_s^\phi) - \int_0^t \int h_{s-\tau^-}(y_s) I(s \in (\tau^-, \tau^+]) H_s(dy) ds)
\end{aligned}$$

is a non-negative local martingale and satisfies

$$\Xi_t^+ = 1 - \int_0^t \int \Xi_s^+ \phi_{s-\tau^-}(y_s) I(s \in (\tau^-, \tau^+]) Z(dy, ds) \quad \text{for } t \geq 0 \text{ a.s.} \tag{3.36}$$

An approximation argument shows that (3.36) continues to hold with $\phi = \hat{\phi}^+$ and $h = h^{(2),+}$. With this choice, and extending $\hat{\phi}_t^+ = 0$ for $t \geq T$, we let $t \to \infty$ to find that

$$\Xi_t^+ \to \exp\left(+u_0^+(\hat{\phi}_0^+) - \mathbf{I}(h^{(2),+}) - \mathbf{I}^{\partial D^+}(h^{(3),+}) \right).$$

Combining (3.33) and (3.36) we have

$$\begin{aligned}
&e^{+u_0^+(\hat{\phi}_0^+) - \mathbf{I}(h^{(2),+}) - \mathbf{I}^{\partial D^+}(h^{(3),+})} \, e^{-\int_0^\infty u_t^-(h_t^{(2),-})dt - \int_0^\infty du_t^{\partial D^-,-}(h_t^{(3),-})} \\
&= e^{-\mu(\hat{\phi}_0^-)} - \int_0^\infty \int \Xi_t^- \Xi_t^+ \hat{\phi}_{t-\tau^-}^+(y_t) I(t \in (\tau^-, \tau^+]) Z(dy, dt) \\
&\quad - \int_0^\infty \int \Xi_t^+ \Xi_t^- \hat{\phi}_t^-(y_t) I(t \leq \tau^-, \tau^- > 0) Z(dy, dt)
\end{aligned} \tag{3.37}$$

(noting that the cross variation of the two stochastic integrals is zero). Note that $\Xi_t^+ \leq \exp(+\|\hat{\phi}^+\|_\infty u_0^+(1))$. The quadratic variation of the first stochastic integral on the right hand side of (3.37) is therefore bounded

$$\|\hat{\phi}^+\|_\infty^2 \exp(+2\|\hat{\phi}^+\|_\infty(\mu(1) + u_\infty^{\partial D^-,-}(1)) u_{[0,\infty)}(1)$$

The finiteness of small positive exponential moments for the total exit measure $u_\infty^{\partial D^-,-}(1)$ implies that the stochastic integral has mean zero if the norm $\|\hat{\phi}^+\|_\infty$ is small, which in turn is satisfied if the norms $\|h^{(2),+}\|_\infty$ and $\|h^{(3),+}\|_\infty$ are small enough. Similarly the other stochastic integral is mean zero. Taking expectations in (3.37) establishes the conditional Laplace functional (3.31) when $\|h^{(2),+}\|_\infty$ and $\|h^{(3),+}\|_\infty$ are sufficiently small. But this is sufficient to characterize the conditional law. □

3.2 Existence and uniqueness via change of measure

Proof of Theorem 2. Many of the change of measure arguments we will use can be found in section IV.1 of Perkins [10]. Fix $f \in \mathcal{B}(D, [0,1])$, and on the path space $(\Omega_D, \mathcal{U}, \mathcal{U}_t, Q_\mu^{D,0})$ use the path space density from (3.18) to set $V_t(x) = f(x) \exp(-U(0,t,x))$, where we use the convention $\exp(-\infty) = 0$. Note that V is $\mathcal{P} \times \mathcal{B}(D)$ measurable and has non-increasing paths as required. Define the stochastic integral

$$M_t^{\beta,\gamma,f} = \int_0^t \int_D (\beta V_s(x) - \gamma) M(dx, ds) \tag{3.38}$$

where $M(ds,dx)$ is the martingale measure on $[0,\infty)\times D$ constructed from U under $Q^{D,0}_\mu$ (as explained in section 3.1.1). A simple version of the arguments in [10] Theorem IV.1.6.(b) implies that the stochastic exponential $\mathcal{E}_t(M^{\beta,\gamma,f})$ is a true martingale, so that there exists a measure $Q^{D,\beta,\gamma}_{\mu,f}$ on the path space $(\Omega_D,\mathcal{U})$ satisfying $dQ^{D,\beta,\gamma}_{\mu,f}/dQ^{D,0}_\mu = \mathcal{E}_t(M^{\beta,\gamma,f})$ on $\mathcal{U}_t$. It is straightforward to check via Girsanov's theorem that the triple $(U_t,U^{\partial D}_t,V_t)$ solve, under $Q^{D,\beta,\gamma}_{\mu,f}$, the extended martingale problem (2.3) and (2.4) for solutions to (1.1). The regularity of $U(s,t,x)$ will imply that (2.2) holds, namely for all bounded measurable ϕ

$$(V_t,\phi)=(f,\phi)-\int_0^t U_s(V_s\phi)ds, \quad Q^{D,\beta,\gamma}_{\mu,f} \text{ a.s.} \tag{3.39}$$

This would be trivial, by the fundamental theorem of calculus, were $U_s(dx)$ to have continuous density in (s,x). We show at the end of this section that it follows in our more singular setting. This completes the proof of part (i) of Theorem 2.

To show uniqueness of solutions, we take a solution (u,v) to (1.1) on D, on a filtered space $(\Omega,(\mathcal{F}_t),\mathcal{F},P)$, with an arbitrary $\mathcal{F}_0$ measurable starting condition. Let $m(dx,dt)$ be the local martingale measure extending the local martingales $m_t(\phi)$. We now reverse the change of measure argument. The local martingale defined by

$$m^{\beta,\gamma,f}_t=\mathcal{E}_t\left(\int\int(\gamma-\beta v_s(x))m(dx,ds)\right)$$

is in fact a true martingale (argue as in [10] Theorem IV.1.6.(a), noting that the deterministic initial conditions there are not needed for this argument). Defining Q by $dQ/dP=m^{\beta,\gamma,f}_t$ on $\mathcal{F}_t$, one finds that (u_t) solves the martingale problem for a DW$(D,0)$ process under Q, and so Q is determined on $\sigma\{u_s: s\le t\}$ by the law of u_0. By Sugitani's results listed in section 3.1, the occupation measures $u_{[s,t]}$ have densities (under P or Q), and moreover that there is a density $(t,x)\to u(s,t,x)$ that is continuous on $\{(s,t,x): 0<s\le t, x\in D\}$. Knowing this one shows that v is given by (see the argument at the end of this section)

$$v_t(x)=v_0(x)e^{-u(0,s,x)} \quad \text{for all } t\ge 0, \text{ for almost all } x,\ P \text{ a.s.} \tag{3.40}$$

where $u(0,s,x)=\lim_{r\downarrow 0}u(r,s,x)$. Now we may replace the change of measure by

$$dQ/dP=\mathcal{E}_t\left(\int\int\left(\gamma-\beta v_0(x)e^{-u(0,s,x)}\right)m(dx,ds)\right) \quad \text{on } \mathcal{F}_t.$$

since the two exponential martingales are equal upto a null set. Inverting one finds

$$dP=\mathcal{E}_t\left(\int\int\left(\beta v_0(x)e^{-u(0,s,x)}-\gamma\right)\tilde m(dx,ds)\right)dQ \quad \text{on } \mathcal{F}_t.$$

where $\tilde m(dx,ds)$ is the local martingale measure determined by the local martingales found in the martingale problem of u under Q. The stochastic exponential is a measurable function of $(v_0,\{u_s: s\le t\})$. Applying the Markov property of u under Q at time $t=0$, we find that P is determined on $\sigma\{u_s: s\le t\}$ by the law of (u_0,v_0). Indeed, for each $t\ge 0$ there is a measurable kernel $p_t(\mu,f,B)$ for $f\in\mathcal{B}(D,[0,1])$, $\mu\in\mathcal{M}(D)$ and Borel $B\subseteq\mathcal{M}(D)$ so that

$$P[u_t\in dB]=\int_{\mathcal{B}}\int_{\mathcal{M}(D)}p_t(\mu,f,B)P[u_0\in d\mu,\ v_0\in df]. \tag{3.41}$$

Under Q, and therefore under P, there exists an exit measure process $u^{\partial D}$ satisfying $u^{\partial D}_t=R_t(\{u_s: s\le t\})$ almost surely. Another Girsanov calculation starting with the extended martingale problem under Q, shows that the exit measures satisfy the extended martingale problem for (1.1) under P. This completes the proof of parts (ii),(iii) and (iv) of Theorem 2. For part (v), note that the measurability of the maps $(\mu,f)\to Q^{D,\beta,\gamma}_{\mu,f}(B)$ follows from the analogous measurability of $Q^{D,0}_\mu$ and the fact that, for $t\ge 0$, the derivative $dP/dQ^{D,0}_\mu$ on $\mathcal{F}_t$ is a measurable map of $(f,\{u_s: s\le t\})$. The strong Markov property for the laws $Q^{D,\beta,\gamma}_{\mu,f}$ can be deduced either from the strong Markov property for the DW laws $Q^{D,\gamma}_\mu$, or more easily from the uniqueness of the one dimensional distributions for the martingale problem (use (3.41) and follow the argument in [10] Theorem II.5.6). □

Remark. It is straightforward to show that uniform integrability of the martingale $t \to \mathcal{E}_t(M^{\beta,\gamma,1})$ under $Q_\mu^{R^d,0}$ for some $\mu \neq 0$ (or that of $t \to \mathcal{E}_t(M^{\beta,0,1})$ under $Q_\mu^{R^d,\gamma}$) is equivalent to certain death for the parameters (β,γ). See the remark in section 7.1.

Proof of (3.39). Suppose $(\nu_s : 0 \le s \le t)$ is a continuous path in $\mathcal{M}(R^d)$. Suppose also that $(n(s,x) : s \le t, x \in R^d)$ is bounded and continuous and that $\int_0^s \nu_r dr = n(s,x)dx$ as measures for $s \le t$. Let T_δ denote the heat semigroup with generator Δ, and write T_δ^* for the dual semigroup acting on measures. For continuous compactly supported ψ, we shall argue that

$$\begin{aligned}\int (1 - e^{-n(t,x)})\psi(x)dx &= \lim_{\delta \downarrow 0} \int (1 - e^{-T_\delta n(t,x)})\psi(x)dx \\ &= \lim_{\delta \downarrow 0} \int_0^t \int e^{-T_\delta n(s,x)} \psi(x) T_\delta^* \nu_s(dx)ds \\ &= \int_0^t \int e^{-n(s,x)} \psi(x) \nu_s(dx)ds.\end{aligned}$$

Since the measures $T_\delta^* \nu_s$ have densities, the paths $s \to T_\delta n(s,x)$ are absolutely continuous for almost all x. So the second equality above follows by expanding $s \to \exp(-T_\delta n(s,x))$ as an integral of its derivative over $[0,t]$ and then interchanging s and x integrals. The third equality holds since $T_\delta n(s,x) \to n(s,x)$ uniformly over $s \le t$ and x in the support of ψ, and since $T_\delta^* \nu_s \to \nu_s$ weakly.

The assumptions above hold, $Q_{\mu,f}^{D,\beta,\gamma}$ almost surely, if we take $\nu_s(dx) = \overline{\psi}(x) U_{\epsilon+s}(dx)$, for some $\epsilon > 0$ and continuous $\overline{\psi}$, with compact support in D, and take the corresponding density $n(s,x) = \overline{\psi}(x) U(\epsilon, \epsilon+s, x)$. Choosing continuous ψ compactly supported in D and $\overline{\psi} \equiv 1$ on the support of ψ we obtain, almost surely,

$$\int e^{-U(\epsilon,t,x)} \psi(x)dx = \int \psi(x)dx - \int_\epsilon^t \int e^{-U(\epsilon,s,x)} \psi(x) U_s(dx)ds.$$

To pass to the limit $\epsilon \downarrow 0$ in this equation note that for $t \ge \epsilon$

$$\int \left| e^{-U(\epsilon,t,x)} - e^{-U(0,t,x)} \right| dx \le \int U(0,\epsilon,x)dx = \int_0^\epsilon U_s(1)ds \to 0 \quad \text{as } \epsilon \downarrow 0,$$

and similarly

$$\begin{aligned}&\left| \int_\epsilon^t \int e^{-U(\epsilon,s,x)} \psi(x) U_s(dx)ds - \int_\epsilon^t \int e^{-U(0,s,x)} \psi(x) U_s(dx)ds \right| \\ \le\ & \|\psi\|_\infty \int_\epsilon^t U_s(U(0,\epsilon))ds \\ \le\ & \|\psi\|_\infty \int U(0,t,x) U(0,\epsilon,x)dx\end{aligned}$$

which converges in $\mathcal{L}^2$ to zero, as $\epsilon \downarrow 0$, by (3.22). This leads to

$$\int e^{-U(0,t,x)} \psi(x)dx = \int \psi(x)dx - \int_0^t \int e^{-U(0,s,x)} \psi(x) U_s(dx)ds$$

for continuous compactly supported ψ. We may extend to general bounded measurable $\psi : D \to R$ by monotone class arguments, and choosing $\psi = f\phi$ for $\phi \in C_b^0(\overline{D})$ shows that (3.39) holds. □

Proof of (3.40). First, we may use a density argument to ensure $\langle v_t, \phi \rangle = \langle v_0, \phi \rangle - \int_0^t u_s(v_s \phi)ds$ for all $t \ge 0$ and all continuous ϕ with compact support in D, P a.s. Now we can argue pathwise. Fix a sample point ω so that the above happens and so that also $(r,s,x) \to u(r,s,x)$ is continuous over $x \in D, 0 < r \le s \le t$. Choose a partition $0 < s = t_0 < t_1 < \ldots < t_N = t$ with $\max_i(t_i - t_{i-1}) = \delta$. Choosing $\phi(x) = \exp(-u(t_{n-1}, t, x))\psi(x)$, for some continuous $\psi \geq 0$ with compact support $A \subseteq D$, we find

$$\left\langle v_{t_n}, e^{-u(t_{n-1},t)}\psi \right\rangle = \left\langle v_{t_{n-1}}, e^{-u(t_{n-1},t)}\psi \right\rangle - \int_{t_{n-1}}^{t_n} u_s \left(v_s e^{-u(t_{n-1},t)}\psi \right) ds.$$

A little rearrangement leads to

$$\left\langle v_{t_n}, e^{-u(t_n,t)}\psi \right\rangle = \left\langle v_{t_{n-1}}, e^{-u(t_{n-1},t)}\psi \right\rangle - E_n^1 + E_n^2$$

where $E_n^i \geq 0$ are given by

$$E_n^1 = \int_{t_{n-1}}^{t_n} u_s \left(e^{-u(t_{n-1},t)}(v_s - v_{t_n})\psi \right) ds \leq \|\psi\|_\infty \left\langle u(t_{n-1}, t_n) I(A), v_{t_{n-1}} - v_{t_n} \right\rangle$$

and, using $0 \leq e^z - 1 - z \leq 2z^2$ for $z \in [0, 1]$,

$$E_n^2 = \left\langle v_{t_n} \psi e^{-u(t_{n-1},t)}, e^{u(t_{n-1},t_n)} - 1 - u(t_{n-1}, t_n) \right\rangle \leq 2\|\psi\|_\infty \left\langle I(A), u^2(t_{n-1}, t_n) \wedge 1 \right\rangle.$$

Now sum over the partition to find

$$\langle v_t, \psi \rangle = \left\langle v_s, e^{-u(s,t)}\psi \right\rangle - \sum_n E_n^1 + \sum_n E_n^2.$$

The error bounds above, and the fact that $\max\{u(s', t', x) : s \leq s' \leq t' \leq t, |s' - t'| \leq \delta, x \in A\} \to 0$ as $\delta \to 0$, imply that $\sum_n E_n^i \to 0$ as $\delta \to 0$. Thus $v_t(x) = v_s(x) \exp(-u(s, t, x))$ for almost all x. The non-increasing paths $t \to v_t(x)$ ensures that $v_t(x) = v_s(x) \exp(-u(s, t, x))$ for all $0 < s < t$, for almost all x. Since $\lim_{s \downarrow 0} v_s(x) = v_0(x)$ for almost all x (use the non-increasing paths of v and (2.2)) we reach the conclusion (3.40). □

We conclude this section with the proof of local extinction of solutions.

Lemma 9. *Suppose that $\gamma > 0$. Then, for compact $A \subseteq R^d$,*

$$Q_{\mu,f}^{R^d,\beta,\gamma} \left[U_{[t,\infty)}(A) = 0 \right] \to 1 \qquad \text{as } t \to \infty.$$

Proof. Although not directly used, we first give a weaker fixed t extinction result whose proof is similar to the support results in section 3.3. Fix $R > 0$ and let $0 \leq \psi_R \leq 1$ be smooth and satisfy $\{\psi_R > 0\} = B_R = \{x : |x| < R\}$. Let $\phi_t^{(M)}(x)$ be the solution, for $x \in R^d$, $t \geq 0$, of

$$\partial_t \phi = \Delta\phi - \gamma\phi - \frac{\phi^2}{2} + \beta\phi I(t \leq 1), \quad \phi_0 = M\psi_R.$$

Then stochastic calculus shows that, under $Q_{\mu,f}^{R^d,\beta,\gamma}$,

$$d\left(e^{-U_s(\phi_{t-s}^{(M)})} \right) \geq -\beta e^{-U_s(\phi_{t-s}^{(M)})} U_s(V_s \phi_{t-s}^{(M)}) I(s \leq t-1) ds \quad + \text{ martingale increments}$$

so that,

$$\begin{aligned} Q_{\mu,f}^{R^d,\beta,\gamma} \left[U_t(B_R) = 0 \right] &= \lim_{M \to \infty} Q_{\mu,f}^{R^d,\beta,\gamma} \left[e^{-MU_t(\psi_R)} \right] \\ &\geq e^{-\mu(\phi_t)} - \beta Q_{\mu,f}^{R^d,\beta,\gamma} \left[\int_0^\infty U_s(V_s \phi_{t-s}) I(s \leq t-1) ds \right], \end{aligned} \tag{3.42}$$

where $\phi = \lim_{M \to \infty} \phi^{(M)}$. As in the extinction results in section 3.1.4, a comparison argument shows that ϕ_t is bounded for all $t > 0$. Moreover, then $\|\phi_t\|_\infty \to 0$ as $t \to \infty$, so that the term $\exp(-\mu(\phi_t)) \to 1$. Similarly, for fixed (s,ω) we have $U_s(V_s\phi_{t-s}) \to 0$ as $t\to\infty$. To justify the conclusion that $Q^{R^d,\beta,\gamma}_{\mu,f}[U_t(B_R)=0] \to 0$ as $t\to\infty$ we will dominate the second term on the right hand side of (3.42) by

$$Q^{R^d,\beta,\gamma}_{\mu,f}\left[\int_0^\infty U_s(V_s\phi^*)ds\right] = Q^{R^d,\beta,\gamma}_{\mu,f}\left[\int \phi^*(x)f(x)(1-e^{-U(0,\infty,x)})dx\right] \le \langle \phi^*,1\rangle$$

where $\phi^*(x) = \sup_{t\ge 1}\phi_t(x)$, and where we are argue as in (3.39) for the first equality. But, for $\gamma>0$ and where T_t is convolution with the heat kernel $p_t(x)$,

$$\begin{aligned}\phi^*(x) &\le \sup_{t\ge 0} e^{-\gamma t}T_t\phi_1(x)\\ &\le \sup_t\left\{\int_{|y|\le 1}\phi_1(x+y)e^{-\gamma t}p_t(y)dy\right\} + \int_{|y|\ge 1}\phi_1(x+y)\sup_t\{e^{-\gamma t}p_t(y)\}dy\\ &\le \sup\{\phi_1(z): |z-x|\le 1\} + C(\gamma)\int \phi_1(x+y)e^{-\gamma^{1/2}|y|}dy.\end{aligned}$$

The fact that $\langle\phi^*,1\rangle<\infty$ can now be deduced from the boundedness and exponential decay of ϕ_1, which follow from comparison arguments as in [3] Lemma 3.1.

The stronger conclusion of the lemma follows by a similar argument using a slightly more complicated test function. Suppose that ψ_R satisfies in addition $\psi_R = 1$ on B_{R-1}. Fix $t_0,t_1\ge 0$ and redefine ϕ as the solution to

$$\partial_t\phi = \Delta\phi - \gamma\phi - \frac{\phi^2}{2} + M\psi_R I(t\le t_1) + \beta\phi\psi_{R+2}I(t\le t_1+1), \qquad \phi_0 = 0.$$

Some calculus shows that, under $Q^{R^d,\beta,\gamma}_{\mu,f}$,

$$\begin{aligned}&d(\exp(-U_s(\phi_{t_0+t_1-s}) - MU_{[t_0,t_0\vee s]}(\psi_R)))\\ &= \exp(-U_s(\phi_{t_0+t_1-s}) - MU_{[t_0,t_0\vee s]}(\psi_R))\,(-\beta U_s(V_s\phi_{t_0+t_1-s}) + \beta U_s(\phi_{t_0+t_1-s}\psi_{R+2})I(s\ge t_0-1))\,dt\end{aligned}$$

up to martingale increments. Taking expectations we have

$$\begin{aligned}&Q^{R^d,\beta,\gamma}_{\mu,f}\left[e^{-MU_{[t_0,t_0+t_1]}(\psi_R)}\right]\\ &\ge e^{-\mu(\phi_{t_0+t_1})} - \beta Q^{R^d,\beta,\gamma}_{\mu,f}\left[\int_0^{t_0+t_1} U_s(\phi_{t_0+t_1-s}V_s) - U_s(\phi_{t_0+t_1-s}\psi_{R+2})I(s\ge t_0-1)ds\right]. \qquad (3.43)\end{aligned}$$

There is a unique non-negative solution $\overline{\phi}$ (see the arguments of [7]) to

$$\Delta\overline{\phi} = \gamma\overline{\phi} + \frac{\overline{\phi}^2}{2} - \beta\overline{\phi}\psi_{R+2} \quad \text{on } B_R^c \text{ and } \overline{\phi}(x)\uparrow\infty \text{ as } |x|\downarrow R.$$

A comparison argument shows that $\phi_t(x)\le\overline{\phi}(x)$ for $t\le t_1+1$ and $|x|>R$. Also $\phi_{t_1+t}\le\lambda_t$ for $t>0$ where

$$\dot{\lambda}_s = -\gamma\lambda_s - \frac{\lambda_s^2}{2} + \beta\lambda_s I(s\le 1) \quad \text{and } \lambda_s\uparrow\infty \text{ as } s\downarrow 0.$$

Moroever, as in the fixed t argument,

$$\phi_{t_1+1+s} \le e^{-\gamma s}T_s\phi_{t_1+1} \le e^{-\gamma s}T_s(\overline{\phi}\wedge\lambda_1) := \overline{\phi}_s.$$

Using these bounds in (3.43) we obtain for $t_0>1$

$$\begin{aligned}&Q^{R^d,\beta,\gamma}_{\mu,f}\left[e^{-MU_{[t_0,t_0+t_1]}(\psi_R)}\right]\\ &\ge e^{-\lambda_{t_0}\mu(1)} - \beta Q^{R^d,\beta,\gamma}_{\mu,f}\left[\int_0^{t_0-1} U_s(V_s\overline{\phi}_{t_0-s-1})ds + \int_{t_0-1}^\infty U_s(V_s\overline{\phi}I(B^c_{R+1}))ds\right].\end{aligned}$$

Now we let $M,t_1\to\infty$ to obtain the same upper bound for $Q^{R^d,\beta,\gamma}_{\mu,f}[U_{[t_0,\infty)}(B_R^c)=0]$. As $t_0\to\infty$ we have $\lambda_{t_0}\to 0$. The conclusion of the lemma follows as in the fixed t result using the domination $\int_0^\infty U_s(V_s\overline{\phi}^*)ds \le \langle\overline{\phi}^*,1\rangle$ where $\overline{\phi}^* = \max\{\overline{\phi}I(B^c_{R+1}), \sup_s\overline{\phi}_s\}$. The integrability $\langle\overline{\phi}^*,1\rangle<\infty$ follows as in the fixed t result using the exponential decay of the function $\overline{\phi}$. □

3.3 Existence of the critical curve $\Psi(\beta)$

For a $\mathrm{DW}(D,\gamma)$ process the death time time $\tau = \inf\{t : U_t = 0\}$ is finite, and then $U_{[0,\infty)} = U_{[0,\tau]}$ and $U_\infty^{\partial D} = U_\tau^{\partial D}$ are finite measures, $Q_\mu^{D,\gamma}$ almost surely. Using the change of measure from section 2.2 the same holds for the law $Q_{\mu,f}^{D,\beta,\gamma}$ when D is a bounded domain. Indeed the change of measure martingale $M^{\beta,0,f}$ from (3.38) satisfies

$$\begin{aligned}
\left[M^{\beta,0,f}\right]_t &= \beta^2 \int_0^t \int_D f^2(x) e^{-2U(0,s,x)} U_s(dx) ds \\
&= \frac{\beta^2}{2} \int_D f^2(x) \left(1 - e^{-2U(0,t,x)}\right) dx
\end{aligned}$$

by arguing as in the proof of (3.39). This is bounded independently of t, for a bounded domain D, so that $\mathcal{E}_t(M^{\beta,0,f})$ is a uniformly integrable martingale, ensuring the above almost sure properties carry over to the law $Q_{\mu,f}^{D,\beta,\gamma}$.

A similar argument leads to the following lemma, which shows that either certain death or possible life must occur, for each pair of parameter values β, γ.

Lemma 10. *If $Q_{\mu,1}^{R^d,\beta,\gamma}[U_t \neq 0$ for all $t]$ is strictly positive for some $\mu \in \mathcal{M}(R^d)$ then it is strictly positive for all non-zero $\mu \in \mathcal{M}(R^d)$.*

Proof. Step 1: We claim that the laws of solutions $Q_{\mu,f}^{R^d,\beta,\gamma}$ and $Q_{\mu,g}^{R^d,\beta,\gamma}$ are mutually absolutely continuous on the entire sigma field $\mathcal{U}_\infty = \sigma\{U_t : t \geq 0\}$, provided that f and g differ only on a compact set. To see this, consider the Radon-Nikodym derivative between the two laws, which is the stochastic exponential $\mathcal{E}(M)$ arising from the martingale $M_t = \beta \int_0^t \int_{R^d} (f-g)(x) e^{-U_{[0,s]}(x)} M(dx, ds)$. Then, if $0 \leq f, g \leq 1$ are supported inside the compact set A,

$$\begin{aligned}
[M]_t &= \beta^2 \int_0^t \int_{R^d} (f-g)^2(x) e^{-2U_{[0,s]}(x)} U_s(dx) ds \\
&\leq \beta^2 \int_0^t \int_A e^{-2U_{[0,s]}(x)} U_s(dx) ds \\
&= \frac{\beta^2}{2} \int_A \left(1 - e^{-2U_{[0,t]}(x)}\right) dx \leq \frac{\beta^2}{2} |A|.
\end{aligned} \tag{3.44}$$

The exponential martingale is therefore uniformly integrable which establishes the claim.

Step 2: A result of Evans and Perkins [6] shows that, for any fixed $t > 0$, the laws $Q_\mu^{D,0}[U_t \in \cdot]$ and $Q_{\tilde{\mu}}^{D,0}[U_t \in \cdot]$ of two $\mathrm{DW}(R^d, 0)$ processes started at non-zero μ, $\tilde{\mu}$ are mutually absolutely continuous on $\mathcal{M}(R^d)$. The change of measure ideas used to show existence and uniqueness imply that the same absolute continuity holds for $Q_{\mu,f}^{R^d,\beta,\gamma}[U_t \in \cdot]$ and $Q_{\tilde{\mu},f}^{R^d,\beta,\gamma}[U_t \in \cdot]$.

Now we combine the two steps. Suppose μ, $\tilde{\mu}$ are non-zero and compactly supported, and that started at $(\mu, 1)$ the solutions may survive. Then

$$\begin{aligned}
0 &< Q_{\mu,1}^{R^d,\beta,\gamma}[U_t \neq 0 \text{ for all } t] \\
&= \int_{\mathcal{B}(R^d,[0,1]) \times \mathcal{M}(R^d)} Q_{\nu,g}^{R^d,\beta,\gamma}[U_t \neq 0 \text{ for all } t] \, Q_{\mu,1}^{R^d,\beta,\gamma}[U_{t_0} \in d\nu, \, f e^{-U(0,t_0)} \in dg]
\end{aligned}$$

by the Markov property at time $t_0 > 0$. By the compact support property of solutions, $f \exp(-U(0,t_0))$ is identically one outside a compact set, almost surely. So, using step one,

$$\begin{aligned}
0 &< \int_{\mathcal{B}(R^d,[0,1]) \times \mathcal{M}(R^d)} Q_{\nu,1}^{R^d,\beta,\gamma}[U_t \neq 0 \text{ for all } t] \, Q_{\mu,1}^{R^d,\beta,\gamma}[U_{t_0} \in d\nu, \, f e^{-U(0,t_0)} \in dg] \\
&= \int_{\mathcal{M}(R^d)} Q_{\nu,1}^{R^d,\beta,\gamma}[U_t \neq 0 \text{ for all } t] \, Q_{\mu,1}^{R^d,\beta,\gamma}[U_{t_0} \in d\nu].
\end{aligned}$$

Now using step two above we obtain $0 < \int_{\mathcal{M}} Q^{R^d,\beta,\gamma}_{\nu,1}[U_t \neq 0 \text{ for all } t]\, Q^{R^d,\beta,\gamma}_{\tilde{\mu},1}[U_{t_0} \in d\nu]$ and undoing the steps we find that the process with initial conditions $(\tilde{\mu}, 1)$ may survive.

To remove the restrictions that $\mu, \tilde{\mu}$ be compactly supported, recall that the solutions are compactly supported at any time $t > 0$ almost surely. Hence, if solutions may live from some general $\mu \neq 0$, they may, by applying the Markov property at time $t > 0$, live from some, and hence all, non-zero compactly supported initial conditions. Any solution started at non-zero $\tilde{\mu}$ will have positive probability of being alive and compactly supported at small times $t > 0$ (use the continuity of the total mass process) and hence, by the Markov property again, may live for all time. □

The next two lemmas give a characterization of certain death, which allows us to reduce the question of possible life/certain death to the study of the total exit measures solutions on bounded domains. The existence of a non-decreasing critical curve follows from this, since these exit measures are stochastically monotone in β, γ.

Lemma 11. *Suppose $\mu \in \mathcal{M}(R^d)$ is compactly supported. Then $Q^{R^d,\beta,\gamma}_{\mu,f}$ almost surely*

$$\{U_t = 0 \text{ for large } t\} = \{U_{[0,\infty)} \text{ is compactly supported}\}.$$

Proof. The inclusion $\{U_t = 0 \text{ for large } t\} \subseteq \{U_{[0,\infty)} \text{ is compactly supported}\}$ follows from the compact support property (3.16). Let $D_L = (-L, L)^d$. We claim for any L, P a.s., there exists a $t > 0$ so that either $U_t(dx) = 0$ or $U_t(D_L^c) > 0$ must occur. This implies the opposite inclusion.

Applying the first and second moment bounds (3.12,3.13), and the simple estimate $P[Z > 0] \geq (E[Z])^2/E[Z^2]$ for a non-negative variable Z, we may find $\epsilon_1 > 0$, depending only L, β, γ, so that $Q^{D,\beta,\gamma}_{\mu,f}[U_1(D_L^c) > 0] \geq \epsilon_1$ for all $f \in \mathcal{B}(D, [0,1])$ and $\mu(1) \geq 1$. Using the extinction probability (3.14) we may find $\epsilon_2 > 0$, depending only on β, γ, so that $Q^{D,\beta,\gamma}_{\mu,f}[U_1 = 0] \geq \epsilon_2$ for all $f \in \mathcal{B}(D, [0,1])$ and $\mu(1) \leq 1$. Applying the Markov property at time t, we obtain

$$Q^{R^d,\beta,\gamma}_{\mu,f}\left[U_{t+1}(dx) = 0 \text{ or } U_{t+1}(D_L^c) > 0 \,|\, \mathcal{U}_t\right] \geq \epsilon_1 \wedge \epsilon_2.$$

Iterating this estimate over the integers $t = 1, 2, \ldots$ and applying Borel-Cantelli completes the argument. □.

Lemma 12. *Suppose $\mu \in \mathcal{M}$ is compactly supported in D. Then*

$$Q^{R^d,\beta,\gamma}_{\mu,1}\left[U_{[0,\infty)}(D^c) = 0\right] = Q^{D,\beta,\gamma}_{\mu,1}[U^{\partial D}_\infty = 0].$$

Proof. We may construct a coupling of solutions (u^-, v^-) and (u^+, v^+) as follows: (u^-, v^-) is a solution to (1.1) on D with initial conditions $(\mu, 1)$; $u^{\partial D,-}$ is the associated exit measure process; $\tau = \inf\{t : u^{\partial D,-}_t > 0\}$; and conditional on $\sigma\{u^-, v^-\}$ the process (u^+, v^+) has the law of a solution on R^d started at the random initial condition $u^+_0 = u^-_\tau$ and $v^+_0 = v^-_\tau$. One way to do this is to use the measurability of $(\mu, f) \to Q^{D,\beta,\gamma}_{\mu,f}$ and construct a skew-product measure on the product space $\Omega_D \times \Omega_{R^d}$. We may also suppose that $v^-_t = \exp(-u^-(0,t))$ and $v^+_t = \exp(-u^-(0,\tau) - u^+(0,t))$ for all $t \geq 0$. Now define $(u_t, v_t) = (u^-_t, v^-_t)$ for $t \leq \tau$ and $(u_t, v_t) = (u^+_{t-\tau}, v^+_{t-\tau})$ for $t \geq \tau$. Then v_t is a measurable function of $(u_s : s \leq t)$ and this can be used to check that, with respect to the filtration $\sigma\{u_s : s \leq t\}$, the process v is predictable and non-increasing. For $\phi \in C^2_b(R^d)$, we may apply the extended martingale problem for (u^-, v^-) to the restriction of ϕ on $\overline{D}$. Combining with the martingale problem for (u^+, v^+) we find that (u, v) is a solution on R^d with initial conditions $(\mu, 1)$. Moreover on the set $\{\tau = \infty\}$ we have that $u_{[0,\infty)}(D^c) = u^-_{[0,\infty)}(D^c) = 0$. Thus

$$Q^{R^d,\beta,\gamma}_{\mu,1}\left[U_{[0,\infty)}(D^c) = 0\right] \geq P\left[\tau = \infty\right] = Q^{D,\beta,\gamma}_{\mu,1}\left[U^{\partial D}_\infty = 0\right].$$

For the converse inclusion we construct (u^-, v^-) a solution to (1.1) on R^d with initial conditions $(\mu, 1)$; $\tau = \inf\{t : u^-_t(D^c) > 0\}$; and conditional on $\sigma\{u^-, v^-\}$ a process (u^+, v^+) with the law of a solution on D started at the random initial condition $u^+_0 = u^-_\tau$ and $v^+_0 = v^-_\tau$. A result of Perkins (see [10] Theorem III.5.1 and use absolute continuity) implies that the paths of $t \to u^-_t(D^c)$ are almost surely continuous. So we may define (u_t, v_t) as above and the process u is almost surely continuous. A test function $\phi \in C_0^2(\overline{D})$ may be extended to $\tilde{\phi} \in C_b^2(R^d)$. Applying the martingale problem for (u^-, v^-) to $\tilde{\phi}$ and the martingale problem for (u^+, v^+) to ϕ, we find that (u, v) is a solution on D with initial conditions $(\mu, 1)$. Moreover, choosing $u_t^{\partial D} = u_{(t-\tau)_+}^{\partial D,+}$ on $t > \tau$ and $u_t^{\partial D} = 0$ for $t \leq \tau$ one finds the extended martingale problem is solved. Then on the set $\{\tau = \infty\}$ we have that $u_\infty^{\partial D} = 0$ and thus

$$Q_{\mu,1}^{D,\beta,\gamma}\left[U_\infty^{\partial D} = 0\right] \geq P\left[\tau = \infty\right] = Q_{\mu,1}^{R^d,\beta,\gamma}\left[U_{[0,\infty)}(D^c) = 0\right]$$

completing the proof. □

Using the above results and the comparison results stated in section 2.2 we will now deduce the existence of an non-decreasing critical curve Ψ as in Theorem 1.

Corollary 13. *There exists an non-decreasing function $\Psi(\beta) : [0, \infty) \to [0, \infty]$ so that for $0 \leq \gamma < \Psi(\beta)$ possible life occurs and for $\gamma > \Psi(\beta)$ certain death occurs.*

Proof. Lemmas 10, 11 and 12 show, for $\mu \neq 0$, that $Q_{\mu,1}^{R^d,\beta,\gamma}[U_t = 0 \text{ for large } t \geq 0] = 1$ if and only if $Q_{\delta_0,1}^{R^d,\beta,\gamma}[U_t = 0 \text{ for large } t \geq 0] = 1$ if and only if $Q_{\delta_0,1}^{R^d,\beta,\gamma}[U_{[0,\infty)} \text{ is compactly supported}] = 1$ if and only if

$$\sup_n Q_{\delta_0,1}^{D_n,\beta,\gamma}[U_\infty^{\partial D_n} = 0] = 1.$$

But Lemma 5 shows that these probabilities are non-increasing in β and non-decreasing in γ. The result follows by setting $\Psi(\beta) = \sup\{\gamma \geq 0 : \sup_n Q_{\delta_0,1}^{D_n,\beta,\gamma}[U_\infty^{\partial D_n} = 0] < 1\}$. □

Remark. The change of measure (3.38) can be used to show that $(\beta, \gamma) \to Q_{\mu,1}^{D,\beta,\gamma}[U_\infty^{\partial D} = 0]$ is continuous when D is bounded, and hence that $Q_{\mu,1}^{R^d,\beta,\gamma}[U_t = 0 \text{ for large } t]$ is lower semicontinuous in β, γ. This does not seem to have any immediate implications for the continuity of Ψ.

4 Approximations

The proof of the decomposition results in section 2.2 is rather clear for particle systems that approximate our reaction diffusion system. Our first proofs used a full particle approximation with population and nutrient particles living on discrete lattices. We later realized that the key to establishing the comparisons was to discretise the effect of the nutrient, and we here present an approximation where the reaction with the nutrient occurs in a finite number of discrete 'packages' but the population still evolves as a continuum SPDE. This intermediate approximation makes passage to the limit easier to establish. This passage to the limit is broadly similar to many in the literature, and the points of most interest are perhaps (i) that the nutrient interaction is singular as a function of the population measure u, and requires tightness of the occupation densities; (ii) that convergence of the exit measures follows quite simply from convergence of the population measures, via the extended martingale problem (2.4).

4.1 Construction of the approximation

The approximations will depend on a parameter N, which we will suppress in the notation in this section. Fix a domain $D \subseteq R^d$ and partition it as a disjoint union of sets $D = \cup D_j$, where each set D_j has diameter at most N^{-1}. Choose a finite number of functions $(\psi_k : 1 \leq k \leq K)$, each of the form $N^{-1} I(x \in D_j)$ for some D_j in the partition. Each ψ_k will represent a small package of nutrient which may be triggered to produce new population. The function $f = \sum_k \psi_k \leq 1$ will be our approximation to the initial nutrient level.

Choose a probability space equipped with the following independent families: an i.i.d. family of rate one exponential variables e_k for $k \geq 1$; independent DW(D,γ) processes $(u_{k,t} : t \geq 0)$, for $k \geq 0$, with initial conditions

$$u_{0,0} = \mu, \quad \text{and} \quad u_{k,0} = \beta \psi_k(x) dx \;\; \text{for } k \geq 1.$$

Let $(u^{\partial D}_{k,t} : t \geq 0)$ be the associated exit measures processes. Given realizations of these variables, the approximation can be constructed pathwise. We will list as $\mathcal{S}_t \subseteq \{1, \ldots, K\}$ the labels of those nutrient packages that have been triggered by time t, and denote by $\tau_k \in [0,\infty]$ the time at which the kth nutrient package is triggered. Thus $\mathcal{S}_0 = \emptyset$ and $\mathcal{S}_t = \{k : \tau_k \leq t\}$. The approximation will satisfy

$$u_t = u_{0,t} + \sum_{k \in \mathcal{S}_t} u_{k,t-\tau_k}, \quad u^{\partial D}_t = u^{\partial D}_{0,t} + \sum_{k \in \mathcal{S}_t} u^{\partial D}_{k,t-\tau_k}, \quad v_t = \sum_{k \notin \mathcal{S}_t} \psi_k. \tag{4.1}$$

Moreover we want the approximation process to be triggered in such a way that

$$\tau_k = \inf \left\{ t : u_{[0,t]}(\hat{\psi}_k) > e_k \right\} \quad \text{for } k \geq 1$$

where $\hat{\psi}_k = \psi_k / \langle \psi_k, 1 \rangle$. This uniquely specifies a process, which can be constructed pathwise by defining the triggering times τ_k and the processes u_t, v_t inductively over the intervals between triggers (and where, on a null set, packages may be simultaneously triggered).

We now derive a martingale problem for the approximation. On the intervals $[\tau_k, \tau_{k+1})$ the approximation process evolves as a finite sum of DW processes. So it satisfies the following martingale problem: for $\phi \in C^2_b(\overline{D})$

$$u_t(\phi) = \mu(\phi) + \int_0^t u_s \left(\Delta \phi - \gamma \phi\right) ds - u^{\partial D}_t(\phi) + \sum_{s \leq t} D_s u_s(\phi) + m_t(\phi), \tag{4.2}$$

where $m_t(\phi)$ is a continuous martingale with quadratic variation $\int_0^t u_s(\phi^2) ds$ and where $D_s u_s(\phi)$ is the jump $u_s(\phi) - u_{s-}(\phi)$ at time s. We can compensate the jump term as

$$\sum_{s \leq t} D_s u_s(\phi) = \beta \int_0^t \sum_{k \notin \mathcal{S}_s} \langle \phi, \psi_k \rangle u_s(\hat{\psi}_k) ds + E^{(1)}_t(\phi) \tag{4.3}$$

where $E^{(1)}_t(\phi)$ is a martingale with jumps bounded by $\beta N^{-d-1} \|\phi\|_\infty$ and with predictable brackets process given by

$$\begin{aligned} d\langle E^{(1)}(\phi) \rangle_t &= \beta^2 \sum_{k \notin \mathcal{S}_t} \langle \phi, \psi_k \rangle^2 u_t(\hat{\psi}_k) dt \\ &\leq \beta^2 \|\phi\|^2_\infty N^{-d-1} \sum_{k \notin \mathcal{S}_t} u_t(\psi_k) dt \leq \beta^2 \|\phi\|^2_\infty N^{-d-1} u_t(1) dt. \end{aligned} \tag{4.4}$$

The term N^{-d-1} arises from bounding $\langle 1, \psi_k \rangle \leq N^{-d-1}$ (since $0 \leq \psi_k \leq N^{-1}$ and the diameter of the support of ψ_k is at most N^{-1}). Examining the variation of the function ϕ over the support of ψ_k, we can approximate

$$\left| \langle \phi, \psi_k \rangle u_s(\hat{\psi}_k) - u_s(\psi_k \, \phi) \right| \leq N^{-1} \|\nabla \phi\|_\infty u_s(\psi_k)$$

and rewrite the compensator in (4.3) as $\beta \int_0^t u_s(v_s \phi) ds$ up to an error of size $O(N^{-1})$. Combining these estimates we have

$$u_t(\phi) = \mu(\phi) + \int_0^t u_s \left(\Delta \phi + \beta v_s \phi - \gamma \phi\right) ds - u^{\partial D}_t(\phi) + m_t(\phi) + E^{(1)}_t(\phi) + E^{(2)}_t(\phi) \tag{4.5}$$

where the second error term $E^{(2)}_t(\phi)$ is controlled by

$$\begin{aligned} \left| E^{(2)}_t(\phi) - E^{(2)}_t(\phi) \right| &= \left| \beta \int_s^t \sum_{k \notin \mathcal{S}_r} \left(\langle \phi, \psi_k \rangle u_r(\hat{\psi}_k) - u_r(\phi \psi_k) \right) dr \right| \\ &\leq \beta \left((N^{-1} \|\nabla \phi\|_\infty) \wedge \|\phi\|_\infty \right) \int_s^t u_r(1) dr. \end{aligned} \tag{4.6}$$

Taking $\phi = 1$ in (4.5) we find that

$$u_t(1) \leq \mu(1) + (\beta - \gamma) \int_0^t u_s(1) ds + m_t(1) + E_t^{(1)}(1).$$

From this, via the standard argument (localizing, applying Burkholder's inequality to control the martingales and Gronwall's inequality), one can derive the moment bound, for any $p > 0$ adnd $T < \infty$,

$$E\left[\sup_{t \leq T} |u_t(1)|^p\right] \leq C(p, \beta, \gamma, T) \left(1 + (\mu(1))^p\right). \tag{4.7}$$

We now consider the martingale decomposition of $t \to v_t(x)$. For $x \in D$ let D_x be the partition element that contains x, let $\psi_x(y) = N^{-1} I(y \in D_x)$ and $\hat{\psi}_x = \psi_x / \langle \psi_x, 1 \rangle$. Then a nutrient package satisfies $\psi_k(x) > 0$ only if $\psi_k = \psi_x$. The process $t \to v_t(x)$ is a pure jump process with compensator given as in

$$\begin{aligned} v_t(x) &= f(x) - \int_0^t \sum_{k \notin \mathcal{S}_s} \psi_k(x) u_s(\hat{\psi}_k) ds + m_t^v(x) \\ &= f(x) - \int_0^t v_s(x) u_s(\hat{\psi}_x) ds + m_t^v(x) \end{aligned} \tag{4.8}$$

for a martingale $m_t^v(x)$ with jumps bounded by N^{-1} and with predictable brackets process

$$d\langle m^v(x) \rangle_t = \sum_{k \notin \mathcal{S}_t} \psi_k^2(x) u_t(\hat{\psi}_k) dt \leq N^{-1} v_t(x) u_t(\hat{\psi}_x) dt. \tag{4.9}$$

We can solve the equation (4.8) for $v_t(x)$ as

$$v_t(x) = e^{-u_{[0,t]}(\hat{\psi}_x)} \left(f(x) + \int_0^t e^{-u_{[0,s]}(\hat{\psi}_x)} dm_s^v(x) \right). \tag{4.10}$$

4.2 Tightness of the approximations

To pass to the limit in the martingale problems for the approximation we require tightness estimates. We tried deriving these from known tightness estimates for each of the DW processes used as building blocks, however the random start times caused some difficulties in combining the estimates, so we proceed by repeating Sugitani's estimates for our discretized approximation.

We now include the dependence on N and write the approximation constructed in section 4.1 as $(u^{(N)}, v^{(N)})$.

Proposition 14. *Let D be a bounded domain. Suppose the approximation $(u^{(N)}, v^{(N)})$ has initial conditions $u_0^{(N)} = \mu^{(N)}$ and $v_0^{(N)} = f^{(N)} \leq 1$ satisfying that $\mu^{(N)} \to \mu$ weakly and $f^{(N)} \to f$ in $L^1(D)$ for some $f \in \mathcal{B}(D, [0,1])$. Then*

(i) the laws of $(u^{(N)})$ are tight in $D([0,\infty), \mathcal{M}(D))$ and limit points are continuous,

(ii) for any $\delta > 0$, the laws of the occupation densities $(u^{(N)}(\delta, t+\delta, x) : t \geq 0, x \in D)$ are tight in $C([0,\infty), C(D))$

where $C(D)$ is the space of continuous functions with the topology of uniform convergence on compacts.

Proof. The moment bound (4.7) allows us the uniform control on total mass moments:

$$E\left[\sup_{t \leq T} |u_t^{(N)}(1)|^p\right] \leq C(p, T, \beta, \gamma) \left(1 + (\mu^{(N)}(1))^p\right). \tag{4.11}$$

To control the moments of the approximate densities, the following simple upper bound for $u^{(N)}$ is useful. Triggering all the nutrient packages $(u_k : 1 \leq k \leq K_N)$ used in the construction of $u^{(N)}$ at time zero gives the process $\tilde{u}^{(N)}_t = \sum_{k=0}^{K_N} u_{k,t}$. This satisfies

$$u^{(N)}(0,t,x) \leq \tilde{u}^{(N)}(0,t,x) \quad \text{for all } x \in D \text{ and } t \geq 0. \tag{4.12}$$

Moroever $\tilde{u}^{(N)}$ has the law $Q^{D,\gamma}_{\mu^{(N)}+\beta f^{(N)}dx}$ of a DW(D,γ) process.

Now fix $\phi \in C^2_0(\overline{D})$. Using the decomposition (4.5), we consider the increment $|u^{(N)}_t(\phi) - u^{(N)}_s(\phi)|$ for $0 \leq s < t$. This leads, via the total mass moment bounds and Burkholder's inequality, to the estimate

$$E\left[\sup_{r\in[s,t]} \left|u^{(N)}_r(\phi) - u^{(N)}_s(\phi)\right|^p\right] \leq C\left(|t-s|^{p/2} + N^{-(d+1)p}\right) \quad \text{for all } N \text{ and } s,t \in [0,T], \tag{4.13}$$

where $C < \infty$ depends only on $\beta, \gamma, \phi, p, T$ and $\sup_N \mu^{(N)}(1)$ (we suppress this dependence below). The term $N^{-(d+1)p}$ comes from the size of the jumps in the Burkholder inequality, for a cadlag martingale M, in the form $E[\sup_{s\leq t}|M_s|^p] \leq C_p E[[M]^{p/2}_t] + C_p E[\sup_{s\leq t}|D_s M|^p]$. The estimate (4.13) gives, via a Chebychev inequality,

$$P\left[\sup_{s\in[t,t+N^{-(d+1)}]} \left|u^{(N)}_s(\phi) - u^{(N)}_t(\phi)\right| \geq N^{-1/2}\right] \leq CN^{-dp/2}.$$

By summing over the grid $t_j = jN^{-(d+1)}$ and applying Borel Cantelli, we find that

$$\left|u^{(N)}_s(\phi) - u^{(N)}_t(\phi)\right| \leq 2N^{-1/2} \quad \text{whenever } 0 \leq s,t \leq T \text{ and } |s-t| \leq N^{-(d+1)}, \text{ for all large } N, \ P \text{ a.s.} \tag{4.14}$$

We claim that (4.13) is sufficient to imply that $u^{(N)}(\phi)$ is tight in $D([0,\infty),R)$ and has continuous limit points. To see this we smooth the sample paths by setting $\overline{u}^{(N)}_t = N^{d+1} u^{(N)}_{[t,t+N^{-(d+1)}]}$. Then $t \to \overline{u}^{(N)}(\phi)$ is continuous and

$$\left|\overline{u}^{(N)}_t(\phi) - \overline{u}^{(N)}_t(\phi)\right| \leq 2N^{(d+1)}|t-s|\|\phi\|_\infty \sup_{r\leq T+1} u^{(N)}_r(1) \quad \text{for all } N \text{ and } s,t \in [0,T].$$

This together with (4.13) implies that $E[|\overline{u}^{(N)}_t(\phi) - \overline{u}^{(N)}_s(\phi)|^p] \leq C|t-s|^{p/2}$ for $s,t \in [0,T]$. Hence the laws of $\overline{u}^{(N)}(\phi)$ are tight in $C([0,T],R)$. But (4.14) shows that the error $\sup_{t\leq T}|u^{(N)}_t(\phi) - \overline{u}^{(N)}_t(\phi)| \to 0$ almost surely and $u^{(N)}_t(\phi)$ has the the same continuous limit points as $\overline{u}^{(N)}_t(\phi)$.

We now state a compact containment condition. Let $D_\epsilon = \{x \in D : d(x,D^c) > \epsilon\}$. Choose smooth $\phi_{D,\epsilon} : D \to [0,1]$ satisfying $\phi_{D,\epsilon} = 0$ on $D_{2\epsilon}$ and $\phi_{D,\epsilon} = 1$ on $D \setminus D_\epsilon$. We also suppose that $\phi_{D,\epsilon}$ is decreasing in ϵ. We claim that for all $\delta > 0$ there exists $\epsilon(\delta) > 0$ satisfying

$$\lim_{\delta\to 0} \limsup_{N\to\infty} P\left[\sup_{t\leq T} u^{(N)}_t(\phi_{D,\epsilon(\delta)}) \geq \delta\right] = 0. \tag{4.15}$$

We postpone the proof of this to the end of the section. This condition controls the amount of mass near the boundary and allows us to extend the tightness of $u^{(N)}_t(\phi)$ above to all $\phi \in C^0_b(\overline{D})$ by an approximation argument. It also is the compact containment hypothesis in [10] Theorem II.4.1, from which we obtain the tightness of $u^{(N)}$ in $D([0,\infty),\mathcal{M}(D))$ and the continuity of the limit points.

To establish the tightness of the occupation densities we use the following increment estimate. Under the hypotheses of Proposition 14, we claim there exist $0 < \alpha_i < \infty$ and finite C, depending on $\beta, \gamma, p, \delta, \delta', T, D$ and $\sup_N \mu^{(N)}(1) + (f^{(N)}, 1)$, so that

$$E\left[|u^{(N)}(\delta,s,x) - u^{(N)}(\delta,t,y)|^p\right] \leq C\left(|t-s|^{\alpha_1 p} + |x-y|^{\alpha_2 p}\right) \tag{4.16}$$

for all $\delta \le s,t \le T$ and $x,y \in D \setminus D_{\delta'}$. This implies tightness of $(t,x) \to u^{(N)}(\delta,t,x)$ in $C([\delta,\infty),C(D))$, where $C(D)$ is the space of continuous functions with the topology of uniform convergence on compacts.

To establish (4.16) we use a Green's function representation for the density (note the density exists since it exists for each component part $u_{k,t}$). It is convenient to break the approximation into two parts

$$u_t^{(N)} = u_{0,t} + \hat{u}_t^{(N)} \quad \text{where} \quad \hat{u}_t^{(N)} = \sum_{k \in \mathcal{S}_t} u_{k,t-\tau_k}. \tag{4.17}$$

For the process $\hat{u}^{(N)}$ we may consider $\hat{u}^{(N)}(0,t,x)$ since the nutrient packages do not start at a singular initial condition. Indeed combining the Green's function representation (3.19) for each $u_{k,t-\tau_k}$ and summing over $k \in \mathcal{S}_t$ we obtain

$$\hat{u}^{(N)}(0,t,x) = \int_0^t \int_D G^{D,x}_{[0,t-s]}(z) \left(-\gamma \hat{u}^{(N)}_s(dz) + \hat{m}^{(N)}(dz,ds) \right) + \sum_{s \le t} D_s u_s(G^{D,x}_{[0,t-s]}) \tag{4.18}$$

where $\hat{m}^{(N)}$ is the martingale measure associated to $\hat{u}^{(N)}$. We now consider increments of this representation. There are many terms, so we choose to illustrate the idea on one key noise term and on the jump term. We shall also state all the underlying estimates on the kernel $G^{D,x}_{[0,t]}$. First

$$G^{D,x}_{[0,t]}(z) \le G^{R^3,x}_{[0,t]}(z) \le C|x-z|^{-1} \quad \text{for } t>0 \text{ and } x,z \in R^3$$

and the analogous bounds $G^{R^2,x}_{[0,t]}(z) \le C(T)(1 \vee \ln(1/|x-z|)$ and $G^{R,x}_{[0,t]}(z) \le C(T)$ when $t \le T$. Also

$$|G^{D,x}_{[0,t]}(z) - G^{D,x}_{[0,t']}(z)| \le C(d,\delta)|t-t'||x-z|^{-d}$$

for all t,t' and $x,z \in D$ and

$$|G^{D,x}_{[0,t]}(z) - G^{D,y}_{[0,t]}(z)| \le C(d,\delta,T)|x-y|(1+|x-z|^{1-d})$$

for $t,t' \le T$ and $x,y \in D_\delta$, $z \in D$ satisfying $|x-z| \ge 2|x-y|$. These estimates are explicit calculations when $D = R^d$. One way to obtain them for general D is via a suitable coupling argument.

Consider the term $K(t,x) = \int_0^t \int G^{D,x}_{[0,t-s]}(z)\hat{m}^{(N)}(dz,ds)$. The time increment $K(t+s,x) - K(t,x)$ can be split into two terms, defined by

$$K_1 = \int_0^t \int G^{D,x}_{[0,t+s-r]}(z) - G^{D,x}_{[0,t-r]}(z)\hat{m}^{(N)}(dz,dr), \quad K_2 = \int_t^{t+s} \int G^{D,x}_{[0,t-r]}(z)\hat{m}^{(N)}(dz,dr).$$

Using a Burkholder inequality, and the kernel estimates above we find, when $d=3$,

$$E[|K_1|^p] \le C(p) E\left[\left| \int_D \hat{u}^{(N)}(0,t,z) \left(|z-x|^{-2} \wedge s^2|z-x|^{-6} \right) dz \right|^{p/2} \right].$$

A comparison as in (4.12) combined with the moments (3.21) shows that $\sup_z E[\hat{u}^{(N)}(0,t,z)^{p/2}] \le C(p,\beta,T)$ when $t \le T$ and $z \in D$. This leads to $E[|K_1|^p] \le C(p,\beta,T,D)s^{p/4}$. Similarly, again when $d=3$ and $t+s \le T$, apply Holder's inequality to find

$$\begin{aligned}
E[|K_2|^p] &\le C(p) E\left[\left| \int_0^T I(r \in [t,t+s]) \hat{u}^{(N)}_r(|x-\cdot|^{-2}) \right|^{p/2} \right] \\
&\le C(p) E\left[\left| \int_0^T I(r \in [t,t+s]) \hat{u}^{(N)}_r(dz) \right|^{p/8} \left| \int_0^T \hat{u}^{(N)}_r(|x-\cdot|^{-8/3}) \right|^{3p/8} \right] \\
&\le C(p)|s|^{p/8} E\left[|\sup_{t \le T} \hat{u}^{(N)}_t(1)|^{p/8} \left| \int \hat{u}^{(N)}(0,T,z)|x-z|^{-8/3} dz \right|^{3p/8} \right] \\
&\le C(p,\beta,T,D)|s|^{p/8}.
\end{aligned}$$

In this second estimate we do not have the optimal power of s, but have shown how to rely only on the simple moments estimates for $\sup_{t\le T}\hat{u}_t^{(N)}(1)$ and $u^{(N)}(0,T,z)$.

The jump term in the representation (4.18) can also be easily handled, since the sum of all jumps in the measure $t\to u_t$ is at most βdx. For example, an increment in x is bounded using the kernel estimates above, when $d=3$ and $|x-y|\le 1/2$,

$$\begin{aligned}
&\left|\sum_{s\le t} D_s u_s(G^{D,x}_{[0,t-s]}) - D_s u_s(G^{D,y}_{[0,t-s]})\right| \\
&\quad\le\ \beta\int_D \sup_{s\le t}|G^{D,x}_{[0,t-s]}(z)-G^{D,y}_{[0,t-s]}(z)|dz \\
&\quad\le\ C\beta\int_{B(x,|x-y|^{1/2})}|x-z|^{-1}+|y-z|^{-1}dz + C\beta|x-y|\int_{D\setminus B(x,|x-y|^{1/2})}1+|x-z|^{-2}dz \\
&\quad\le\ C(\beta,D)|x-y|.
\end{aligned}$$

Other terms in the Green's function representation of the increment $\hat{u}(0,t,x)$, and in all dimensions $d=1,2,3$, can be controlled by similar estimates.

From (4.17) we have $u^{(N)}(\delta,t,x)=u_0(\delta,t,x)+\hat{u}^{(N)}(\delta,t,x)$ and we have controlled the increments of $\hat{u}^{(N)}(0,t,x)$, and hence of $\hat{u}^{(N)}(\delta,t,x)=\hat{u}^{(N)}(0,t,x)-\hat{u}^{(N)}(0,\delta,x)$. The increment estimates for $u_0(\delta,t,x)$ are similar, except that one restricts to $\delta>0$ and uses (3.20). One also needs estimates on the fixed time measure $u_{0,\delta}$ to control the increments of $u_{0,\delta}(G^{D,x}_{[0,t-\delta]})$. For this, use the method of Lemma III.3.6 in Perkins [10] to show, for $p\in[0,2)$ and suitable $c_0=c_0(p)>0$, that $E[\exp(c_0u_{0,\delta}(|x-\cdot|^{-p})]\le C(d,\sup_n\mu_n(1),\delta,p)<\infty$, which is sufficient. $\Box$

Proof of the compact containment (4.15). We start by showing that

$$\sup_N E\left[u^{(N)}_{[0,T]}(\phi_{D,\epsilon})\wedge 1\right]\to 0\quad\text{as }\epsilon\downarrow 0. \tag{4.19}$$

Indeed, arguing as in (3.28) we have

$$Q^{R^d,\gamma}_{\mu}\left[U_{[0,T]}(\phi_{D,\epsilon})\wedge 1\right]\downarrow 0\quad\text{as }\epsilon\downarrow 0. \tag{4.20}$$

Moreover the continuity of the laws $\mu\to Q^{R^d,\gamma}_{\mu}$ on $C([0,\infty),\mathcal{M}(R^d))$ ensures that the expectation in (4.20) is a continuous function of μ. Therefore the limit in (4.20) is uniform over the compact set $K_{D,L}$ of measures supported in $\overline{D}$ and with total mass $\mu(1)\le L$. By the coupling (4.12) and the simple comparison (3.29) we find

$$E\left[u^{(N)}_{[0,T]}(\phi_{D,\epsilon})\wedge 1\right]\le Q^{D,\gamma}_{\mu^{(N)}+\beta f^{(N)}dx}\left[U_{[0,T]}(\phi_{D,\epsilon})\wedge 1\right]\le Q^{R^d,\gamma}_{\mu^{(N)}+\beta f^{(N)}dx}\left[U_{[0,T]}(\phi_{D,\epsilon})\wedge 1\right]$$

which leads to (4.19).

To control the supremum $\sup_{t\le T}u^{(N)}(\phi_{D,\epsilon})$, we will establish a modulus of continuity that is uniform in N and ϵ. Let $G^D_t\phi$ denote the action of the Green's function as defined by $G^D_t\phi(x)=\int_D G^{D,x}_t(y)\phi(y)dy$. Then, arguing as above, we have

$$\sup_N\sup_{t\le T}\mu^{(N)}(G^D_t\phi_{D,\epsilon})\to 0\quad\text{as }\epsilon\to 0. \tag{4.21}$$

For $\phi\in\mathcal{B}(D,[0,1])$, the measurable functions from $\overline{D}$ to $[0,1]$, define

$$X_t(N,\phi)=u^{(N)}_t(\phi)-\mu^{(N)}(G^D_t\phi)\quad\text{and}\quad \overline{X}_t(N,\phi)=N^{d+1}\int_t^{t+N^{-(d+1)}}X_s(N,\phi)ds.$$

We may extend the decomposition (4.5) to time dependent test functions as in section 3.1.1. For smooth ϕ compactly supported inside D, using the test function $(s,x)\to G^D_{t-s}\phi(x)$ in (4.5), over the interval $s\in[0,t]$, leads to the Green's function representation for $X_t(N,\phi)$. This representation in turn can be used to reach an increment estimate

$$E[|X_t(N,\phi) - X_s(N,\phi)|^p] \le C\left(|t-s|^{p/2} + N^{-(d+1)p}\right) \quad \text{for all } N \text{ and } s,t \in [0,T],$$

where C depends on p, T but not on ϕ or N. This requires (see Perkins [12] Corollary 5 for this argument) only a smoothing property of the Green's kernel, namely we use:

$$\sup_{x\in D} \int_D |G_u^{D,x}(y) - G_t^{D,x}(y)| dy \le C(D) t^{-1}(u-t) \quad \text{for all } 0 < t < u.$$

This estimate holds by direct calculation for the case where $D = R^d$, and thereby when D is a box, since then the Green's kernel is a finite combination of reflected copies of the free space Green's kernel. For domains with a smooth boundary, it can be derived from Corollary 5 of Davies [1].

The smoothed version $\overline{X}_t(N,\phi)$ then satisfies $E[|\overline{X}_t(N,\phi) - \overline{X}_s(N,\phi)|^p] \le C|t-s|^{p/2}$ for all N and $s,t \in [0,T]$. The argument from Perkins [12] Theorem 1 then shows that for all $\phi \in \mathcal{B}(D,[0,1])$, there exists a random $c(\phi,N,T) > 0$ so that

$$\left|\overline{X}_t(N,\phi) - \overline{X}_s(N,\phi)\right| \le |t-s|^{1/3} \quad \text{whenever } s,t \in [0,T] \text{ and } |t-s| \le c(\phi,N,T) \tag{4.22}$$

and moreover that $P[c(\phi,N,T) \le \delta] \to 0$ as $\delta \downarrow 0$ uniformly over N and $\phi \in \mathcal{B}(D,[0,1])$. The point here is that the estimates on $c(\phi,N,T)$ depend only on total moment bounds and on $\|\phi\|_\infty$.

It is straightforward to combine (4.19), (4.21) and (4.22) to see that there exists $\epsilon(\delta) > 0$ so that

$$\lim_{\delta\to 0} \sup_N P\left[\sup_{t\le T} \overline{u}_t^{(N)}(\phi_{D,\epsilon(\delta)}) \ge 2\delta\right] = 0. \tag{4.23}$$

Indeed, consider the set

$$\left\{\sup_{t\le T} \overline{u}_t^{(N)}(\phi_{D,\epsilon}) \ge 2\delta,\ c(\phi_{D,\epsilon}, N, T+1) \ge \delta^3\right\}.$$

If $\delta > 1$ and $\sup_N \sup_{t\le T+1} \mu^{(N)}(G_t^D(\phi_{D,\epsilon})) \le \delta^4/8$, then on this set we have $\sup_{t\le T} \overline{X}_t(N,\phi_{D,\epsilon}) \ge \delta$ and the modulus of continuity ensures that $\int_0^T \overline{X}_t(N,\phi_{D,\epsilon})dt \ge \delta^4/4$, and thence that $\int_0^T \overline{u}_t^{(N)}(\phi_{D,\epsilon})dt \ge \delta^4/8$. But this has small probability by (4.19) and Markov's inequality. So (4.23) follows by first choosing δ so that that $P[c(\phi_{D,\epsilon}, N, T{+}1) \le \delta^3]$ is small, uniformly in N and ϵ, and then choosing ϵ so that $\sup_N \sup_{t\le T+1} \mu^{(N)}(G_t^D(\phi_{D,\epsilon}) \le \delta^4/8$ and $\sup_N E[\overline{u}_{[0,T]}^{(N)}(\phi_{D,\epsilon}) \wedge 1]$ is small.

To remove the time smoothing in 4.23), we would like to apply (4.14). However, since $\phi_{D,\epsilon}$ does not lie in $C_0^2(\overline{D})$, there is an extra exit measure term when deriving the increment estimate (4.13). However the exit measure is non-negative and we can still deduce the one sided estimate $u_t^{(N)}(\phi) \le u_s^{(N)}(\phi) + 2N^{-1/2}$ whenever $0 \le s < t \le T$ and $|t-s| \le N^{-(d+1)}$, for all large N, P a.s, and this is sufficient to deduce from (4.23) the desired compact containment estimate. □

4.3 Passage to the limit

The aim now is to prove Theorem 15 below, showing that the approximations constructed in section 4.1 converge in law to solutions of (1.1). The method is to pass to the limit in the approximate martingale problem (4.5) and confirm that any limit point of the sequence of approximations must be a solution to (1.1). Many steps in the argument are quite standard (see, for example, the arguments in Proposition II.4.2 in [10]), so we concentrate on those that concern the interaction term and the exit measures.

Since we wish to let $N \to \infty$, we again include this dependence in the notation. Assume the hypotheses of the tightness Proposition 14. For any subsequence, we may choose a sub-subsequence where the approximations and their occupation densities converge in distribution. By a Skorokhod embedding, we may choose versions of these approximations that converge almost surely. Without changing notation for the version and labelling the convergent sub-subsequence still as N, we may suppose there exists a limit process u, with continuous paths in $\mathcal{M}(D)$ and a continuous field $(u(s,t,x) : 0 < s \leq t, x \in D)$ so that, almost surely,

$$\sup_{s \leq t} \left| u_s^{(N)}(\phi) - u_s(\phi) \right| \to 0, \qquad \text{for all } \phi \in C_b^0(\overline{D}) \text{ and } t < \infty, \tag{4.24}$$

$$\sup_{x \in A, s \in [\delta,t]} \left| u^{(N)}(\delta,s,x) - u(\delta,s,x) \right| \to 0, \qquad \text{for all } 0 < \delta \leq t < \infty \text{ and compact } A \subseteq D. \tag{4.25}$$

Note that $u(s,t,x)$ must act as a density for $u_{[s,t]}(dx)$ for $0 < s \leq t$. Since also $s \to u(s,t,x)$ is non-increasing we may set $v_t(x) = f(x)\exp(-u(0,t,x)) = \lim_{s \downarrow 0} f(x)\exp(-u(s,t,x))$.

Fix $\phi \in C_0^2(\overline{D})$. Consider the martingale problem (4.5) for $u^{(N)}$ at a fixed $t \geq 0$, noting that $u_t^{\partial D,(N)}(\phi) = 0$. The terms $u_t^{(N)}(\phi)$, $\mu^{(N)}(\phi)$ and $\int_0^t u_s^{(N)}(\Delta\phi - \gamma\phi)ds$ converge pathwise as $N \to \infty$ by (4.24). The error terms $E_t^{(1,N)}(\phi)$ and $E_t^{(2,N)}(\phi)$ converge to zero (in L^2 and pathwise respectively) using the estimates (4.4) and (4.6) and the moment bounds (4.11). To handle the key term $\int_0^t u_s^{(N)}(v_s^{(N)}\phi)ds$ we approximate, using (4.10),

$$\int_0^t u_s^{(N)}(v_s^{(N)}\phi)ds = \int_0^t u_s^{(N)} \left(f^{(N)} e^{-u_{[0,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds + E_t^{(3,N)}(\phi)$$

where the error term is bounded by

$$\left| E_t^{(3,N)}(\phi) \right| = \left| \int_0^t u_s^{(N)} \left(\phi e^{-u_{[0,s]}^{(N)}(\hat{\psi}_\cdot)} \tilde{m}_s^{v,N} \right) ds \right| \leq \|\phi\|_\infty \int_D u^{(N)}(0,t,x) \sup_{s \leq t} \tilde{m}_s^{v,N}(x)\, dx \tag{4.26}$$

and $\tilde{m}_t^{v,N}(x) = \int_0^t \exp(-u_{[0,s]}^{(N)}(\hat{\psi}_x))dm_s^{v,N}(x)$. Doob's inequality, and the estimate (4.9), shows that

$$E\left[\sup_{s \leq t} |\tilde{m}_t^{v,N}(x)|^2 \right] \leq 2N^{-1} \int_0^t E\left[e^{-2u_{[0,s]}^{(N)}(\hat{\psi}_x)} u_s^{(N)}(\hat{\psi}_x) \right] ds \leq N^{-1}.$$

The comparison (4.12) shows that $E\left[(u^{(N)}(0,t,x))^2 \right] \leq Q_{\mu^{(N)} + \beta I_D dx}^{D,\gamma} \left[(U(0,t,x))^2 \right]$. Applying Cauchy-Schwarz to (4.26), and using the the estimate (3.22), we find that $E(|E_t^{(3,N)}(\phi)|) \to 0$ as $N \to \infty$.

For $\delta \in (0,t)$ we write

$$\begin{aligned}
&\int_0^t u_s^{(N)} \left(f^{(N)} e^{-u_{[0,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds \\
= \quad & \int_\delta^t u_s^{(N)} \left(f e^{-u_{[\delta,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds + \int_\delta^t u_s^{(N)} \left((f - f^{(N)}) e^{-u_{[\delta,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds \\
& + \int_0^\delta u_s^{(N)} \left(f^{(N)} e^{-u_{[0,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds + \int_\delta^t u_s^{(N)} \left(f^{(N)} (e^{-u_{[0,s]}^{(N)}(\hat{\psi}_\cdot)} - e^{-u_{[\delta,s]}^{(N)}(\hat{\psi}_\cdot)}) \phi \right) ds \\
= \quad & \int_\delta^t u_s^{(N)} \left(f e^{-u_{[\delta,s]}^{(N)}(\hat{\psi}_\cdot)} \phi \right) ds + E_t^{(4,N)}(\phi) + E_t^{(5,N)}(\phi) + E_t^{(6,N)}(\phi).
\end{aligned} \tag{4.27}$$

The error term $E_t^{(5,N)}(\phi)$ is bounded by $\|\phi\|_\infty \int_0^\delta u_s^{(N)}(1)ds$ and converges to zero, uniformly in N, as $\delta \to 0$. The error term $E_t^{(6,N)}(\phi)$ is bounded by

$$\|\phi\|_\infty \int_0^t u_s^{(N)}(u_{[0,\delta]}^{(N)}(\hat{\psi}_\cdot))ds = \|\phi\|_\infty \int_D u^{(N)}(0,t,x) u_{[0,\delta]}^{(N)}(\hat{\psi}_x) dx.$$

The approximate density $u_{[0,\delta]}^{(N)}(\hat{\psi}_x)$ satisfies $\int_D (u_{[0,\delta]}^{(N)}(\hat{\psi}_x))^2 dx \leq \int_D (u^{(N)}(0,\delta,x))^2 dx$. This, together with Cauchy-Schwarz as above, shows that $E_t^{(6,N)}(\phi)$ converges to zero in L^1, uniformly in N, as $\delta \to 0$. The error term $E_t^{(4,N)}(\phi)$

is bounded by

$$\|\phi\|_\infty \int_0^t u_s^{(N)} \left(|f^{(N)} - f| \right) = \|\phi\|_\infty \int_D u^{(N)}(0,t,x)|f^{(N)}(x) - f(x)|dx.$$

Since $f^{(N)}$ are bounded we have $f^{(N)} \to f$ in $L^2(D)$ and combined with (3.22) we see that $E_t^{(4,N)}(\phi) \to 0$ in L^2.

Finally, the first term on the right hand side of (4.27) converges pathwise, for fixed δ, to $\int_\delta^t u_s(f \exp(-u(\delta,s))\phi)$ as $N \to \infty$. This follows from (4.24,4.25) when $f \in C_b^0(D)$, and for general measurable f we can approximate in L^2 by continuous $\tilde{f}$ and control the error as for the error term $E_t^{(4,N)}(\phi)$. The limit $\int_\delta^t u_s(f \exp(-u(\delta,s))\phi)ds$ is, by repeating the above approximations, close, for small δ, to $\int_0^t u_s(v_s \phi)ds$. The conclusion is that the key term $\int_0^t u_s^{(N)}(v_s^{(N)}\phi)ds$ converges, in probability, to $\int_0^t u_s(v_s\phi)ds$.

We may now define a continuous process $m_t(\phi)$, for $\phi \in C_0^2(\overline{D})$, by the formula (2.1). The convergence of all other terms ensures that $m_t^{(N)}(\phi) \to m_t(\phi)$ in probability. Moreover, standard arguments yield that $m_t(\phi)$ is a martingale with respect to $\sigma\{u_s : s \le t\}$, and with the correct quadratic variation (note that we have uniform moment control by (4.11)). Thus the limiting process (u,v) is a solution to the martingale problem for (1.1) started from (μ, f) with respect to its natural filtration $\sigma\{u_s : s \le t\}$. We have almost proved the following convergence result.

Theorem 15. *Let D be a bounded domain. Suppose the approximation $(u^{(N)}, v^{(N)})$ has initial $u_0^{(N)} = \mu^{(N)}$ and $v_0^{(N)} = f^{(N)} \le 1$ satisfying that $\mu^{(N)} \to \mu$ weakly and $f^{(N)} \to f$ in $L^1(D)$ for some $f \in \mathcal{B}(D,[0,1])$. Then,*

(i) the laws $(u^{(N)}, u^{\partial D,(N)})$ converges in distribution on $D([0,\infty), \mathcal{M}(D)) \times C([0,\infty), \mathcal{M}(\partial D))$ to the limit $Q_{\mu,f}^{D,\beta,\gamma}$;

(ii) for any $T < \infty$, the law of the triple $(u_{[0,T]}^{(N)}, u_T^{\partial D,(N)}, v_T^{(N)})$ on $\mathcal{M}(D) \times \mathcal{M}(\partial D) \times L^1(D)$ converges to the law of $(U_{[0,T]}, U_T^{\partial D}, f \exp(-U(0,T)))$ under $Q_{\mu,f}^{D,\beta,\gamma}$.

Completion of the proof. The convergence of the exit measures can be deduced from the extended martingale problem for test functions $\phi \in C_b^2(\overline{D})$. Indeed, return to the sub-subsequence studied before the statement of the theorem. Since (u,v) solves (1.1) there is a continuous exit measure process $u^{\partial D}$ solving the extended martingale problem. Using an approximation argument by $\phi_n \in C_0^2(\overline{D})$, it follows that $m_t^{(N)}(\phi) \to m_t(\phi)$ in probability for all $\phi \in C_b^2(\overline{D})$. Choose $\phi \in C_b^2(\overline{D})$ so that $h = \phi|_{\partial D}$ is non-negative on ∂D. Then passing to the limit in the extended approximate martingale problem (4.5) for $u^{(N)}$, we have shown convergence of all but one of the terms, so that this last term $u_t^{\partial D,(N)}(h)$ must also converge in probability to $u_t^{\partial D}(h)$. The fact that $t \to u_t^{(N),\partial D}(h)$ and $t \to u_t^{\partial D}(h)$ are non-decreasing and continuous imply, at least along a further subsequence and for all t, that $\sup_{s \le t} |u_s^{(N),\partial D}(h) - u_s^{\partial D}(h)| \to 0$ in probability. A further subsequence argument allows us to conclude the same for a countable dense set in $C_b^0(\partial D)$, and this implies that $u^{\partial D,(N)} \to u^{\partial D}$ in $C([0,\infty), \mathcal{M}(\partial D))$. The uniqueness in law of $(u, u^{\partial D})$ for solutions to (1.1) implies the convergence of $(u^{(N)}, u^{\partial D,(N)})$ in part (i).

Based on (4.10), the proof of the L^1 convergence of part (ii) uses similar, but slightly simpler, tricks as above when dealing with the convergence of the key term $\int_0^t u_s^{(N)}(v_s^{(N)}\phi)ds$, and the details are omitted. □

5 Proof of the decomposition results

This section contains the proofs of the decomposition results stated in section 2.2. In section 5.1 we show that the exit and occupation measures of the approximate discrete nutrient process, as constructed in section 4.1, can be done in two stages. In section 5.2, five different examples of this two-stage construction lead to five decomposition results for the approximation processes. Passage to the limit is done in section 5.3.

5.1 A two-stage construction of the approximations

The construction of the approximation process given in section 4.1 is a pathwise construction. The approximation is constructed as a deterministic procedure, which we call the *basic construction*, applied to fixed realizations of the non-interacting DW(D,γ) processes $(u_k : k = 0, 1, \ldots)$ and exponential variables $(e_k : k = 1, 2, \ldots)$. In this section we show that the occupation and exit measures from this construction can be built up in two stages, where each stage applies the basic construction to a certain set of variables. This is described in a somewhat abstract manner, and the reader might want to look ahead at one of the five examples in section 5.2 to be convinced it is a natural idea. For the rest of this subsection we act pathwise, supposing a single realization has been fixed of the underlying variables.

Integrating over t in (4.1), or letting $t \to \infty$, we obtain

$$u_{[0,\infty)} = u_{0,[0,\infty)} + \sum_{k \in \mathcal{S}_\infty} u_{k,[0,\infty)} \quad \text{and} \quad u^{\partial D}_\infty = u^{\partial D}_{0,\infty} + \sum_{k \in \mathcal{S}_\infty} u^{\partial D}_{k,\infty}, \tag{5.1}$$

where $\mathcal{S}_\infty = \lim_{t\to\infty} \mathcal{S}_t$ is the set of labels of nutrient packages that are ever triggered. So the total occupation and exit measures are determined by the set $\mathcal{S}_\infty$ and the total occupation and exit measures of the non-interacting DW processes. Moreover for $k \in \mathcal{S}_\infty$ we have

$$u_{0,[0,\infty)}(\hat{\psi}_k) + \sum_{j \in \mathcal{S}_\infty} u_{j,[0,\infty)}(\hat{\psi}_k) > e_k$$

while for the $k \notin \mathcal{S}_\infty$ the converse inequality holds. This is exactly the condition that $\mathcal{S}_\infty$ is a fixed point of the mapping T defined in the abstract lemma below, once the choices $A = \{1, \ldots, K\}$ and

$$e(k) = e_k, \quad f(k) = u_{0,[0,\infty)}(\hat{\psi}_k), \quad M(j,k) = u_{j,[0,\infty)}(\hat{\psi}_k) \quad \text{for } j, k \in A$$

have been made.

Lemma 16. *Suppose A is a finite set, and fix $e, f : A \to [0,\infty)$ and $M : A \times A \to [0,\infty)$. Define, for $B \subseteq A$,*

$$T(B) = \left\{ a \in A : f(a) + \sum_{b \in B} M(b,a) > e(a) \right\}. \tag{5.2}$$

Then there is a unique smallest fixed point $S \subseteq A$ of T, that is $T(S) = S$ and S is contained in any other fixed point. Moreover if $B \subseteq S$ then $T^n(B)$ equals S for large n.

Proof. Since $M(a,b) \geq 0$ we see that if $B \subseteq B'$ then $T(B) \subseteq T(B')$. So $T^n(\emptyset)$ increases to a limit S which must be a fixed point of T. Any other fixed point S' contains $T^n(\emptyset)$ for all n and hence contains S and the uniqueness of the smallest fixed point is clear. If $B \subseteq S$ then $T^n(\emptyset) \subseteq T^n(B) \subseteq T^n(S) = S$ and the result follows since $T^n(\emptyset) = S$ for large n. □

It is straightforward to check that $\mathcal{S}_\infty$ is the smallest fixed point of T. Indeed, let k_m be the label of the mth nutrient package to be triggered, and let τ_m be the time it is triggered; then the definition of τ_m shows, when $\tau_m < \infty$, that $k_m \in T(\{k_1, \ldots, k_{m-1}\})$. Inductively, $\{k_1, \ldots, k_m\} \subseteq T^m(\emptyset)$ must hold and letting $m \to \infty$ shows that $\mathcal{S}_\infty$ is contained in the smallest fixed point of T.

Example 0. To illustrate the use of this lemma, we give here the analogue of the comparison Lemma 6 for the approximation processes. Suppose that $\sum_{k=1}^L \psi_k = f$ and $\sum_{k=L+1}^K \psi_k = g$. Then running the basic construction on the processes $(u_k : k = 0, 1, \ldots, K)$ and exponential variables $(e_k : k = 1, \ldots, K)$ we obtain the approximation process $(u, u^{\partial D}, v)$ started at μ and with initial nutrient level $f + g$, and the triggered set $\mathcal{S}_\infty$ that is the smallest fixed point of a mapping T. We now set $\tilde{u}_0 = u_0 + \sum_{k=L+1}^K u_k$. Now running the basic construction on the realizations of $(\tilde{u}_0, u_1, u_2, \ldots, u_L)$ and the exponential variables $(e_k : k = 1, \ldots, L)$ we obtain an approximation process $(\tilde{u}, \tilde{u}^{\partial D}, \tilde{v})$ started at $\mu + \beta g\, dx$ and with initial nutrient level f, and the triggered set $\tilde{\mathcal{S}}_\infty$ that is the smallest fixed point of a corresponding mapping $\tilde{T}$. Moreover, for $A \subseteq \{1, \ldots, L\}$,

$$\begin{aligned}\tilde{T}(A) &= \left\{ k \in \{1,\ldots,L\} : \tilde{u}_{0,[0,\infty)}(\hat{\psi}_k) + \sum_{j=1}^{L} I(j \in A) u_{j,[0,\infty)}(\hat{\psi}_k) > e_k \right\} \\ &= \left\{ k \in \{1,\ldots,L\} : u_{0,[0,\infty)}(\hat{\psi}_k) + \sum_{j=1}^{K} I(j \in A \cup \{L+1,\ldots,K\}) u_{j,[0,\infty)}(\hat{\psi}_k) > e_k \right\} \\ &= T(A \cup \{L+1,\ldots,K\}) \setminus \{L+1,\ldots,K\}.\end{aligned}$$

Choosing $A = \emptyset$ and then iterating we find that $T^n(\emptyset) \setminus \{L+1,\ldots,K\} \subseteq \tilde{T}^n(\emptyset)$ and hence, by Lemma 16, that $\mathcal{S}_\infty \setminus \{L+1,\ldots,K\} \subseteq \tilde{\mathcal{S}}_\infty$. Combining this with (5.1) shows that $(u_{[0,\infty)}, u^{\partial D}_\infty)$ is smaller than $(\tilde{u}_{[0,\infty)}, \tilde{u}^{\partial D}_\infty)$.

We now describe the two-stage procedure, which will build up the set $\mathcal{S}_\infty$ of triggered nutrient packages in two steps. We suppose there is a splitting of each of the DW processes $(u_k : k = 0, \ldots, K)$, and their exit measures, into two parts, u^-_k and u^+_k, each with continuous measure valued paths, and satisfying

$$\left(u_{k,[0,\infty)}, u^{\partial D}_{k,\infty}\right) = \left(u^-_{k,[0,\infty)} + u^+_{k,[0,\infty)}, u^{\partial D,-}_{k,\infty} + u^{\partial D,+}_{k,\infty}\right). \tag{5.3}$$

In stage one we apply the basic construction using the processes $(u^-_k : k = 0, 1, \ldots, K)$ and the exponential variables $(e_k : k = 1, \ldots, K)$ to create a process (u^-, v^-) and a set $\mathcal{S}^-_\infty$ of triggered nutrient packages. In particular

$$u^-_{[0,\infty)} = u^-_{0,[0,\infty)} + \sum_{k \in \mathcal{S}^-_\infty} u^-_{k,[0,\infty)}. \tag{5.4}$$

We then define $\hat{u}_{0,t} = u^+_{0,t} + \sum_{k \in \mathcal{S}^-_\infty} u^+_{k,t}$ and, for $k \in \{1, \ldots, K\} \setminus \mathcal{S}^-_\infty$,

$$\hat{u}_{k,t} = u_{k,t} \quad \text{and} \quad \hat{e}_k = e_k - u^-_{0,[0,\infty)}(\hat{\psi}_k) - \sum_{j \in \mathcal{S}^-_\infty} u^-_{j,[0,\infty)}(\hat{\psi}_k). \tag{5.5}$$

In stage two we run the basic construction on the processes $\hat{u}_0$ and $(\hat{u}_k : k \in \{1, \ldots, K\} \setminus \mathcal{S}^-_\infty)$, with the nutrient packages triggered using the values $(\hat{e}_k : k \in \{1, \ldots, K\} \setminus \mathcal{S}^-_\infty)$. This leads to a second process (u^+, v^+) and a second set of triggered nutrient packages $\mathcal{S}^+_\infty \subseteq \{1, \ldots, K\} \setminus \mathcal{S}^-_\infty$ satisfying

$$u^+_{[0,\infty)} = u^+_{0,[0,\infty)} + \sum_{k \in \mathcal{S}^-_\infty} u^+_{k,[0,\infty)} + \sum_{k \in \mathcal{S}^+_\infty} u_{k,[0,\infty)}. \tag{5.6}$$

Adding (5.4) and (5.6) and comparing to (5.1), we see that $u_{[0,\infty)} = u^-_{[0,\infty)} + u^+_{[0,\infty)}$ will hold provided that the equality $\mathcal{S}_\infty = \mathcal{S}^-_\infty \cup \mathcal{S}^+_\infty$ holds. A similar argument shows that this equality is also sufficient to ensure that the total exit measures satisfy $u^{\partial D}_\infty = u^{\partial D,-}_\infty + u^{\partial D,+}_\infty$.

To verify this equality we use the following second abstract lemma. With the choices

$$e(k) = e_k, \quad f^\pm(k) = u^\pm_{0,[0,\infty)}(\hat{\psi}_k), \quad M^\pm(j,k) = u^\pm_{j,[0,\infty)}(\hat{\psi}_k) \quad \text{for } j,k \in A$$

note that the definition of $\hat{f}, \hat{e}, \hat{M}$ in the lemma corresponds to the definition of $\hat{u}_0, \hat{e}_k$ in the stage two construction. The conclusion of the lemma then implies that the equality $\mathcal{S}_\infty = \mathcal{S}^-_\infty \cup \mathcal{S}^+_\infty$ does indeed hold.

For the statement of the lemma we denote the unique smallest fixed point in Lemma 16 by $S = S(A, e, f, M)$ to indicate its dependence.

Lemma 17. *Suppose A is a finite set, and fix $e, f^-, f^+ : A \to [0,\infty)$ and $M^-, M^+ : A \times A \to [0,\infty)$. Set $M = M^- + M^+$ and $f = f^- + f^+$. Then the fixed point $S(A, e, f, M)$ can be broken into two subsets as follows. First let $S^- = S(A, e, f^-, M^-)$. Then define, for $a \in A \setminus S^-$,*

$$\hat{f}(a) = f^+(a) + \sum_{b \in S^-} M^+(b,a), \quad \hat{e}(a) = e(a) - f^-(a) - \sum_{b \in S^-} M^-(b,a)$$

and write $\hat{M}$ for the restriction of M to $(A\backslash S^-)\times(A\backslash S^-)$. Then $\hat{e}\geq 0$ and if S^+ is the fixed point $S(A\backslash S^-,\hat{e},\hat{f},\hat{M})$ we have

$$S(A,e,f,M)=S^-\cup S^+.$$

Proof. Let T (respectively T^- and T^+) be the map defined by (5.2) used for the definition of S (respectively S^- and S^+). The fact that $\hat{e}(a)\geq 0$ for $a\in A\setminus S^-$ follows from the fact that S^- is a fixed point of T^-.

Since $f\geq f^-$ and $M\geq M^-$ we see that $T^-(B)\subseteq T(B)$. (Note in particular $S^-\subseteq T^n(S^-)$ for all $n\geq 1$.) Then $(T^-)^n(\emptyset)\subseteq T^n(\emptyset)$ for all n and hence $S^-\subseteq S$.

For $B\subseteq A\setminus S^-$ we have

$$\begin{aligned}
T^+(B) &= \left\{a\in A\setminus S^- : \hat{f}(a)+\sum_{b\in B}\hat{M}(b,a)>\hat{e}(a)\right\}\\
&= \left\{a\in A\setminus S^- : f^+(a)+\sum_{b\in S^-}M^+(b,a)+\sum_{b\in B}M(b,a)>e(a)-f^-(a)-\sum_{b\in S^-}M^-(b,a)\right\}\\
&= \left\{a\in A\setminus S^- : f(a)+\sum_{b\in B\cup S^-}M(b,a)>e(a)\right\}\\
&= T(B\cup S^-)\setminus S^-.
\end{aligned}$$

Applying this first to $B=\emptyset$ and iterating, we obtain $(T^+)^n(\emptyset)=T^n(S^-)\backslash S^-$. The equality here uses $S^-\subseteq T^n(S^-)$. Using the final statement of Lemma 16, we may let $n\to\infty$ to obtain $S^+=S\setminus S^-$ as desired. □

5.2 Comparison theorems for the approximations

Notation We denote the law of the approximation $(u_t,u_t^{\partial D}:t\geq 0)$ constructed in section 4.1, on the space $D([0,\infty),\mathcal{M}(D))\times C([0,\infty),\mathcal{M}(\partial D))$, by $Q^{N,D,\beta,\gamma}_{\mu,f}$.

We now give five examples of the two stage construction of the last section, each leading to a decomposition theorem for the discrete nutrient approximation process. In each example we describe the processes u_k^-,u_k^+ used for the splitting (5.3). We let

$$\mathcal{G}^-=\sigma\{u_0^-\}\vee\sigma\{\mathcal{S}_\infty^-\}\vee\sigma\{u_k^-I(k\in\mathcal{S}_\infty^-):1\leq k\leq K\}$$

be the information gained by observing the stage one process.

Example 1. Decomposition for the initial condition μ. Suppose that $\mu=\mu^-+\mu^+$. Take independent DW(D,γ) processes $(u_{0,t}^-)$ and $(u_{0,t}^+)$, with initial conditions μ^- and μ^+. Then $u_{0,t}=u_{0,t}^-+u_{0,t}^+$ is still a DW(D,γ) process. Using independent DW(D,γ) processes $(u_k:1\leq k\leq K)$ started at $u_{k,0}=\psi_k$, we set

$$u_{k,t}^-=u_{k,t}\quad\text{and}\quad u_{k,t}^+=0\quad\text{for } 1\leq k\leq K \text{ and } t\geq 0$$

so that the splitting (5.3) holds trivially. The first stage leads to a process $(u^-,u^{\partial D,-})$ with law $Q^{N,D,\beta,\gamma}_{\mu^-,f}$. The memorylessness property of exponentials implies that, conditional on $\mathcal{G}^-$, the variables $(\hat{e}_k:k\in\{1,\ldots,K\}\setminus\mathcal{S}_\infty^-)$ defined in (5.5) remain independent rate one exponential variables. The processes $\hat{u}_k$ for $k\in\{0,1,\ldots,K\}\setminus\mathcal{S}_\infty^-$ are, conditionally on $\mathcal{G}^-$, independent DW(D,γ) processes. Note that $v_\infty^-=\sum_{k\notin\mathcal{S}_\infty^-}\psi_k$ is $\mathcal{G}^-$ measurable. The second stage therefore produces a process $(u^+,u^{\partial D,+})$ which, conditional on $\mathcal{G}^-$, has law $Q^{N,D,\beta,\gamma}_{\mu^+,v_\infty^-}$.

Example 2. Decomposition for the initial condition f. We split the set of nutrient package labels into two parts $\{1,\ldots,L\}$ and $\{L+1,\ldots,K\}$ and set $f^- = \sum_{k=1}^L \psi_k$, $f^+ = \sum_{k=L+1}^K \psi_k$. Then we form the splitting (5.3) out of the independent DW(D,γ) processes $(u_k : 0 \le k \le K)$ started at $\mu, \psi_1, \ldots, \psi_K$ as follows: for $t \ge 0$

$$u^-_{k,t} = \begin{cases} u_{k,t} & \text{for } 0 \le k \le L, \\ 0 & \text{for } L+1 \le k \le K, \end{cases} \quad \text{and} \quad u^+_{k,t} = \begin{cases} 0 & \text{for } 0 \le k \le L, \\ u_{k,t} & \text{for } L+1 \le k \le K. \end{cases}$$

The first stage of the construction produces a process $(u^-, u^{\partial D,-})$ with law $Q^{N,D,\beta,\gamma}_{\mu,f^-}$. Note that $u^+_0 = \hat{u}_{0,0} = \beta \sum_{k=L+1}^K \psi_k I(k \in \mathcal{S}^-_\infty)$ and $v^+_0 = \sum_{k \notin \mathcal{S}^-_\infty} \psi_k$ are measurable with respect to the sigma field $\mathcal{G}^-$. Moreover, as above, the variables $(\hat{e}_k, \hat{u}_k : k \in \{1,\ldots,K\} \setminus \mathcal{S}^-_\infty)$ are, conditionally upon $\mathcal{G}^-$, independent exponential variables and DW(D,γ) processes. The second stage therefore produces a process $(u^+, u^{\partial D,+})$ which, conditional on $\mathcal{G}^-$, has law $Q^{N,D,\beta,\gamma}_{u^+_0,v^+_0}$.

Example 3. Spatial Markov property. We fix domains $D^- \subseteq D^+$. We will choose the finite partition $D^+ = \cup D_j$ used in the construction of the approximation so that each D_j is either a subset of D^- or a subset of $D^+ \setminus D^-$. Then a subset, which we list as $(\psi_k : 1 \le k \le L)$, will satisfy $\sum_{k=1}^L \psi_k = fI(D^-)$ and $\sum_{k=L+1}^K = fI(D^+ \setminus D^-)$. We apply the spatial Markov Lemma 8 to find processes $(u_k, u^-_k, u^+_k : 0 \le k \le K)$ so that $(u_k : 0 \le k \le K)$ are independent DW(D^+,γ) processes with initial conditions $\mu, \psi_1, \ldots, \psi_K$; $(u^-_k : 0 \le k \le K)$ are independent DW(D^-,γ) processes with initial conditions $\mu I(D^-), \psi_1, \ldots, \psi_L, 0, \ldots, 0$; and, conditional on $\sigma\{u^-_k : 0 \le k \le L\}$, $(u^+_k : 0 \le k \le K)$ are independent DW(D^+,γ) processes with initial conditions $u^+_{0,0} = \mu I(D^+ \setminus D^-) + u^{\partial D^-,-}_{0,[0,\infty)}|_{D^+}$, $u^+_{k,0} = u^{\partial D^-,-}_{k,\infty}|_{D^+}$ for $1 \le k \le L$ and $u^+_{k,0} = \psi_k$ for $L+1 \le k \le K$. In addition, the Lemma 8 ensures the splitting

$$\left(u_{k,[0,\infty)}, u^{\partial D^+}_{k,\infty}\right) = \left(u^-_{k,[0,\infty)} + u^+_{k,[0,\infty)}, u^{\partial D^-,-}_\infty|_{\partial D^+} + u^{\partial D^+,+}_{k,\infty}\right) \quad \text{for } 0 \le k \le K.$$

This gives a suitable splitting as in (5.3) for the two part construction, where the exit measures are all evaluated as measures on ∂D^+. The first stage of the construction produces a process $(u^-, u^{\partial D^-,-})$ with law $Q^{N,D^-,\beta,\gamma}_{\mu I(D^-), fI(D^-)}$. The second stage produces a process $(u^+, u^{\partial D^+,+})$ which, conditional on $\mathcal{G}^-$, has law $Q^{N,D^+,\beta,\gamma}_{u^+_0,v^+_0}$, where $u^+_0 = \mu I(D^+ \setminus D^-) + u^{\partial D^-,-}_{[0,\infty)}|_{D^+}$ and $v^+_0 = fI(D^+ \setminus D^-) + v^-_\infty I(D^-)$.

Example 4. Decomposition for the birth rate β. Fix $\beta = \beta^- + \beta^+$. Take independent DW$(D,\gamma)$ processes $(u^-_k, u^+_k : 0 \le k \le K)$ with initial conditions $u^-_{0,0} = \mu$, $u^+_{0,0} = 0$, $u^-_{k,0} = \beta^- \psi_k$ and $u^+_{k,0} = \beta^+ \psi_k$. Set $u_k = u^-_k + u^+_k$ so that $(u_k : 0 \le k \le K)$ satisfy the splitting (5.3) and are themselves DW(D,γ) processes. Then the first stage of the construction produces a process $(u^-, u^{\partial D,-})$ with law $Q^{N,D,\beta^-,\gamma}_{\mu,f}$. The second stage produces a process $(u^+, u^{\partial D,+})$ which, conditional on $\mathcal{G}^-$, has law $Q^{N,D,\beta,\gamma}_{u^+_0,v^+_0}$, where $u^+_0 = \beta^+(f - v^-_\infty)\,dx$ and $v^+_0 = v^-_\infty$.

Example 5. Decomposition for the death rate γ. Fix $\gamma^- = \gamma + \gamma^+$. Lemma 18 below ensures we can find processes $(u_k, u^-_k, u^+_k : 0 \le k \le K)$ satisfying the splitting (5.3) and so that $(u_k : 0 \le k \le K)$ are independent DW(D,γ) processes with initial conditions $\mu, \psi_1, \ldots, \psi_K$; $(u^-_k : 0 \le k \le K)$ are independent DW(D,γ^-) processes with initial conditions $\mu, \psi_1, \ldots, \psi_K$; and, conditional on $\sigma\{u^-_k : 0 \le k \le K\}$, $(u^+_k : 0 \le k \le K)$ are independent DW(D,γ) processes with initial conditions $(\gamma^+ u^-_{k,[0,\infty)} : 0 \le k \le K)$. Then the first stage of the construction produces a process $(u^-, u^{\partial D,-})$ with law $Q^{N,D,\beta,\gamma^-}_{\mu,f}$. The second stage produces a process $(u^+, u^{\partial D,+})$ which, conditional on $\mathcal{G}^-$, has law $Q^{N,D,\beta,\gamma}_{u^+_0,v^+_0}$, where $u^+_0 = \gamma^+ u^-_{[0,\infty)}$ and $v^+_0 = v^-_\infty$.

Lemma 18. *Suppose $\gamma^- = \gamma + \gamma^+$. There exists a coupling of three processes: u a DW(γ, D) process with initial condition μ; u^- a DW(γ^-, D) process with initial condition μ; and u^+ which, conditional on $\sigma\{u^-\}$, is a DW(γ, D) process with initial condition $\gamma^+ u^-_{[0,\infty)}$; and moreover these processes satisfy the splitting*

$$\left(u_{[0,\infty)}, u^{\partial D}_\infty\right) = \left(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u^{\partial D,-}_\infty + u^{\partial D,+}_\infty\right).$$

Proof. Construct a coupling of two processes $(u^-, \tilde{u})$ as follows. Let u^- be a $\mathrm{DW}(D, \gamma^-)$ process with initial condition μ. Conditionally on $\sigma\{u^-\}$, let $\tilde{u}$ be a DW process on the space $D \times R$, with spatial motion that is Brownian on the first component D and zero on the second component R, with annihilation rate γ and with initial condition $\gamma^+ u_r(dx) I(r \geq 0) dr$. If we define

$$u_t^+(dx) = \tilde{u}_t(dx \times R) \quad \text{and} \quad u_t^{\partial D,+}(A) = u_t^{\partial(D\times R)}(A \times R) \text{ for } A \subseteq \partial D$$

then conditionally on $\sigma\{u^-\}$ the process u^+ is a $\mathrm{DW}(D, \gamma)$ process started at $\gamma^+ u^-_{[0,\infty)}$ and with exit measure $u^{\partial D,+}$. Define measures I on $[0,\infty) \times D$ and $I^{\partial D}$ on $[0,\infty) \times \partial D$ by

$$\begin{aligned} I(h^{(2)}) &= I^-(h^{(2)}) + I^+(h^{(2)}) = \int_0^\infty u_s^-(h_s^{(2)}) ds + \int_0^\infty \int_{D\times R} h_{s+r}^{(2)}(x) \tilde{u}_s(dx, dr)\, ds, \\ I^{\partial D}(h^{(3)}) &= I^{\partial D,-}(h^{(3)}) + I^{\partial D,+}(h^{(3)}) = \int_0^\infty d_s u_s^{\partial D}(h_s^{(3)}) + \int_0^\infty \int_{\partial(D\times R)} h_{s+r}^{(3)}(x) d_s \tilde{u}_s(dx, dr) \end{aligned}$$

for bounded measurable $h^{(2)} : [0,\infty) \times D \to R$ and $h^{(3)} : [0,\infty) \times \partial D \to R$. We claim that $(I(dt,dx), I^{\partial D}(dt,dx))$ has the same law as $(U_t(dx)dt, d_t U_t^{\partial D}(dx))$ under $Q_\mu^{D,\gamma}$. This allows us to define the the required process $(u, u^{\partial D})$ using $(I, I^{\partial D})$ and the required splitting follows from the definitions of I and u^+.

To establish the claim on the law of $(I, I^{\partial D})$ we fix smooth non-negative $h_s^{(2)}(x), h_s^{(3)}(x)$ that vanish for $s \geq t$. It is enough to check that $E[\exp(-I(h^{(2)}) - I^{\partial D}(h^{(3)}))] = \exp(-\mu(\phi_t))$ where $(\phi_s : s \in [0,t])$ is the solution to the log-Laplace equation (3.7). We start by calculating the conditional expectation $E[\exp(-I(h^{(2)}) - I^{\partial D}(h^{(3)}))|\sigma\{u^-\}]$. Note the almost sure limits

$$I^+(h^{(2)}) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_0^\infty \int_{D\times[\frac{k}{N},\frac{k+1}{N})} h_{s+\frac{k}{N}}^{(2)}(x) \tilde{u}_s(dx,dr) ds = \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_0^\infty X_{K,s}(h_{s+\frac{k}{N}}^{(2)}) ds$$

and

$$I^{\partial D,+}(h^{(3)}) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_0^\infty \int_{\partial(D\times R)} I_{r\in[\frac{k}{N},\frac{k+1}{N})} h_{s+\frac{k}{N}}^{(3)}(x) d_s \tilde{u}_s(dx,dr) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_0^\infty d_s X_{K,s}^{\partial D}(h_{s+\frac{k}{N}}^{(2)})$$

where $(X_{k,t} : t \geq 0)$, for $k = 0, 1, \ldots$, are defined by $X_{k,t}(A) = \tilde{u}_t(A \times [\frac{k}{N}, \frac{k+1}{N}))$ and are, conditionally on $\sigma\{u^-\}$, independent $\mathrm{DW}(D,\gamma)$ processes with initial conditions $X_{k,0} = \gamma^+ u^-_{[\frac{k}{N},\frac{k+1}{N}]}$. Using the Laplace functional (3.6) of $(X_k, X_k^{\partial D})$ we find

$$\begin{aligned} E[e^{-I^+(h^{(2)}) - I^{\partial D,+}(h^{(3)})}|\sigma\{u^-\}] &= \lim_{N\to\infty} \exp(-\gamma^+ \sum_{k=0}^{N-1} u^-_{[\frac{k}{N},\frac{k+1}{N}]}(\phi_{t-\frac{k}{N}})) \\ &= e^{-\gamma^+ \int_0^t u_s^-(\phi_{t-s}) ds}. \end{aligned}$$

Then we complete the expectation as

$$E[e^{-I(h^{(2)}) - I^{\partial D}(h^{(3)})}] = E[e^{-\int_0^t u_s^-(h^{(2)} + \gamma^+ \phi_{t-s}) ds - \int_0^t d_s u_s^{\partial D,-}(h_s^{(3)})}]$$

which can be calculated again using the Laplace functional of $(u^-, u^{\partial D,-})$. It is straightforward to check the required log-Laplace function is solved by ϕ and this proves the claim. □

Remark. A more natural splitting for the final example is to let (u_k^-, u_k^+) solve, for each k, the system

$$\begin{aligned} \partial_t u^- &= \Delta u^- - \gamma^- u^- + \sqrt{u^-}\, \dot{W}^-, \\ \partial_t u^+ &= \Delta u^+ + \gamma^+ u^- - \gamma u^+ + \sqrt{u^+}\, \dot{W}^+, \end{aligned}$$

with $u_0^+ = 0$ and with orthogonal martingale terms. Then u_k^- is a $\mathrm{DW}(D, \gamma-)$ process and $u_k = u_k^- + u_k^+$ is a $\mathrm{DW}(D,\gamma)$ process. The first stage of the construction produces a process (u^-, v^-) with law $Q_{\mu,f}^{N,D,\beta,\gamma^-}$. The second stage produces a process (u^+, v^+) which, conditional on $\mathcal{G}^-$, has the law of the approximation process with parameters β, γ, but with an extra immigration term $\gamma^+ u^-$. This leads to a corresponding decomposition for solutions with immigration to (1.1), but we shall not make any use of it.

5.3 Completion of the proof of the comparison results

This section contains the proofs of the decomposition lemmas 4, 5, 6 and 7, using the comparison results for the approximations from section 5.2 with the convergence of the approximations established in section 5.1.

We start with a simple estimate controlling the death time $\tau = \inf\{t : U_t = 0\}$ for the approximation processes on a bounded domain uniformly in N. The change of measure argument given at the beginning of section 3.3 can be used to show the limiting death time has a tail $Q^{D,\beta,\gamma}_{\mu,f}[\tau \geq t] \leq C(\beta, p, |D|, \mu) t^{-p}$ for any $p < 1$. This argument does not apply directly to the approximations since they are not absolutely continuous with respect to a DW process. However, the argument below follows the same main idea. It is combined with a first moment argument that leads to a suboptimal bound, but which is sufficient for our needs.

Lemma 19. *There exists $C(\beta) < \infty$ so that for all $t \geq e$ and $N \geq 1$*

$$Q^{N,D,\beta,\gamma}_{\mu,f} [\tau \geq t] \leq C(\beta) \left((\mu(1))^{1/2} + \langle f, 1 \rangle \right) \ln(t)^{-1}.$$

Proof. Take an approximation process u with law $Q^{N,D,\beta,\gamma}_{\mu,f}$. Taking $\phi = 1$ in the martingale problem (4.5) and $\lambda_s = -2/s$, so that $\dot{\lambda}_s = -\lambda_s^2/2$, we find for fixed $t > 0$ and $s \in [0, t)$

$$\begin{aligned} de^{-\lambda_{t-s} u_s(1)} &= e^{-\lambda_{t-s} u_s(1)} \left(u_s(1)(\dot{\lambda}_{t-s} + \frac{1}{2}\lambda^2_{t-s} - \gamma \lambda_{t-s}) ds + \lambda_{t-s} du^{\partial D}_s(1) - \beta \lambda_{t-s} u_s(v_s) ds \right) \\ &\quad + \sum_{s \leq t} D_s(e^{-\lambda_{t-s} u_s(1)}) + \lambda_{t-s} e^{-\lambda_{t-s} u_{s-}(1)} D_s u_s(1) + \text{martingale increments} \\ &\geq -\beta \lambda_{t-s} e^{-\lambda_{t-s} u_s(1)} u_s(v_s) ds + \text{martingale increments}. \end{aligned}$$

Let $m(dx, ds)$ be the continuous martingale measure constructed from the martingales $m_t(\phi)$ in (4.5) and, as in the change of measure arguments, set $M_t = \int_0^t \int v_s(x) m(dx, ds)$. Then Ito's lemma, using the cross variation,

$$\begin{aligned} d\left[\mathcal{E}_s(-\beta M), e^{-\lambda_{t-s} u_s(1)} \right]_s &= \beta \lambda_{t-s} e^{-\lambda_{t-s} u_s(1)} \mathcal{E}_s(-\beta M) d\left[M_s, m_s(1) \right]_s \\ &= \beta \lambda_{t-s} e^{-\lambda_{t-s} u_s(1)} \mathcal{E}_s(-\beta M) u_s(v_s) ds \end{aligned}$$

shows that $s \to \mathcal{E}_s(-\beta M)(1 - \exp(-\lambda_{t-s} u_s(1))$ is a non-negative supermartingale on $s \in [0, t)$. Taking expectations at $s_n \uparrow t$, we find

$$\begin{aligned} E\left[\mathcal{E}_t(-\beta M) I(\tau > t) \right] &\leq \lim_{n \to \infty} E\left[\mathcal{E}_{s_n}(-\beta M)(1 - e^{-\lambda_{t-s_n} u_{s_n}(1)}) \right] \\ &\leq \left(1 - e^{-\lambda_t \mu(1)} \right) \leq 2\mu(1)/t. \end{aligned}$$

Let $\sigma_K = \inf\{s : \int_0^s u_s(v_s) ds \geq K\}$. Then, using Cauchy-Schwarz, and $v_s \leq f \leq 1$,

$$\begin{aligned} \left(P\left[\tau > t, \sigma_K > t \right] \right)^2 &\leq E\left[I(\tau > t) \mathcal{E}_t(-\beta M) \right] E\left[I(\sigma_K > t) \mathcal{E}_t^{-1}(-\beta M) \right] \\ &\leq \frac{2\mu(1)}{t} E\left[I(\sigma_K > t) \mathcal{E}_t(+\beta M) e^{\beta^2 \int_0^t u_s(v_s^2) ds} \right] \leq \frac{2\mu(1)}{t} e^{\beta^2 K}. \end{aligned} \tag{5.7}$$

To control σ_K we use a crude first moment bound. From (4.8) we obtain

$$E[(v_t, 1)] = \langle f, 1 \rangle - \int_0^t \int_D E[v_s(x) u_s(\hat{\psi}_x)] dx ds = \langle f, 1 \rangle - \int_0^t E[u_s(v_s)] ds$$

(using the fact that v is constant on the partition sets D_j). Then $P[\sigma_K \leq t] \leq \langle f, 1 \rangle / K$ by Markov's inequality. Combining this with (5.7) and the choice $K = c \ln t$, for small $c = c(\beta)$, leads to the desired bound. □

We may characterize the total occupation and exit measures via the Laplace functional, defined by

$$\Phi_{\phi,\psi}(\nu_1,\nu_2) = e^{-\nu_1(\phi)-\nu_2(\psi)}$$

for $\nu_1 \in \mathcal{M}(D)$, $\nu_2 \in \mathcal{M}(\partial D)$, $0 \le \phi \in C_b^0(D)$ and $0 \le \psi \in C_b^0(\partial D)$. Fix $\mu \in \mathcal{M}(D)$ and $f \in \mathcal{L}^1(D)$, for a bounded domain D. Choose approximations processes with initial nutrient levels $1 \ge f^{(N)} \to f$ in L^1 and $\mu^{(N)} \to \mu$ weakly. Theorem 15 (ii) implies that

$$\begin{aligned}\lim_{N\to\infty} Q^{N,D,\beta,\gamma}_{\mu^{(N)},f^{(N)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right] &= \lim_{N\to\infty}\lim_{T\to\infty} Q^{N,D,\beta,\gamma}_{\mu^{(N)},f^{(N)}}\left[\Phi_{\phi,\psi}(U_{[0,T]},U^{\partial D}_T)\right] \\ &= \lim_{T\to\infty}\lim_{N\to\infty} Q^{N,D,\beta,\gamma}_{\mu^{(N)},f^{(N)}}\left[\Phi_{\phi,\psi}(U_{[0,T]},U^{\partial D}_T)\right] \\ &= \lim_{T\to\infty} Q^{D,\beta,\gamma}_{\mu,f}\left[\Phi_{\phi,\psi}(U_{[0,T]},U^{\partial D}_T)\right] \\ &= Q^{D,\beta,\gamma}_{\mu,f}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]. \end{aligned} \tag{5.8}$$

The interchange of limits is justified by the above lemma, which gives uniform (in N) control on $Q^{N,D,\beta,\gamma}_{\mu^{(N)},f^{(N)}}[U_T = 0]$, and the fact that $U_{[0,\infty)} = U_{[0,T]}$ and $U^{\partial D}_\infty = U^{\partial D}_T$ on the set $\{U_t = 0\}$.

Proof of Lemma 6. Returning to Remark 0 in section 5.1 we see that, taking approximations $f^{(N)}, g^{(N)} \le 1$ converging to f, g as above, and noting that $g^{(N)}(x)dx \to g(x)dx$ in $\mathcal{M}(D)$,

$$Q^{N,D,\beta,\gamma}_{\mu,f^{(N)}+g^{(N)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right] \ge Q^{N,D,\beta,\gamma}_{\mu+\beta g^{(N)}dx,f^{(N)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right].$$

Passing to the limit as above we find that

$$Q^{D,\beta,\gamma}_{\mu,f+g}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right] \ge Q^{D,\beta,\gamma}_{\mu+\beta g\,dx,f}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right].$$

This inequality between the Laplace functionals implies the desired stochastic domination.

Proof of Lemma 4 part (i). Consider the decomposition u^-, u^+ given in section 5.2 example 1, with initial conditions $\mu = \mu^- + \mu^-$ and $f^{(N)} \to f$ in $L^1(D)$. Then pathwise

$$\Phi_{\phi,\psi}(u_{[0,\infty)},u^{\partial D}_\infty) = \Phi_{\phi,\psi}(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u^{\partial D,-}_\infty + u^{\partial D,+}_\infty). \tag{5.9}$$

As above, the expectation of the left hand side converges, as $N \to \infty$, to $Q^{D,\beta,\gamma}_{\mu,f}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]$. Using the law of (u^+, v^+) conditional on $\mathcal{G}^-$, the limit of the expectation of the right hand side can be written as

$$\begin{aligned}&\lim_{N\to\infty} E\left[\Phi_{\phi,\psi}(u^-_{[0,\infty)},u^{\partial D,-}_\infty)\, Q^{N,D,\beta,\gamma}_{\mu^+,v^-_\infty}\left[\Phi(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right] \\ &= \lim_{N\to\infty}\lim_{T\to\infty} E\left[\Phi_{\phi,\psi}(u^-_{[0,T]},u^{\partial D,-}_T)\, Q^{N,D,\beta,\gamma}_{\mu^+,v^-_T}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right] \\ &= \lim_{T\to\infty} Q^{D,\beta,\gamma}_{\mu^-,f}\left[\Phi_{\phi,\psi}(U_{[0,T]},U^{\partial D}_T)\, Q^{D,\beta,\gamma}_{\mu^+,fe^{-U(0,T)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right] \\ &= Q^{D,\beta,\gamma}_{\mu^-,f}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\, Q^{D,\beta,\gamma}_{\mu^-,fe^{-U(0,\infty)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right].\end{aligned}$$

The interchange in limits follows as above, using $v^-_T = v^-_\infty$ on $\{u^-_T = 0\}$. The $N \to \infty$ limit holds using the convergence of $(u^-_{[0,T]}, u^{\partial D,-}, v^-_T)$ given by Theorem 15 (ii) and (5.8) above. The final identity between Laplace functionals implies the desired result.

Proof of Lemma 5. The proofs of both parts of this lemma follow closely the argument used for the proof of Lemma 4 part (i) above. For example, for part (i) take the decomposition u^-, u^+ given in section 5.2 example 4, so that (5.9) holds, and with initial conditions μ and $f^{(N)} \to f$ in L^1. Then argue as above, noting that

$$\begin{aligned}&\lim_{N\to\infty} E\left[\Phi_{\phi,\psi}(u^-_{[0,T]},u^{\partial D,-}_T) Q^{N,D,\beta,\gamma}_{\beta^+(f-v^-_T)dx,v^-_T}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right] \\ &= Q^{D,\beta^-,\gamma}_{\mu,f}\left[\Phi_{\phi,\psi}(U_{[0,T]},U^{\partial D}_T)\, Q^{D,\beta,\gamma}_{\beta^+f(1-e^{-U(0,T)})dx,fe^{-U(0,T)}}\left[\Phi_{\phi,\psi}(U_{[0,\infty)},U^{\partial D}_\infty)\right]\right]\end{aligned}$$

using the fact that if v_T^- converges in $L^1(D)$ then $\beta^+(f - v_T^-)dx$ converges in $\mathcal{M}(D)$. Part (ii) is entirely similar starting with example 5 in section 5.2.

Proof of Lemma 7. We can follow the previous argument closely, but we write this out since we have to be a little careful about the two domains. To indicate that we use the joint Laplace functional acting on measures on D^+ and ∂D^+, we denote it by $\Phi^+_{\phi,\psi}$. Take the decomposition u^-, u^+ given in section 5.2 example 3, so that $\Phi^+_{\phi,\psi}(u_{[0,\infty)}, u_\infty^{\partial D^+}) = \Phi^+_{\phi,\psi}(u^-_{[0,\infty)} + u^+_{[0,\infty)}, u_\infty^{\partial D^-,-}|_{\partial D^+} + u_\infty^{\partial D^+,+})$. Taking expectations, the left hand side converges, as $N \to \infty$, to $Q^{D^+,\beta,\gamma}_{\mu,f}\left[\Phi^+_{\phi,\psi}(U_{[0,\infty)}, U_\infty^{\partial D^+})\right]$. For this proof only, set $\mu^- = \mu I(D^-)$, $\mu^+ = \mu I(D^+ \backslash D^-)$, $f^- = fI(D^-)$, $f^+ = fI(D^+ \setminus D^-)$ and $f^{(N),-} = f^{(N)}I(D^-)$, $f^{(N),+} = f^{(N)}I(D^+ \setminus D^-)$ The limit of the right hand side can be evaluated as

$$\begin{aligned}
&\lim_{N\to\infty} E\left[\Phi^+_{\phi,\psi}(u^-_{[0,\infty)}, u_\infty^{\partial D^-,-}|_{\partial D^+})\, Q^{N,D^+,\beta,\gamma}_{\mu^+ + u_\infty^{\partial D^-,-}|_{D^+}, f^{(N),+} + v_\infty^- I(D^-)}\left[\Phi^+(U_{[0,\infty)}, U_\infty^{\partial D^+})\right]\right] \\
&= \lim_{N\to\infty}\lim_{T\to\infty} E\left[\Phi^+_{\phi,\psi}(u^-_{[0,T]}, u_T^{\partial D^-,-}|_{\partial D^+})\, Q^{N,D^+,\beta,\gamma}_{\mu^+ + u_T^{\partial D^-,-}|_{D^+}, f^{(N),+} + v_T^- I(D^-)}\left[\Phi^+_{\phi,\psi}(U_{[0,\infty)}, U_\infty^{\partial D^+})\right]\right] \\
&= \lim_{T\to\infty} Q^{D^-,\beta,\gamma}_{\mu^-,f^-}\left[\Phi^+_{\phi,\psi}(U_{[0,T]}, U_T^{\partial D^-}|_{\partial D^+})\, Q^{D^+,\beta,\gamma}_{\mu^+ + U_T^{\partial D^-}|_{D^+}, f^+ + fe^{-U(0,T)}I(D^-)}\left[\Phi^+_{\phi,\psi}(U_{[0,\infty)}, U_\infty^{\partial D^+})\right]\right] \\
&= Q^{D^-,\beta,\gamma}_{\mu^-,f^-}\left[\Phi^+_{\phi,\psi}(U_{[0,\infty)}, U_\infty^{\partial D^-}|_{\partial D^+})\, Q^{D^+,\beta,\gamma}_{\mu^+ + U_\infty^{\partial D^-}, f^+ + fe^{-U(0,\infty)}I(D^-)}\left[\Phi^+_{\phi,\psi}(U_{[0,\infty)}, U_\infty^{\partial D^+})\right]\right].
\end{aligned}$$

The limits above follow as in the previous examples, once we have established that $u_T^{\partial D^-,-}|_{\partial D^+}$ converges in $\mathcal{M}(\partial D^+)$ and $u_T^{\partial D^-,-}|_{D^+}$ converges in $\mathcal{M}(D^+)$. Both of these follow from the convergence of $u_T^{\partial D^-,-}$ in $\mathcal{M}(\partial D^-)$ under the hypothesis (2.6), since this hypothesis ensures that the limit law of $U_T^{\partial D^-}$ under $Q^{D^-,\beta,\gamma}_{\mu^-,f^-}$ does not charge the discontinuity set $S = (\partial D^- \cap \partial D^+) \cap \overline{(\partial D^- \setminus \partial D^+)}$ (check the first moment of $U_T^{\partial D^-}(S)$ is zero).

Proof of Lemma 4 part (ii). To allow this case to follow from the same argument as the earlier examples we need a slight improvement in Theorem 15 part (ii). Taking the two stage construction from section 5.2 example 3, we need to consider two parts of the approximate density, namely

$$\overline{v}_t^- = \sum_{1\le k\le L} \psi_k I(k \notin \mathcal{S}_t^-) \quad \text{and} \quad \overline{v}_t^+ = \sum_{L<k\le K} \psi_k I(k \notin \mathcal{S}_t^-).$$

Now consider a sequence of models indexed by N in which $f^{(N),\pm} \to f^\pm$ in $L^1(D)$ and where $f = f^- + f^+$. Then Theorem 15 part (ii) can be extended to show that the law of the quadruple $(u^{(N)}_{[0,T]}, u_T^{\partial D,(N)}, \overline{v}_T^{(N),-}, \overline{v}_T^{(N),+})$ on $\mathcal{M}(D) \times \mathcal{M}(\partial D) \times L^1(D) \times L^1(D)$ converges to the law of $(U_{[0,T]}, U_T^{\partial D}, f^- \exp(-U(0,T)), f^+ \exp(-U(0,T)))$ under $Q^{D,\beta,\gamma}_{\mu,f}$. This can be shown as in Theorem 15 once one notes that the analogue of (4.8) holds for $\overline{v}^\pm$, namely that there are martingales $m_t^{v,\pm}$ such that

$$\overline{v}_t^\pm(x) = f(x) - \int_0^t \overline{v}_s^\pm(x) u_s(\hat{\psi}_x) ds + m_t^{v,\pm}(x).$$

The initial conditions for stage two of the construction can be expressed in terms of $\overline{v}^\pm$ as $u_0^+ = \beta(f^+ - \overline{v}_\infty^+)dx$ and $v_0^+ = \overline{v}_\infty^- + \overline{v}_\infty^+$ and the passage to the limit then follows the same lines as previous examples.

6 Life

6.1 Embedded oriented percolation processes

In the proofs of possible life, the main step is to construct a discrete one-dimensional oriented percolation (OP) $(\omega(j,k) : (j,k) \in \mathcal{L})$ where $\mathcal{L} = \{(j,k) : j,k \in Z,\ j \ge 1,\ j+k \text{ is even}\}$, so that life occurs if the percolation is supercritical. The variables $\omega(j,k)$ will be defined in terms of the exit measures for solutions to (1.1) on a series of boxes. We will choose a parameter $L>0$ controlling the length scale. In the grid of points $x^L_{j,k}=(3jL,2kL)$ in $d=2$ or $x^L_{j,k}=(3jL,2kL,0)$ in $d=3$, the x_1 direction will play the usual role of time, and the x_2 direction the role of space for the comparison OP process. For this argument we therefore need to be in dimension $d\geq 2$.

We will freeze the mass of the solution as it exits the boxes $D(n)=(-3nL,3nL)^d$ for $n=1,2,\ldots$ We define exit measures $u^{n,\partial D(n)}_\infty$ inductively. Choose an initial condition μ supported in $D(1)$ and let (u^1,v^1) be a solution on $D(1)$ started from $(\mu,1)$. Suppose that (u^j,v^j) have been defined for $j=1,\ldots,n$. Then, conditional on $\sigma\{u^j:j=1,\ldots,n\}$ we let (u^{n+1},v^{n+1}) be a solution on $D(n+1)$ started from

$$u^{n+1}_0=u^{n,\partial D(n)}_\infty, \quad \text{and} \quad v^{n+1}_0(x)=\begin{cases} v^n_\infty(x) & x\in D(n),\\ 1 & x\in D(n+1)\setminus D(n).\end{cases}$$

Repeated use of the spatial Markov property in Lemma 7 shows that $u^{n,\partial D(n)}_\infty$ has the same law as the exit measure $U^{\partial D(n)}_\infty$ under the law $Q^{D(n),\beta,\gamma}_{\mu,1}$.

In the hyperplane $x_1=0$ we define the box $I_L=\{x:|x|\leq L, x_1=0\}$. Define, for $j,k\in\mathcal{L}$,

$$\tilde\omega(j,k)=\begin{cases} 1 & \text{if } u^{j,\partial D(j)}_\infty\left(x^L_{j,k}+I_L\right)>M,\\ 0 & \text{otherwise.}\end{cases}$$

Set $\tilde\omega(0,k)=0$ if $k\neq 0$ and $\tilde\omega(0,0)=I(\mu(I_L)\geq M)$. Define for $(j,k)\in\mathcal{L}$,

$$\omega(j,k)=\begin{cases} 1 & \text{if } \tilde\omega(j-1,k-1)=\tilde\omega(j-1,k+1)=0,\\ \tilde\omega(j,k) & \text{otherwise.}\end{cases}$$

We aim to show, for suitable choice of the length and mass scales L and M, that $(\omega(j,k):(j,k)\in\mathcal{L})$ is a 3-dependent oriented site percolation with density at least $1-\epsilon$. We recall the definition of this from Durrett [4]: whenever $(j_n,k_n)\in\mathcal{L}$, for $1\leq n\leq N$, satisfy $|j_n-j_m|\geq 3$ or $|k_n-k_m|\geq 3$ for all $n\neq m$, then

$$P[\omega(j_n,k_n)=0 \text{ for } 1\leq n\leq N]\leq \epsilon^N. \tag{6.1}$$

We write $(0,0)\to(j,k)$ when there exist a sequence of points $0=k_0,k_1,\ldots,k_{j-1},k_j=k$ so that so that $|k_m-k_{m-1}|=1$ for $0<m\leq j$ and satisfying $\omega(m,k_m)=1$ for $1\leq m\leq j$. Then let $\mathcal{C}_0=\{(j,k):(0,0)\to(j,k)\}$ be the cluster of sites connected to the origin. This definition is sufficient for the key property of percolation to hold: Theorem 4.1 of [4] shows that when $\epsilon\leq\epsilon_0=6^{-196}$ then $P[|\mathcal{C}_0|<\infty]\leq 1/20$, where $|\mathcal{C}_0|$ is the cardinality of $\mathcal{C}_0$. In particular, if the initial condition is such that $\tilde\omega(0,0)=1$, that is $\mu(I_L)\geq M$, then

$$Q^{D(n),\beta,\gamma}_{\mu,1}[U^{\partial D(n)}_\infty\neq 0]=P[u^{n,\partial D(n)}_\infty\neq 0]\geq P[|\mathcal{C}_0|=\infty]$$

is bounded away from zero, and by Lemmas 11 and 12, possible life occurs.

It remains to check the hypothesis (6.1) of being a 3-dependent oriented site percolation. We do this inductively. By conditioning on $\mathcal{G}_M=\sigma\{u^j:j=1,\ldots,M\}$ (and setting $\mathcal{G}_0$ to be the trivial sigma field) it suffices to show an estimate of the form

$$P[\omega(M+1,k_n)=0 \text{ for } 1\leq n\leq N_0\,|\,\mathcal{G}_M]\leq\epsilon^{N_0}.$$

whenever $|k_n-k_m|\geq 3$ for all $n\neq m$. If $\tilde\omega(M,k_n-1)=\tilde\omega(M,k_n+1)=0$ for some n then the conditional expectation is zero. So we may also restrict to the set where, for each n, there exists $\tilde k_n\in\{k_n-1,k_n+1\}$ for which $\tilde\omega(M,\tilde k_n)=1$. We let $D(M,n)$ be the box $x^L_{M,\tilde k_n}+D_{3L}$. Note that the boxes $D(M,n)$ for $n=1,\ldots,N_0$ are disjoint and contained in $D(M+1)$.

We now apply the spatial Markov property Lemma 7 to the pairs of domains $D^-=D(M,n)\subseteq D(M+1)=D^+$, for $n=1,\ldots,N_0$ in succession. Since the domains $D(M,n)$ are disjoint, this amounts to constructing, conditional on $\sigma\{u^j:j=1,\ldots,M\}$, independent solutions $(u^{M+1,n},v^{M+1,n})$ on $D(M,n)$ with initial condition

$$u^{M+1,n}_0=u^{M,\partial D(M)}_\infty I(D(M,n)), \quad \text{and} \quad v^{M+1,n}_0(x)=\begin{cases} v^M_\infty(x) & x\in D(M,n)\cap D(M),\\ 1 & x\in D(M,n)\setminus D(M),\end{cases}$$

for $n = 1, \ldots, N_0$. Then an exit measure $\tilde{u}_\infty^{M+1,\partial D(M+1)}$ can be constructed by running one further solution, conditionally on these N_0 processes, that starts with the combined exit measures of $(u^{M+1,n} : n = 1, \ldots, N_0)$ on their domains and any part of $u_\infty^{M,\partial D(M)}$ that is unused, and run until the exit from $D(M+1)$. This leads to a construction of an exit measure $\tilde{u}_\infty^{M+1,\partial D(M+1)}$ that, by the spatial Markov property, is equal in law to $u_\infty^{M+1,\partial D(M+1)}$, but that is pathwise larger than the sum of the exit measures $\sum_{n=1}^{N_0} u_\infty^{M+1,n,\partial D(M,n)}$ restricted to $\partial D(M+1)$. This in turn implies, since the solutions $(u^{M+1,n}, v^{M+1,n})$ are conditionally independent, that

$$P[\omega(M+1,k_n) = 0 \text{ for } 1 \le n \le N_0 \,|\, \mathcal{G}_M] \le \prod_{n=1}^{N_0} P[u_\infty^{M+1,n,\partial D(M,n)}(x^L_{M+1,k_n} + I_L) \le M \mid \mathcal{G}_M]$$

By translational invariance, and the monotonicity in the initial measure, the condition (6.1) will then follow from an estimate of the form

$$Q_{\mu,f}^{D_{3L},\beta,\gamma}[U_\infty^{\partial D_{3L}}(x^L_{1,\pm 1} + I_L) \le M] \le \epsilon$$
$$\text{whenever } f \ge I(x_1 \ge 0) \text{ and } \mu \text{ is supported on } I_L \text{ with } \mu(1) \ge M. \qquad (6.2)$$

By monotonicity in f it is enough to consider just the case $f = I(x_1 \ge 0)$. We shall check this block estimate in the next two subsections for two different regions of parameter values (β, γ).

6.2 Proof that $\liminf_{\beta \to 0} \Psi(\beta)/\beta^2 > 0$ in dimension $d = 3$

After applying the scaling in Lemma 3 with the choices $a = c = L\beta^{-1}, b = L^2\beta^{-2}, e = 1$, solutions to (1.1) with the parameter values $(\beta, L^{-2}\beta^2)$ in $d = 3$ become solutions to the rescaled equation (6.3) studied in the following lemma. Undoing this scaling, the lemma shows that the block estimate (6.2) holds for solutions to (1.1) when $\beta \in (0, 1]$ and with the choices

$$\gamma = L_0^{-2}(\epsilon)\beta^2, \; L = L_0(\epsilon)\beta^{-1}, \text{ and } M = L_0^2(\epsilon)\beta^{-2}.$$

Choosing $\epsilon = \epsilon_0$, the value from in section 6.1 that ensures the OP is supercritical, the OP comparison implies possible life for the parameter values $(\beta, L_0^{-2}(\epsilon_0)\beta^2)$, and thus $\liminf \Psi(\beta)/\beta^2 \ge L_0^{-2}(\epsilon_0)$ as $\beta \to 0$.

Lemma 20. *Consider solutions, in $d = 3$, on the domain $D_3 = (-3, 3)^3$ to the equation*

$$\begin{cases} \partial_t u = \Delta u + L^2\beta^{-1}uv - u + \sqrt{u}\,\dot{W}, \\ \partial_t v = -L\beta^{-1}uv, \quad v_0(x) = I(x \in H), \end{cases} \qquad (6.3)$$

where $H = \{x : x_1 \ge 0\}$. For any $\epsilon > 0$ there exists $L_0 = L_0(\epsilon) \ge 1$ so that, whenever (u, v) is a solution to (6.3) with parameters L_0 and $\beta \in (0, 1]$, and with an initial condition μ supported in $I_1 = \{x : x_1 = 0, |x| \le 1\}$ and satisfying $\mu(1) \ge 1$,

$$P\left[u_\infty^{\partial D_3}(x_{1,\pm 1} + I_1) \le 1\right] \le \epsilon.$$

where $x_{1,1} = (3, 2, 0)$ and $x_{-1,1} = (-3, 2, 0)$.

Remark. The intuition for the lemma comes from taking L large, ignoring for a moment the diffusion, killing and noise terms and reducing to the ODE system $\partial_t x = L^2\beta^{-1}xy, \; \partial_t y = -L\beta^{-1}xy$, where, for L large, the solution converges fast to $x_\infty = x_0 + Ly_0$. Thus the nutrient leads to a large build-up of mass, which in turn leads to a high probability that the exit measure is large. The corresponding scaling in dimension $d = 2$ fails to produce this large build up of mass.

Proof. We first construct a DW process u^- on D_3 with law $Q_\mu^{D_3,1}$. Then, conditional on $\sigma\{u^-\}$, we construct a solution (u^+, v^+) to (6.3) on D_3 with initial conditions

$$u_0^+ = LI(x \in H)(1 - e^{-L\beta^{-1}u^-(0,\infty,x)})dx \quad \text{and} \quad v_0^+ = I(x \in H)e^{-L\beta^{-1}u^-(0,\infty,x)}.$$

Lemma 5, after a suitable linear scaling, shows that $u_\infty^{\partial D_3,-} + u_\infty^{\partial D_3,+}$ has the same law as the total exit measure $u_\infty^{\partial D_3}$ of a solution to (6.3) started at μ. Lemma 5 also implies that the exit measure $u_\infty^{\partial D_3,+}$ is stochastically larger than the exit measure $U_\infty^{\partial D_3}$ where U has the law $Q^{D_3,1}_{u_0^+}$ of a DW process, conditionally on $\sigma\{u-\}$. Choose smooth $h : \partial D_3 \to [0,1]$ so that $\{h > 0\} = x_{1,1} + I_1$ (the case of $x_{1,-1}$ is symmetric). Then

$$E\left[e^{-u_\infty^{\partial D_3}(x_{1,1}+I_1)}\right] \le E\left[e^{-u_\infty^{\partial D_3}(h)}\right] \le E\left[Q^{D_3,1}_{u_0^+}[e^{-U_\infty^{\partial D_3}(h)}]\right] = E\left[e^{-u_0^+(w)}\right]$$

where, using the DW Laplace functional from (3.8), $\Delta w = \frac{1}{2}w^2 + w$ on D_3 and $w = h$ on ∂D_3. Let $c_0 = \inf\{w(x) : x \in D_2 \cap H\}$. Then a comparison argument shows that $c_0 \in (0,\infty)$. Using the initial condition of u_0^+ we continue, for $\beta \le 1$ and $L \ge 1$,

$$\begin{aligned} E\left[e^{-u_\infty^{\partial D_3}(x_{1,1}+I_1)}\right] &\le Q^{D_3,1}_{\mu}\left[\exp\left(-L\int_{H\cap D_3}\left(1-e^{-L\beta^{-1}U(0,\infty,x)}\right)w(x)dx\right)\right] \\ &\le Q^{D_3,1}_{\mu}\left[\exp\left(-Lc_0\int_{H\cap D_2}\left(1-e^{-U(0,\infty,x)}\right)dx\right)\right] =: p(L,\mu). \end{aligned}$$

We will show that $p(L,\mu)$ converges to zero, as $L \to \infty$, uniformly over μ as stated in the lemma. Then a Chebychev argument finishes the proof.

It is not difficult to see that if $\mu(1) \neq 0$ and μ is supported on I_1 then $U_{[0,1]}(D_2 \cap H) > 0$, $Q^{D_3,1}_\mu$ almost surely. It is possible to argue this directly from the Laplace functional, but it follows also from results in the literature. Indeed the projection of the process onto $\{x_2 = \ldots = x_d = 0\}$ is a DW process started at $c\delta_0$, so one can use absolute continuity and the strong law of the logarithm for the closed support shown in Tribe ([14]). Since $x \to U(0,\infty,x)$ is lower semi-continuous, the set $\{x \in D_2 \cap H : U(0,\infty,x) > 0\}$ is open and almost surely non-empty by the above. This implies that $p(L,\mu) \to 0$ as $L \to \infty$ for each non-zero μ. Lemma 4 shows that $\mu \to p(L,\mu)$ is non-increasing, so we may restrict to the set $\mathcal{M}_1(I)$ of measures supported in I and with total mass one. Note that $\mathcal{M}_1(I)$ is compact in the weak topology. Since $L \to p(L,\mu)$ is also non-increasing, the required uniform convergence follows by a Dini argument if we can show that $\mu \to p(L,\mu)$ is upper semi-continuous. This will be true since $p(L,\mu)$ is the decreasing limit of $p(L,\mu,\epsilon)$ as $\epsilon \downarrow 0$, where

$$p(L,\mu,\epsilon) = Q^D_{\mu,1}\left[\exp\left(-Lc_0\int_{D_2\cap H}\left(1-e^{-U(\epsilon,\epsilon^{-1},x)}\right)dx\right)\right]$$

Increment estimates (similar to those in (4.16) show that the laws $Q^D_{\mu,1}(U(\epsilon,\epsilon^{-1}) \in df)$ on $C(\overline{D_2},R)$ are tight, uniformly over $\mu \in \mathcal{M}_1(I)$. This in turn implies that the map $\mu \to Q^D_{\mu,1}(U(\epsilon,\epsilon^{-1}) \in df)$ and hence also $\mu \to p(L,\mu,\epsilon)$ is continuous, which finishes the proof. □

6.3 Proof that $\lim_{\beta\to\infty}\beta^{-1}\Psi(\beta) = 1$.

We fix $\kappa \in (0,1)$ and show there exists $\beta_0 = \beta_0(\kappa)$ so that possible life occurs whenever $\beta \ge \beta_0$ and $\gamma = \kappa\beta$. Since $\Psi(\beta) \le \beta$ this implies the desired limit.

Lemma 21. *Fix $\kappa \in (0,1)$ and smooth $\eta : \overline{D_3} \to [0,1]$ compactly supported inside $D_3 \cap \{x : x_1 > 0\}$ and so that $\eta = 1$ on $D_2 \cap \{x : x_1 \ge 1\}$. Consider, in dimensions $d \ge 2$, solutions on the domain $D_3 = (-3,3)^d$ to the equation for a DW process given by*

$$\partial_t u = \Delta u + \theta\left(\frac{1+\kappa}{2}\eta - \kappa\right)u + \sqrt{\sigma u}\,\dot{W}. \tag{6.4}$$

Given $\epsilon > 0$ we may choose $\theta_0(\epsilon), \sigma_0(\epsilon) > 0$ so that for any solution u to (6.4) with parameters σ_0 and θ_0, and initial condition $\mu \in \mathcal{M}_1(I_1)$,

$$P\left[u_1^{\partial D_3}(x_{1,\pm1} + I_1) \ge 1\right] \ge 1 - \epsilon$$

where $x_{1,\pm1} = (3, \pm2, \ldots)$.

Proof. Choose $h : \partial D_3 \to [0,1]$ so that $\{h > 0\} = x_{1,1} + I_1$ (the case $x_{1,-1}$ being symmetric). Using the Laplace functional from (3.6) and scaling, we have $E[\exp(-u_1^{\partial D_3}(h))] = \exp(-\mu(\phi_1))$ where

$$\left\{ \begin{array}{l} \partial_t \phi = \Delta\phi + \theta\left(\frac{1+\kappa}{2}\eta - \kappa\right)\phi - \frac{\sigma}{2}\phi^2 \quad \text{on } D_3, \\ \quad \phi = h \text{ on } \partial D_3 \text{ for } t \geq 0, \quad \text{ and } \phi_0 = 0 \text{ on } D_3. \end{array} \right. \tag{6.5}$$

As $\sigma \to 0$ the solution ϕ_1 converges to $\overline{\phi_1}$ where $\overline{\phi}$ is the solution of (6.5) with $\sigma = 0$. Since ϕ is decreasing in the parameter σ, the convergence is uniform over $x \in I_1$.

We claim that $\overline{\phi}_1(x) \to \infty$ as $\theta \to \infty$ and that the convergence is uniform over $x \in I_1$. One way to see this is via the Feynman-Kac representation for $\overline{\phi}$, namely

$$\overline{\phi}_1(x) = E_x\left[h(X_\tau)I(\tau \leq 1)e^{\theta\int_0^\tau\left(\frac{1+\kappa}{2}\eta(X_s)-\kappa\right)ds}\right]$$

where X is a Brownian motion (with generator Δ) and $\tau = \inf\{t : X_t \in \partial D_3\}$. Note that if $X_s \in D_2 \cap \{x : x_1 \geq 1\}$ then $(\frac{1+\kappa}{2}\eta(X_s) - \kappa) \geq (1-\kappa)/2$. Now we use the support theorem for Wiener measure. There is an open set of paths $O_x \subseteq C([0,1], R^d)$ where $f \in O_x$ satisfy $f_0 = x$, f exits D_3 before time 1 and $\int_0^\tau(\frac{1+\kappa}{2}\eta(f_s) - \kappa)ds > (1-\kappa)/4$. Then $\overline{\phi}_1(x) \geq \exp(\theta(1-\kappa)/4)E_x[h(X_\tau)I(X_\cdot \in O_x)]$. Moroever O_x can be chosen so that $E_x[h(X_\tau)I(X_\cdot \in O_x)]$ is bounded below uniformly over $x \in I_1$, establishing the claim. Then we may choose $\theta_0(\epsilon)$ and then $\sigma_0(\epsilon)$ so that

$$\begin{aligned} P\left[u_1^{\partial D_3}(x_{1,1} + I_1) \leq 1\right] &\leq P\left[u_1^{\partial D_3}(h) \leq 1\right] \\ &\leq e\, e^{-\mu(\phi_1)} \quad \text{by Markov's inequality} \\ &\leq e\, e^{-\mu(\overline{\phi}_1)} + \frac{\epsilon}{2} \quad \text{by the suitable choice of } \sigma_0(\epsilon) \\ &\leq \epsilon \quad \text{by the suitable choice of } \theta_0(\epsilon) \end{aligned}$$

which completes the proof. □

Lemma 22. *Consider, in dimensions $d = 2$ or 3, solutions on the domain D_3 to the equation*

$$\left\{ \begin{array}{l} \partial_t u = \Delta u + \theta u\,(v - \kappa) + \sqrt{\sigma u}\,\dot{W}, \\ \partial_t v = -\delta uv, \quad v_0 = I(x_1 \geq 0). \end{array} \right. \tag{6.6}$$

Given $\epsilon > 0$ we may choose $\theta_1(\epsilon), \sigma_1(\epsilon), \delta_1(\epsilon) > 0$ so that for any solution (u,v) to (6.6), with parameter values θ_1, σ_1 and $\delta \leq \delta_1$, and with initial condition μ supported on I_1 satisfying $\mu(1) \geq 1$,

$$P\left[u_\infty^{\partial D_3}(x_{1,\pm 1} + I_1) \geq 1\right] \geq 1 - \epsilon$$

where $x_{1,\pm1} = (3, \pm 2, \ldots)$.

Proof. By the monotonicity in μ given in Lemma 4, it is enough to consider $\mu \in \mathcal{M}_1(I_1)$. Fix $\theta_1 = \theta_0(\epsilon/2)$ and $\sigma_1 = \sigma_0(\epsilon/2)$ from the previous lemma. Let η be as in the previous lemma and let K be the closed support of η. We consider a solution to (6.6) on path space and set

$$\tau_\delta = \inf\{t : \delta U(0,t,x) \geq (1-\kappa)/2 \text{ for some } x \in K\}.$$

Note that $V(t,x) = \exp(-\delta U(0,t,x)) \geq 1 - (1-\kappa)/2$ for $t \leq \tau_\delta$ and $x \in K$. This implies that

$$V(t,x) - \kappa \geq \frac{1+\kappa}{2}\eta(x) - \kappa \quad \text{for } t \leq \tau_\delta \text{ and } x \in D_3.$$

Informally, upto the time τ_δ the process dominates a solution to (6.4). More carefully, we may expand the process $\exp(-u_s(\phi_{1-s}) - u_s^{\partial D_3}(\phi_s))$ over the interval $s \in [0, \tau_\delta \wedge 1]$, with ϕ as in the previous lemma, to obtain an upper bound on the the Laplace functional $E[\exp(-u_1^{\partial D_3}(h))I(\tau_\delta \geq 1)]$. This shows that $P[u_\infty^{\partial D_3}(x_{1,\pm1} + I_1) \leq 1, \tau_\delta \geq 1] \leq \epsilon/2$ for all $\mu \in \mathcal{M}_1(I_1)$ (and for any choice of $\delta > 0$).

It remains to estimate $P[\tau_\delta < 1]$. Informally, we can control this by choosing δ small and the fact that the process is dominated by a $\mathrm{DW}(D_3, -\theta_1)$ process. Rather than use the natural pathwise comparison, which is messy (although possible) to establish in this non-Lipschitz setting, we exploit a change of measure argument. Let Q_μ be the law of $(u, u^{\partial D_3})$ on path space Ω_{D_3} of the DW process satisfying $\partial_t u = \Delta u + \sqrt{\sigma_1 u}\,\dot{W}$ starting at μ. Set

$$M_t = \sigma_1^{-1}\theta_1 \int_0^t \int \left(e^{\delta U(0,s,x)} - \kappa\right) Z(dx, ds)$$

and let $\mathcal{E}_t(M)$ be the associated stochastic exponential. Arguing as in section 3.2, under the the measure $dP|_{\mathcal{U}_t} = \mathcal{E}_t(M)Q_\mu$, the process U is a solution to (6.6). Then

$$P[\tau_\delta < 1] = Q_\mu\left[\mathcal{E}_{\tau_\delta\wedge 1}(M) I(\tau_\delta < 1)\right] \leq \left(Q_\mu\left[\mathcal{E}_{\tau_\delta\wedge 1}(pM)e^{((p-1)/2)[M]_{\tau_\delta\wedge 1}}\right]\right)^{1/p} \left(Q_\mu[\tau_\delta < 1]\right)^{1/q} \tag{6.7}$$

for a dual pair $p^{-1} + q^{-1} = 1$. Note that $[M]_t \leq \sigma_1^{-2}\theta_1^2 U_{[0,t]}(1)$ for $t \leq \tau_\delta$. Reversing the change of measure, under the law $\mathcal{E}(pM)dQ_\mu$ the process has a mass creation at most at rate $p\theta_1$ up to time τ_δ. For small enough $\theta > 0$ the exponential moments $Q_\mu^{D_3,-p\theta_1}[\exp(\theta U_{[0,1]}(1))]$ are finite, and bounded uniformly over $\mu \in \mathcal{M}_1(I_1)$. Thus we may choose $p = p(\epsilon) > 1$ so that the first term on the right hand side of (6.7) is bounded uniformly over $\mu \in \mathcal{M}_1(I_1)$, and it remains to estimate $Q_\mu[\tau_\delta < 1]$. Note that

$$Q_\mu[\tau_\delta < 1] \leq Q_\mu\left[\min\{1, 2\delta(1-\kappa)^{-1}\sup_{x\in K} U(0,1,x)\}\right] := p(\delta, \mu).$$

The continuity of the occupation density $U(0,1,x)$ for $x \in K$ (the finite spead of support ensures that the solution does not hit K immediately) implies that $\sup_{x\in K} U(0,1,x)$ is finite, and so that $p(\delta,\mu) \downarrow 0$ as $\delta \downarrow 0$. But $p(\mu,\delta)$ is the deceasing limit of $p(\delta,\mu,\epsilon)$ as $\epsilon \downarrow 0$ where

$$p(\delta,\mu,\epsilon) = Q_\mu\left[\min\{1, 2\delta(1-\kappa)^{-1}\sup_{x\in K} U(\epsilon,1,x)\}\right].$$

Arguing as in the proof of Lemma 20, the functions $p(\delta,\mu,\epsilon)$ are continuous in μ and hence $p(\delta,\mu)$ is upper semicontinuous, ensuring that the convergence $p(\delta,\mu) \to 0$ is uniform over $\mathcal{M}_1(I_1)$. This allows us to choose δ_1 to ensure that $P[\tau_\delta < 1] \leq \epsilon/2$ for all $\mu \in \mathcal{M}_1(I_1)$ and completes the proof. □

Now chooose $\epsilon = \epsilon_0$ the value that ensure the OP process is supercritical. Using the scaling lemma 3 for a solution to (1.1) with the choices $b = \theta_1(\epsilon_0)\beta^{-1}$, $a = b^2$, $c^d = ab\sigma_1(\epsilon_0)$ and $e = 1$, the scaled solution $\tilde{u}$ solves (6.6) with the parameter values θ_1, σ_1 and $\delta = \theta_1\beta^{-1}$. Choosing β sufficiently large that $\delta \leq \delta_1(\epsilon_0)$ we see that the block estimate (6.2) holds, ensuring possible life.

Remark. The correct large β asymptotics can be seen from the following formal argument. Choose $\gamma = \beta - \theta\beta^{2/(6-d)}$ for some $\theta > 0$. Applying the scaling in Lemma 3 with the choices $a = \beta^{-(d-2)/(6-d)}$, $b = \beta^{-2/(6-d)}$, $c = \beta^{-1/(6-d)}$ and $e = 1$ we find the scaled equation solves

$$\begin{cases} \partial_t u = \Delta u + \beta^{(4-d)/(6-d)} u\left(v - (1 - \theta\beta^{-(4-d)/(6-d)})\right) + \sqrt{u}\,\dot{W}, \\ \partial_t v = -\beta^{-(4-d)/(6-d)} uv, \quad v_0 = 1. \end{cases}$$

Letting $v = 1 - \beta^{-(4-d)/(6-d)}\hat{v}$ we may rewrite this equation as $\partial_t u = \Delta u + \theta u - u\hat{v} + \sqrt{u}\,\dot{W}$ and $\partial_t \hat{v} = u(1 - \beta^{-(4-d)/(6-d)}\hat{v})$. This suggests that as $\beta \to \infty$ the system converges to the one parameter system

$$\begin{cases} \partial_t u = \Delta u + \theta u - uv + \sqrt{u}\,\dot{W}, \\ \partial_t v = u, \quad v_0 = 0. \end{cases} \tag{6.8}$$

There is monotonicity of total occupation and exit measures in θ and one expects that there is a critical θ_c so that solutions to (6.8) die for $\theta < \theta_c$ and may live for $\theta > \theta_c$. It is therefore reasonable to conjecture that $\lim_{\beta\to\infty} \beta^{-2/(6-d)}(\beta - \Psi(\beta)) = \theta_c$. There is no obvious line of proof for this conjecture, since it is not obvious that a life or death block estimate will hold for θ close to θ_c. However such block estimates should hold for sufficiently large and small θ leading to upper and lower asymptotics of the same order. However note that the block construction we have presented in section 6.2 uses initial nutrient condition $f = I(x_1 \geq 0)$, and the above intuition does not apply as such. One needs a more careful block construction where the nutrient level is controlled, and this in turn seems to require a stronger version of the spatial Markov property. We will present these details in a subsequent paper.

We remark that a similar one parameter scaling limit should hold for the small β asymptotic. Indeed letting $\beta \to 0$ in (6.3) suggest that the small β behaviour when $\gamma = L^{-2}\beta^2$ should approximate the one parameter system

$$\partial_t u = \Delta u + L\delta_{\sigma(x)} - u + \sqrt{u}\,\dot{W} \quad \text{where } \sigma(x) = \inf\{t : U(0,t,x) > 0\}.$$

The analogous conjecture is that that this equation makes sense, that it has has a critical value L_c, and that the limit $\Psi(\beta)/\beta^2$ exists and equals L_c.

7 Death

7.1 Death in dimension $d = 1$

We will show that death is certain when $\gamma = 0, \beta \geq 0$ and $d = 1$.

Lemma 23. *There are closed bounded intervals $I(\mu) \subseteq (\infty, \infty)$, indexed by $\mu \in \mathcal{M}(R)$, and probabilities $p_\beta > 0$, for $\beta \geq 0$, so that when μ is compactly supported*

$$Q^{R,\beta,0}_{\mu,1}[U_{[0,\infty)}(I(\mu)^c) = 0] \geq p_\beta.$$

By arguing as in the proof of Lemma 12, one may check that

$$Q^{R,\beta,0}_{\mu,f}[U_{[0,\infty)}(I^c) = 0] = Q^{(-M,M),\beta,0}_{\mu,f|_{(-M,M)}}[U_{[0,\infty)}(I^c) = 0] \tag{7.1}$$

whenever I is a closed interval supporting μ and satisfying $I \subseteq (-M, M)$. We define stopping times iteratively as follows. Let $\tau_1 = \inf\{t : U_t(I(\mu)^c) > 0\}$ (with $\inf\{\emptyset\} = +\infty$). Supposing $\tau_n < \infty$ we set

$$\tau_{n+1} = \inf\{t \geq \tau_n : U_t(I(U_{\tau_n})^c) > 0\}.$$

If $\tau_n = \infty$ then we let $\tau_{n+1} = \infty$. Set $p(\mu, f) = Q^{R,\beta,0}_{\mu,f}[U_{[0,\infty)}(I(\mu)^c) > 0]$. The decomposition in Lemma 4 and (7.1) show that $f \to p(\mu, f)$ is non-decreasing. By the lemma above, $Q^{R,\beta,0}_{\mu,1}[\tau_1 < \infty] = p(\mu, 1) \leq 1 - p_\beta$. By the strong Markov property at τ_n we have on the set $\{\tau_n < \infty\}$

$$\begin{aligned} Q^{R,\beta,0}_{\mu,1}[\tau_{n+1} < \infty | \mathcal{U}_{\tau_n}] &= p(U_{\tau_n}, \exp(-U(0,\tau_n))) \\ &\leq p(U_{\tau_n}, 1) \leq 1 - p_\beta. \end{aligned}$$

Then iterating the lemma shows that

$$Q^{R,\beta,0}_{\mu,1}[U_{[0,\infty)} \text{ is not compactly supported}] \leq Q^{R,\beta,0}_{\mu,1}[\tau_n < \infty] \leq (1 - p_\beta)^n$$

completing the proof of death by the characterization in Lemma 11.

Proof of Lemma 23. Choose a closed bounded interval $\tilde{I} = \tilde{I}(\mu) \subseteq (-\infty, \infty)$ so that $Q^R_{\mu,0}[U_{[0,\infty)}(\tilde{I}^c) = 0] \geq 1/2$. This is possible since paths die out and are compactly supported. Without loss we may assume $\tilde{I}$ has length at least 2. Now set $I = \tilde{I} + [-1, 1]$. Choose M so that $I \subseteq (-M, M)$. By (7.1) we must show $Q^{(-M,M),\beta,1}_{\mu,0}[U_{[0,\infty)}(I^c) = 0] \geq p_\beta$. Construct processes (u^-, u^+, v^+) so that u^- is a DW$((-M, M), 0)$ process started at μ, and conditional on $\sigma\{u^-\}$ the process (u^+, v^+) is a solution to (1.1) on $(-M, M)$ with parameters $(\beta, 0)$ and initial conditions

$u_0^+ = \beta(1-e^{-u^-(0,\infty,x)})dx$ and $v_0^+ = e^{-u^-(0,\infty)}$. Then Lemma 5 (i) shows that $u^-_{[0,\infty)} + u^+_{[0,\infty)}$ has the same law as $U_{[0,\infty)}$ under $Q^{(-M,M),\beta,0}_{\mu,1}$. On the set $\{u^-_{[0,\infty)}(\tilde{I}^c) = 0\}$ we have $u_0^+ \leq \beta I(x \in \tilde{I})dx$. When this occurs, Lemma 6 shows that the total occupation measure $u^+_{[0,\infty)}$ becomes stochastically larger if we replace the initial conditions (u_0^+, v_0^+) by $(\beta I(x\in\tilde{I})dx, I((-M,M)\setminus\tilde{I}))$. Therefore on $\{u^-_{[0,\infty)}(\tilde{I}^c) = 0\}$,

$$\begin{aligned}
P[u^+_{[0,\infty)}(I^c(\mu)) = 0 |\, \sigma\{u^-\}] &= Q^{(-M,M),\beta,0}_{u_0^+,v_0^+}[U_{[0,\infty)}(I^c) = 0] \\
&\geq Q^{(-M,M),\beta,0}_{\beta I(x\in\tilde{I})dx, I((-M,M)\setminus\tilde{I})}[U_{[0,\infty)}(I^c) = 0] \\
&= Q^{R,\beta,0}_{\beta I(x\in\tilde{I})dx, I(\tilde{I}^c)}[U_{[0,\infty)}(I^c) = 0] \quad \text{by (7.1)} \\
&\geq Q^{R,-\beta I(\tilde{I}^c)}_{\beta I(x\in\tilde{I})dx}[U_{[0,\infty)}(I^c) = 0] \\
&\geq \exp(-\beta \int_{\tilde{I}} w(x)dx) \quad \text{using (3.17)}
\end{aligned}$$

provided that w solves

$$\Delta w \leq \tfrac{1}{2} w^2 - \eta w \text{ on } \mathrm{int}(I) \text{ and } w(x) \uparrow \infty \text{ as } x \to \partial I \tag{7.2}$$

with smooth $\eta \geq 0$ satisfying $\eta \geq \beta$ on $I \setminus \tilde{I}$. The proof will be complete if we find such a function w and show that there is an upper bound on $\int_{\tilde{I}} w(x)dx$ that is independent of the length of the interval I.

Choose smooth non-increasing $\overline{\eta} : [0,\infty) \to [0,1]$ so that $\overline{\eta}(x) = 1$ for $x \in [0,1]$ and $\overline{\eta}(x) = 0$ for $x \geq 2$. There is a unique non-increasing $\overline{w}(x) \geq 0$, for $x \in (0,\infty)$ that solves

$$\Delta \overline{w} = \tfrac{1}{2}\overline{w}^2 - 2\beta\overline{\eta}\,\overline{w} \text{ on } (0,\infty), \quad \overline{w}(x) \to \infty \text{ as } x \to 0 \text{ and } \overline{w}(x) \to 0 \text{ as } x \to \infty.$$

Moreover the decay as $x \to \infty$ ensures that $\int_1^\infty \overline{w}(x)dx < \infty$. Suppose $I(\mu) = [a,b]$ and $\tilde{I} = [a+1, b-1]$, where $b \geq a+4$. Then $w(x) = \overline{w}(x-a) + \overline{w}(b-x)$ satisfies the required properties (7.2) with the choice $\eta(x) = \beta\overline{\eta}(x-a) + \beta\overline{\eta}(b-x)$. ☐

Remark. The extinction estimates for a DW(R,γ) can be used to show that $Q^{R,\gamma}_{\delta_0}[U_{[0,\infty)}([-L,L]^c) > 0] \sim \exp(-\gamma^{1/2}L)$ for large L. This in turn yields the finiteness of small positive exponential moments of $\int(1 - \exp(-2U(0,\infty,x))dx$ which implies, via Novikov's criterion, that the exponential martingale $t \to \mathcal{E}_t(M^{\beta,0,1})$ is uniformly integrable for small β and hence certain death. We omit the details since this simple method does not seem to be applicable for large β in $d=1$ or any $\beta > 0$ in dimension $d > 1$.

7.2 A block estimate for death

By Lemmas 11 and 12, to establish certain death for some parameter values (β,γ) it is sufficient to choose a non-zero compactly supported μ and show that

$$\lim_{n\to\infty} Q^{R^d,\beta,\gamma}_{\mu,1}\left[U^{\partial D_n}_\infty = 0\right] = 1. \tag{7.3}$$

It can be simpler to verify the following block estimate, which will imply (7.3) via a simple iteration reminiscent of the more direct branching process comparison used in [9].

Lemma 24. *Suppose that there exists $L, M > 0$ so that for all μ supported in $\overline{D}_L = [-L,L]^d$ and satisfying $\mu(1) \leq M$ the following bounds hold:*

$$Q^{D_{3L},\beta,\gamma}_{\mu,1}\left[U^{\partial D_{3L}}_\infty \neq 0\right] < \epsilon_0 \quad \text{and} \quad Q^{D_{3L},\beta,\gamma}_{\mu,1}\left[U^{\partial D_{3L}}_\infty(1)\right] < \epsilon_0 M, \tag{7.4}$$

where $\epsilon_0 = 1/(4\cdot 3^d)$. Then (7.3) holds for $\mu = M\delta_0$ and hence certain death holds for (β,γ).

Proof. We want to break a measure $\mu \in \mathcal{M}(R^d)$ into submeasures supported on blocks of size L and of mass at most M. We consider translates $x + [-L, L)^d$ for $x \in 2LZ^d$ and break the restriction of μ on $x + [-L, L)^d$ into at most $\mathrm{Int}[M^{-1}\mu(x + [-L, L)^d)]$ parts, each of measure at most M (here $\mathrm{Int}[z]$ is the smallest integer greater than or equal to z). We require only that this splitting is done in a measurable manner. Let $N_{L,M}(\mu)$ be the number of pieces into which μ is thus decomposed, that is

$$N_{L,M}(\mu) = \sum_{x \in LZ^d} \mathrm{Int}\left[M^{-1}\mu(x + [-L, L)^d)\right].$$

We now inductively define a sequence $(X_n : n \geq 0)$ of random measures. Let $X_0 = M\delta_0$. Let $X_1 = u_\infty^{(1),\partial D_{3L}}$ for a solution $(u^{(1)}, v^{(1)})$ to (1.1) on D_{3L} with initial conditions $u_0^{(1)} = X_0$ and $v_0^{(1)} = 1$. The spatial Markov property (Lemma 7) and the monotonicity of exit measures with respect to the initial nutrient level (Lemma 4 (ii)), imply that for $R \geq 3L$

$$Q_{X_0,1}^{D_R,\beta,\gamma}\left[U_\infty^{\partial D_R} = 0\right] \geq E\left[Q_{X_1,1}^{D_R,\beta,\gamma}[U_\infty^{\partial D_R} = 0]\right].$$

Then inductively, provided that $X_{n-1} \neq 0$, we choose one of the submeasures $\tilde{X}_{n-1}$ that form the decomposition of X_{n-1} as in the first paragraph (in some measurable manner), and denote its supporting cube as $x_{n-1} + [-L, L)^d$. Let $D(n) = x_{n-1} + (-3L, 3L)^d$. Conditionally independent of $(u^{(j)}, v^{(j)} : j = 1, \ldots, n-1)$ we run a solution $(u^{(n)}, v^{(n)})$ to (1.1) on $D(n)$ with initial conditions $u_0^{(n)} = \tilde{X}_{n-1}$ and $v_0^{(n)} = 1$. We then set

$$X_n = X_{n-1} - \tilde{X}_{n-1} + u_\infty^{(n),\partial D(n)}. \tag{7.5}$$

When $X_{n-1} = 0$ we set $X_n = 0$.

We apply the decomposition Lemma 5 (i) to the domain $D(n)$ to see that the total exit measure on $\partial D(n)$ started from $(X_{n-1}|_{D(n)}, 1)$ can be constructed in two parts as $u_\infty^{(n),\partial D(n)} + \tilde{u}_\infty^{(n),\partial D(n)}$ where $\tilde{u}^{(n)}$ is a solution on $D(n)$ started at $X_{n-1}|_{D(N)} - \tilde{X}_{n-1}$ and initial nutrient level $\exp(-u^{(n)}(0,\infty))$. The monotonicity of exit measures in the nutrient level shows that $\tilde{u}_\infty^{(n),\partial D(n)}$ is stochastically smaller than the total exit measure of a solution on $D(n)$ started at $(X_n, 1)$. Noting that X_n is certainly supported inside $\overline{D}_{(2n+1)L}$, the spatial Markov property then gives, for $R \geq (2n+1)L$,

$$Q_{X_0,1}^{D_R,\beta,\gamma}\left[U_\infty^{\partial D_R} = 0\right] \geq E\left[Q_{X_n,1}^{D_R,\beta,\gamma}[U_\infty^{\partial D_R} = 0]\right]. \tag{7.6}$$

Using the simple inequality $\mathrm{Int}[z] \leq z + I(z > 0)$, the hypotheses of this lemma imply, when $X_n \neq 0$, that

$$E[N_{L,M}(u_\infty^{(n),\partial D(n)})] \leq 3^d(\epsilon_0 + \epsilon_0) \leq 1/2.$$

Note from (7.5) that $N_{L,M}(X_n) \leq N_{L,M}(X_{n-1}) - 1 + N_{L,M}(u_\infty^{(n),\partial D(n)})$ when $X_{n-1} \neq 0$. Then $n \to N_{L,M}(X_n)$ is a N valued supermartingale whose only possible limit point is zero, which implies that $X_n = 0$ for large n, almost surely. Then (7.6) implies the sufficient condition (7.3) for certain death. □

As an illustration of this block condition, we shall use it to show for any $\beta > 0$ there exists $\gamma < \beta$ so that certain death occurs for (β, γ). This implies that $\Phi(\beta) < \beta$. The following lemma will imply that the block estimates will hold when $\gamma = \beta$, and a perturbation argument will ensure they hold γ sufficiently close to β.

Lemma 25. *Consider solutions in the domain $D_3 = (-3, 3)^d$ to the equation*

$$\begin{cases} \partial_t u = b^{-(4-d)/d}\Delta u + b\beta u(v-1) + \sqrt{u}\,\dot{W}, \\ \partial_t v = -uv, \quad v_0 = 1. \end{cases} \tag{7.7}$$

Then, given $\epsilon > 0$, there exists $b_0 = b_0(\beta, d, \epsilon) > 0$ so that for $b \geq b_0$, and whenever $u_0 = \mu$ is supported in $\overline{D_1}$ and satisfies $\mu(1) \leq 1$,

$$E\left[u_\infty^{\partial D_3}(1)\right] < \epsilon \quad \textit{and} \quad P\left[u_\infty^{\partial D_3} \neq 0\right] < \epsilon.$$

Remark. Note that (7.7) comes from (1.1) via the scaling lemma with the choices $e = 1, a = b, c = b^{2/d}$. Undoing this scaling shows that the block estimates (7.4) hold when $\gamma = \beta$ and $L = b^{2/d}, M = b$ with $b = b_0(\beta, d, \epsilon_0)$.

Proof. We first consider solution to (7.7) on the smaller domain D_2 and estimate the expected total exit measure. Using Lemma 6 we can and shall, at the expense of obtaining a stochastically larger exit measure, change the initial condition to $\mu + \mathbf{I}(D_2)(dx)$ and the initial nutrient to $v_0 = (1-(b\beta)^{-1})I(D_2)$. With these changes the reaction term $b\beta u(v-1)$ is at most $-u$. Let $\phi_{b,r}(x)$ solve $b^{-(4-d)/d}\Delta\phi_{b,r} = \phi_{b,r}$ on the domain D_r with boundary conditions $\phi_{b,r} = 1$ on ∂D_r. Calculus shows for $r > 2$ that $u_t(\phi_{b,r}) + u_t^{\partial D_2}(\phi_{b,r})$ is a non-negative supermartingale. Taking expectations and $r \downarrow 2$ we find $E[u_\infty^{\partial D_2}(1)] \leq \int_{D_2} \phi_{b,2}(x)(\mu(dx) + dx)$. As $b \to \infty$, $\phi_{b,2}(x)$ decreases to zero as $b \to \infty$ for $x \in D_2$ (argue, for example, using the probabilistic representation) and therefore uniformly for $x \in \overline{D_1}$. This implies that $E[u_\infty^{\partial D_2}(1)] \to 0$ as $b \to \infty$, uniformly over μ as in the statement of the lemma.

For a solution to (7.7) on D_3 we apply the spatial Markov property Lemma 7 with the subdomain D_2. Using also extinction estimates as in (3.17) this shows that, when μ supported in $\overline{D_1}$,

$$E[u_\infty^{\partial D_3}(1)] \leq E[u_\infty^{\partial D_2}(1)] \quad \text{and} \quad P[u_\infty^{\partial D_3}(1) > 0] \leq E[1 - e^{-u_\infty^{\partial D_2}(w)}] \leq E[u_\infty^{\partial D_2}(w)]$$

provided that $b^{-(4-d)/d}\Delta w \leq w^2/2$ on D_3 and $\inf\{w(x) : x \in \partial D_r\} \to \infty$ as $r \uparrow 3$. The test function $w(x) = 12\sum_{i=1}^d (x_i+3)^{-2} + (3-x_i)^{-2}$ satisfies these requirements provided $b \geq 1$ and also $\sup\{w(x) : x \in \partial D_2\} < \infty$. The lemma now follows from our control on the expected exit mass $E[u_\infty^{\partial D_2}(1)]$. □

The proof that $\Psi(\beta) < \beta$ will follow from Lemma 24 once we have shown that on a box D, the block estimates $Q_{\mu,1}^{D,\beta,\gamma}[U_\infty^{\partial D} \neq 0]$ and $Q_{\mu,1}^{D,\beta,\gamma}[U_\infty^{\partial D}(1)]$ are continuous as $\gamma \uparrow \beta$, uniformly over μ supported in a certain strict sub-box and with a certain bounded total mass. This follows by a change of measure argument. Indeed the derivative $dQ_{\mu,1}^{D,\beta,\gamma}/dQ_{\mu,1}^{D,\beta,\beta}$ on $\mathcal{U}_t$ is given by the exponential martingale $\mathcal{E}_t((\beta-\gamma)M(1))$ where $M_t(1)$ is the martingale part of $U_t(1)$. Note that $[M(1)]_t = U_{[0,t]}(1)$. The uniform integrability of $\mathcal{E}_t((\beta-\gamma)M(1))$, when $\beta - \gamma$ is small, follows from the finiteness of the exponential moments

$$Q_{\mu,1}^{D,\beta,\beta}\left[\exp(\lambda U_{[0,\infty)}(1))\right] \leq Q_\mu^{D,0}\left[\exp(\lambda U_{[0,\infty)}(1))\right] = e^{\mu(\phi^{(\lambda)})} < \infty$$

where $\phi^{(\lambda)}$ solves, for small enough $\lambda > 0$, $\Delta\phi^{(\lambda)} = -(1/2)(\phi^{(\lambda)})^2 - \lambda$ on D and $\phi^{(\lambda)} = 0$ on ∂D. Then

$$\begin{aligned}\left|Q_{\mu,1}^{D,\beta,\gamma}\left[U_\infty^{\partial D}(1)\right] - Q_{\mu,1}^{D,\beta,\gamma}\left[U_\infty^{\partial D}(1)\right]\right|^2 &\leq \left(Q_{\mu,1}^{D,\beta,\beta}\left[U_\infty^{\partial D}(1)(\mathcal{E}_\infty((\beta-\gamma)M(1)) - 1)\right]\right)^2 \\ &\leq Q_{\mu,1}^{D,\beta,\beta}\left[(U_\infty^{\partial D}(1))^2\right] Q_{\mu,1}^{D,\beta,\beta}\left[(\mathcal{E}_\infty((\beta-\gamma)M(1)) - 1)^2\right].\end{aligned}$$

Then $Q_{\mu,1}^{D,\beta,\beta}[(U_\infty^{\partial D}(1))^2] \leq Q_\mu^{D,0}[(U_\infty^{\partial D}(1))^2]$ can be bounded in terms of the total mass $\mu(1)$ while, when $\beta - \gamma$ is small enough,

$$\begin{aligned}&Q_{\mu,1}^{D,\beta,\beta}\left[(\mathcal{E}_\infty((\beta-\gamma)M(1)) - 1)^2\right] \\ &= Q_{\mu,1}^{D,\beta,\beta}\left[e^{2(\beta-\gamma)M_\infty(1) - (\beta-\gamma)^2 U_{[0,\infty)}(1)}\right] - 1 \\ &\leq \left(Q_{\mu,1}^{D,\beta,\beta}\left[\mathcal{E}_\infty(4(\beta-\gamma)M(1))\right]\right)^{1/2}\left(Q_{\mu,1}^{D,\beta,\beta}\left[e^{6(\beta-\gamma)^2 U_{[0,\infty)}(1)}\right]\right)^{1/2} - 1 \\ &\leq \left(Q_\mu^{D,0}\left[e^{6(\beta-\gamma)^2 U_{[0,\infty)}(1)}\right]\right)^{1/2} - 1 \\ &= \left(\exp(\mu(\phi^{(\lambda)})\right)^{1/2} - 1\end{aligned}$$

where $\lambda = 6(\beta-\gamma)^2$. Since $\phi^{(\lambda)}$ decreases to 0 as $\lambda \downarrow 0$, this shows the required continuity in γ. The argument for $Q_{\mu,1}^{D,\beta,\gamma}[U_\infty^{\partial D} \neq 0]$ is similar.

7.3 Proof that $\Psi(\beta) \le c_1 \beta^2$ in $d = 3$

We sketch a rather simple argument for death, based on the decomposition in Lemma 5 (i). This decomposition suggests we may construct the total occupation measure for a solution to (1.1) in two parts: $u^-_{[0,\infty)}$ from a process u^- with law $Q^{R^d,\gamma}_\mu$, and $u^+_{[0,\infty)}$ from a solution u^+ to (1.1) that conditional on $\sigma\{u^-\}$ has initial condition $u^+_0(dx) = \beta(1 - \exp(-u^-(0,\infty,x))dx$. The expected initial mass $E[u^+_0(1)]$ can be exactly calculated via the Laplace functional of $u^-(0,\infty,x)$ and shown to be bounded by $c_0\beta\gamma^{-1/2}\mu(1)$. This suggests, when $c_0\beta\gamma^{-1/2} < 1$, that this decomposition can be iterated leading to a convergent geometric series and a finite total occupation measure.

We now formalise this argument, working on finite domains and with exit measures.

Lemma 26. *When $c_0\beta\gamma^{-1/2} < 1$ we have*

$$Q^{D_L,\beta,\gamma}_{\mu,1}\left[U^{\partial D_L}_\infty(1)\right] \le \mu(\phi^{(L)}) + \mu(1)\sum_{k=1}^\infty (c_0\beta\gamma^{-1/2})^k$$

where ϕ solves $\Delta\phi^{(L)} = \gamma\phi^{(L)}$ on D_L and $\phi^{(L)} = 1$ on ∂D_L, and $c_0 = \int_{R^3}\psi(x)dx$ where $(\psi(x) : R^3 \setminus \{0\})$ is the maximal solution to

$$\Delta\psi = \tfrac{1}{2}\psi^2 + \psi \text{ on } R^3 \setminus \{0\}, \quad \psi(x) \to 0 \text{ as } |x| \to \infty \text{ and } \psi(x) \to \infty \text{ as } |x| \to 0. \tag{7.8}$$

Remark. The maximal solution ψ can be constructed as follows: non-negative solutions $\psi^{(\epsilon)}$ to $\Delta\psi^{(\epsilon)} = (1/2)(\psi^{(\epsilon)})^2 + \psi^{(\epsilon)}$ on $\{x : 0 < \epsilon < |x| < \infty\}$, with boundary conditions $\psi^{(\epsilon)}(x) \to 0$ as $|x| \to \infty$ and $\psi^{(\epsilon)}(x) \to \infty$ as $|x| \downarrow \epsilon$ exist and are unique (take limits through increasing boundary conditions, using the upper bound $C(|x| - \epsilon)^{-2}$, for existence and use the maximum principle for uniqueness.) Moreover $\psi^{(\epsilon)}$ decrease as $\epsilon \downarrow 0$ to function ψ defined on $R^3 \setminus \{0\}$. Interior regularity estimates show that that ψ solves (7.8). The boundary condition for $\psi^{(\epsilon)}$ at $|x| = \epsilon$ imply, by a comparison argument, that ψ is the maximal solution. Finally, another comparison shows that $0 \le \psi \le 4|x|^{-2}$ and that ψ has exponential decay at infinity so that $c_0 = \int \psi(x)dx < \infty$.

Proof. L, β, γ are fixed throughout. For convenience in this proof we let $F(\mu, f) = Q^{D_L,\beta,\gamma}_{\mu,f}[U^{\partial D_L}_\infty(1)]$ and let $\Xi(\mu, d\nu)$ be the the measurable kernel given by the law of $\beta(1 - \exp(-U(0,\infty,x))dx$ on $\mathcal{M}(D_L)$ under $Q^{D_L,\gamma}_\mu$. Then the decomposition in Lemma 5 (i), monotonicity and the exact formula for exit measures of superprocesses (derived, say, from (3.8)) imply

$$\begin{aligned} F(\mu,1) &= Q^{D_L,\gamma}_\mu\left[U^{\partial D_L}_\infty(1)\right] + Q^{D_L,\gamma}_\mu\left[F(\beta(1 - e^{-U(0,\infty,x)}dx, e^{-U(0,\infty,x)})\right] \\ &\le Q^{D_L,\gamma}_\mu\left[U^{\partial D_L}_\infty(1)\right] + Q^{D_L,\gamma}_\mu\left[F(\beta(1 - e^{-U(0,\infty,x)}dx, 1)\right] \\ &= \mu(\phi^{(L)}) + \int F(\nu,1)\,\Xi(\mu, d\nu). \end{aligned} \tag{7.9}$$

Define the range by $\mathcal{R} = \cup_{\delta>0}\overline{\cup_{t\ge\delta}\operatorname{support}(U_t)}$, where support$(\mu)$ is the closed support of μ. Then

$$\begin{aligned} \int \nu(1)\,\Xi(\mu,d\nu) &= \beta\int_{D_L} Q^{D_L,\gamma}_\mu\left[1 - e^{-U(0,\infty,x)}\right] dx \\ &\le \beta\int_{D_L} Q^{D_L,\gamma}_\mu\left[\{x\} \in \mathcal{R}\right] dx = \beta\int_{D_L}\mu(\tilde\psi^{(\gamma,x)})dx \end{aligned}$$

where $\tilde\psi = \tilde\psi^{(\gamma,x)}$ defined for $y \in \overline{D_L} \setminus \{x\}$ is the maximal solution to

$$\Delta\tilde\psi = \tfrac{1}{2}\tilde\psi^2 + \gamma\tilde\psi \text{ on } D_L \setminus \{x\}, \quad \tilde\psi = 0 \text{ on } \partial D_L \text{ and } \tilde\psi(y) \to \infty \text{ as } |y - x| \to 0.$$

The final equality follows using mostly the arguments in [10] Theorem III.5.7 (a) (and constructing $\tilde\psi^{(\gamma,x)}$ as in the above remark). A comparison argument shows that $\tilde\psi^{(\gamma,x)}(y) \le \gamma\psi(\gamma^{1/2}(y - x))$ and this leads to the bound

$\int \nu(1)\,\Xi(\mu, d\nu) \leq c_0\beta\gamma^{-1/2}\mu(1)$. Note that $\phi^{(L)} \leq 1$. We can now interate the inequality (7.9) and the lemma will follow once we have show that the remainder term converges to zero, and this will follow from the fact that $F(\mu,1) \to 0$ when $\mu(1) \to 0$. To see this, note that

$$\begin{aligned} F(\mu,1) &\leq Q^{D_L,\beta,\gamma}_{\mu,1}\left[U^{\partial D_L}_\infty(1) I(\tau > t)\right] + Q^{D_L,\beta,\gamma}_{\mu,1}\left[U^{\partial D_L}_t(1)\right] \\ &\leq \left(Q^{D_L,\beta,\gamma}_{\mu,1}\left[(U^{\partial D_L}_\infty(1))^2\right] Q^{D_L,\beta,\gamma}_{\mu,1}\left[\tau > t\right]\right)^{1/2} + Q^{D_L,\gamma-\beta}_{\mu}\left[U^{\partial D_L}_t(1)\right] \\ &\leq \left(Q^{D_L,\gamma}_{\mu+\beta I(D_L)dx}\left[(U^{\partial D_L}_\infty(1))^2\right] Q^{D_L,\beta,\gamma}_{\mu,1}\left[\tau > t\right]\right)^{1/2} + \mu(\theta_t) \end{aligned}$$

where $\partial\theta = (\beta-\gamma)\theta$ on $[0,t]\times D_L$, $\theta_0 = 0$ and $\theta = 1$ on $[0,t]\times \partial D_L$. Note the term $Q^{D_L,\gamma}_{\mu+\beta I(D_L)dx}\left[(U^{\partial D_L}_\infty(1))^2\right]$ is bounded by $C(L,\beta,\gamma) < \infty$ for $\mu(1) \leq 1$. The term $Q^{D_L,\beta,\gamma}_{\mu,1}[\tau > t]$ converges to zero as $t \to \infty$, uniformly over $\mu(1) \leq 1$, for instance by the estimate in Lemma 19. These together allow us to see that $F(\mu,1) \to 0$ when $\mu(1) \to 0$ completing the proof. □

To complete the proof that $\Psi(\beta) \leq c_1\beta^2$ we will apply the lemma above to a sequence of domains $D(n) := D_{n\gamma^{-1/2}}$, and show that the expected exit measures decay geometrically. Scaling shows that the solution $\phi^{(\gamma^{-1/2})}$ from the above lemma satisfies $\phi^{(\gamma^{-1/2})}(x) = \hat\phi(\gamma^{1/2}x)$ where $\Delta\hat\phi = \hat\phi$ on D_1 and $\hat\phi = 1$ on ∂D_1. Moroever a comparison argument shows that $\phi^{((n+1)\gamma^{-1/2})}(x) \leq \phi^{(\gamma^{-1/2})}(0) = \hat\phi(0) < 1$ for $x \in \partial D(n)$. The lemma therefore implies that

$$Q^{D(n+1),\beta,\gamma}_{\mu,1}\left[U^{\partial D(n+1)}_\infty(1)\right] \leq \mu(1)\left(\hat\phi(0) + (c_0\beta\gamma^{-1/2})(1-c_0\beta\gamma^{-1/2})^{-1}\right) \tag{7.10}$$

whenever μ is supported on $\partial D(n)$. We may now choose $\gamma = c_1\beta^2$ with $c_1 < \infty$ large enough that the right hand side of (7.10) is at most $(1/2)(1+\hat\psi(0))\mu(1) < \mu(1)$. Iterating shows when μ is supported on $\partial D(1)$ that

$$Q^{D(n+1),\beta,c_2\beta^2}_{\mu,1}\left[U^{\partial D(n+1)}_\infty(1)\right] \leq ((1+\hat\phi(0))/2)^n\mu(1).$$

This implies certain death. Indeed the spatial Markov property and the extinction estimate (3.17) show that

$$Q^{D(n),\beta,c_2\beta^2}_{\mu,1}\left[U^{\partial D(n)}_\infty \neq 0\right] \leq Q^{D(n),\beta,c_2\beta^2}_{\mu,1}\left[1 - e^{-U^{\partial D(n-1)}_\infty(w)}\right] \to 0 \quad \text{as } n \to \infty$$

where

$$w(x) = \sum_{i=1}^{3}\left(2\beta + \frac{12}{(x_i + n\gamma^{-1/2})^2}\right) + \left(2\beta + \frac{12}{(n\gamma^{-1/2} - x_i)^2}\right)$$

satisfies $\Delta w \leq (1/2)w^2 - \beta w$ on $D(n)$. Certain death follows from Lemmas 11 and 12.

Remark The approach above can be applied to the case of $d = 2$ and small β. An analysis of the first moment shows that when $\gamma = \exp(-C/\beta)$ the above series construction converges for suitable C and hence certain death. We do not include the details since the argument is quite crude and it seems easier to conjecture that there is death when $\gamma = 0$ for small β. Note however that when $\gamma = 0$ a first moment argument will not show death since the first moment $Q^{D_R,\beta,0}_{\mu,1}[U^{\partial D_R}_\infty(1)] \to \infty$ as $R \to \infty$. We hope to comment on this in a subsequent paper.

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