A STOCHASTIC MOVING BOUNDARY VALUE PROBLEM

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Abstract. We consider a stochastic perturbation of a moving boundary problem proposed by Ludford and Stewart and studied by Caffarelli and Vazquez. We prove existence and uniqueness.

1. Introduction

Moving boundary problems are one of the important areas of partial differential equations. They provide the correct quantitative description of a wide range of physically interesting phenomena where a system has two phases. However, since the boundary between these phases is defined implicitly by the behavior of the rest of the system, they provide deep mathematical challenges in the areas of existence, uniqueness, and regularity.

Our goal here is to study the effect of *noise* on a specific free boundary problem which was introduced by Stewart [Ste85] and subsequently addressed in the mathematics literature (see [CS05, CV95, Vaz96]). Fix a probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ and assume that B is a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$. We consider the SPDE

(1)
$$\begin{aligned} du(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x)dt + \alpha u(t,x)dt + u(t,x)\circ dB_t \qquad x>\beta(t) \\ \lim_{x\searrow\beta(t)} \frac{\partial u}{\partial x}(t,x) &= 1 \\ u(0,x) &= u_\circ(x) \qquad x\in\mathbb{R} \\ \{(t,x)\in\mathbb{R}_+\times\mathbb{R} \mid u(t,x)>0\} &= \{(t,x)\in\mathbb{R}_+\times\mathbb{R} \mid x>\beta(t)\}. \end{aligned}$$

The constant $\alpha \in \mathbb{R}$ is fixed (we shall later see why it is more natural than not to include this term). We also assume that the initial condition $u_\circ \in C(\mathbb{R})$ satisfies some specific properties:

- $u_\circ \equiv 0$ on $\mathbb{R}_-$, $u_\circ > 0$ on $(0,\infty)$, and $\lim_{x\searrow 0} \frac{du_\circ}{dx}(x) = 1$.
- $u_\circ$ and its first three derivatives exist on $(0,\infty)$ and are square-integrable (on $(0,\infty)$).

In (1), $\circ dB_t$ represents Stratonovich integration, and the last line means that the boundary between $u \equiv 0$ and $u > 0$ is the graph of β.

In fact, it is not yet clear that (1) makes sense. Differential equations are pointwise statements. Stochastic differential equations are in fact shorthand representations of corresponding integral equations; pointwise statements typically don't make sense. It will take some work to restate the pointwise stochastic statement in the first line of (1) as a statement about stochastic integrals.

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There has been fairly little written on the effect of noise on moving boundary problems (see [BDP02] and [CLM06]; see also the work on the stochastic porous medium equation in [BDPR09, DPR04a, DPR04b, DPRRW06, Kim06]). We note here that the multiplicative term u in front of the dB_t places this work slightly outside of the purview of the theory of infinite-dimensional evolution equations with Gaussian perturbations. The multiplicative term is in fact a natural nonlinearity. It means that bubbles where u is positive cannot spontaneously nucleate within the region where $u = 0$.

Our major contributions here are to formulate several techniques which can (hopefully) be applied to a number of stochastic moving boundary value problems. In our particular case, where the randomness comes from a single Brownian motion, several transformations (the transformations of Lemmas 3.3 and 3.5 and (17)) can transform the problem into a random nonlinear PDE (see (18)). All of these transformations are not in general available when the noise is more complicated, but most of the techniques we develop here should be. Secondly, the irregularity of the Brownian driving force requires some detailed analysis, no matter what perspective one takes; namely in the analysis of Lemma 3.2 and the iterative bounds of Lemma 4.4.

2. Weak Formulation

To see what we mean by (1), let's replace $\circ dB$ by a smooth path b; the Wong-Zakai result (cf. [KS91, Section 5.2D]) implies that this is reasonable (and that the Stratonovich interpretation is correct when we do so). Let's also assume that there is only one interface. Namely, consider the PDE

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \frac{\partial^2 v}{\partial x^2} + \alpha v(t,x) + v(t,x)b(t) \qquad x > \beta_\circ(t) \\ \lim_{x\searrow\beta_\circ(t)} \frac{\partial v}{\partial x}(t,x) &= 1 \\ v(0,x) &= u_\circ(x). \qquad x \in \mathbb{R} \\ \{(t,x)\in\mathbb{R}_+\times\mathbb{R} \mid v(t,x)>0\} &= \{(t,x)\in\mathbb{R}_+\times\mathbb{R} \mid x>\beta_\circ(t)\}. \end{aligned} \tag{2}$$

This will be our starting point.

Let's see what a weak formulation looks like (see [Fri64, Ch. 8]). Fix $\varphi \in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$. Assume that $\beta_\circ$ is differentiable. Define

$$U_\varphi(t) \overset{\text{def}}{=} \int_{x\in\mathbb{R}} v(t,x)\varphi(t,x)dx = \int_{x=\beta_\circ(t)}^\infty v(t,x)\varphi(t,x)dx.$$

Differentiating, we get that

$$\dot{U}_\varphi(t) = \int_{x=\beta_\circ(t)}^\infty \left\{\frac{\partial v}{\partial t}(t,x)\varphi(t,x) + v(t,x)\frac{\partial \varphi}{\partial t}(t,x)\right\}dx - v(t,\beta_\circ(t))\varphi(t,\beta_\circ(t))\dot{\beta}_\circ(t)$$

and we can use the fact that $v(t,\beta_\circ(t)) = 0$ to delete the last term. We can also use the PDE for v for $x > \beta_\circ(t)$ to rewrite $\frac{\partial v}{\partial t}$. Integrating by parts, we have that

$$\begin{aligned} &\int_{x=\beta_\circ(t)}^\infty \frac{\partial^2 v}{\partial x^2}(t,x)\varphi(t,x)dx \\ &= \lim_{x\searrow\beta_\circ(t)}\left\{-\frac{\partial v}{\partial x}(t,x)\varphi(t,\beta_\circ(t)) + v(t,\beta_\circ(t))\frac{\partial\varphi}{\partial x}(t,\beta_\circ(t))\right\} + \int_{x=\beta_\circ(t)}^\infty v(t,x)\frac{\partial^2\varphi}{\partial x^2}(t,x)dx. \end{aligned}$$

Again we use the fact that $v(t,\beta_\circ(t)) = 0$, and we can also use the boundary condition on $\frac{\partial v}{\partial x}$. Recombining things we get the standard formula that

$$\dot{U}_\varphi(t) = \int_{x\in\mathbb{R}} v(t,x)\left\{\frac{\partial \varphi}{\partial t}(t,x) + \frac{\partial^2\varphi}{\partial x^2}(t,x) + \alpha\varphi(t,x)\right\}dx + \left\{\int_{x\in\mathbb{R}} v(t,x)\varphi(t,x)dx\right\} b(t) - \varphi(t,\beta_\circ(t)).$$

Replacing b by $\circ dB$, we should have the following formulation: that for any $\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$ and any $t>0$,

$$\int_{x\in\mathbb{R}} u(t,x)\varphi(t,x)dx = \int_{x\in\mathbb{R}} u_\circ(x)\varphi(t,x)dx + \int_{r=0}^t\int_{x\in\mathbb{R}} u(t,x)\left\{\frac{\partial \varphi}{\partial t}(r,x) + \frac{\partial^2\varphi}{\partial x^2}(r,x) + \alpha\varphi(r,x)\right\}dx\,dr + \int_{r=0}^t \left\{\int_{x\in\mathbb{R}} u(r,x)\varphi(r,x)dx\right\}\circ dB_r - \int_{r=0}^t \varphi(r,\beta(r))dr.$$

The Ito formulation of this would be that

$$\int_{x\in\mathbb{R}} u(t,x)\varphi(t,x)dx = \int_{x\in\mathbb{R}} u_\circ(x)\varphi(t,x)dx + \int_{r=0}^t\int_{x\in\mathbb{R}} u(t,x)\left\{\frac{\partial \varphi}{\partial t}(r,x) + \frac{\partial^2\varphi}{\partial x^2}(r,x) + \hat{\alpha}\varphi(r,x)\right\}dx\,dr + \int_{r=0}^t \left\{\int_{x\in\mathbb{R}} u(r,x)\varphi(r,x)dx\right\} dB_r - \int_{r=0}^t \varphi(r,\beta(r))dr$$

where $\hat{\alpha} = \alpha + \frac{1}{2}$.

Remark 2.1 Thus the structure of the SPDE (1) is invariant under Ito and Stratonovich formulations; this is the motivation for including α in (1)

We can now formally define a weak solution of (1). In this definition, we allow for blowup. We let $\mathscr{F}_t \overset{\text{def}}{=} \sigma\{B_s;\, 0\le s\le t\}$ for all $t\ge 0$; then B is a Brownian motion with respect to $\{\mathscr{F}_t\}_{t>0}$ and stochastic integration against B will be with respect to this filtration.

Definition 2.2. *A* weak solution *of* (1) *is a predictable path* $\{u(t,\cdot) \mid 0\le t<\tau\}$ *in* $C(\mathbb{R})\cap L^1(\mathbb{R})$, *where* τ *is a predictable stopping time with respect to* $\{\mathscr{F}_t\}_{t>0}$, *such that for any* $\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$ *and any finite stopping time* $\tau'<\tau$,

$$\int_{x\in\mathbb{R}} u(\tau',x)\varphi(t,x)dx = \int_{x\in\mathbb{R}} u_\circ(x)\varphi(\tau',x)dx + \int_{r=0}^{\tau'}\int_{x\in\mathbb{R}} u(t,x)\left\{\frac{\partial \varphi}{\partial t}(r,x) + \frac{\partial^2\varphi}{\partial x^2}(r,x) + \hat{\alpha}\varphi(t,x)\right\}dx\,dr + \int_{r=0}^{\tau'} \left\{\int_{x\in\mathbb{R}} u(r,x)\varphi(r,x)dx\right\} dB_r - \int_{r=0}^{\tau'} \varphi(r,\beta(r))dr$$

and where

$$\{(t,x)\in[0,\tau)\times\mathbb{R} \mid u(t,x)>0\} = \{(t,x)\in[0,\tau)\times\mathbb{R} \mid x>\beta(t)\}.$$

Our main existence and uniqueness theorems are the following. The arguments leading up to these results will come together in Section 4.

Theorem 2.3 (Existence)**.** *A solution of* (1) *exists. Furthermore,* $u(t,\cdot)\in C^2[\beta(t),\infty)$ *for all* $t\in[0,\tau)$ *and*

$$\tau \le \inf\left\{t\ge 0 : \left|\frac{\partial^2 u}{\partial x^2}(t-,\beta(t))\right| = \infty\right\}.$$

Proof. Combine Lemmas 4.6 and 4.7. $\square$

We also have uniqueness.

Theorem 2.4 (Uniqueness)**.** *Suppose that* $\{\tilde{u}_1(t,\cdot); 0 \le t < \tau_1\}$ *and* $\{\tilde{u}_2(t,\cdot); 0 \le t < \tau_2\}$ *are two solutions of* (1)*. Assume that for* $i \in \{1,2\}$*, the map* $x \mapsto u_i(t, x - \beta_i(t))$ *has three generalized square-integrable derivatives on* $(0,\infty)$*. Then* $u_1(t,\cdot) = u_2(t,\cdot)$ *for* $0 \le t < \min\{\tau_1,\tau_2\}$*.*

Proof. The proof follows from Lemma 4.8. □

3. Regularity and a Transformation

The proof of Theorems 2.3 and 2.4 will hinge upon a transformation of (1) into a nonlinear integral equation on a fixed (as opposed to an implicitly defined) domain; we will address this in Subsection 3.2. First, however, let's make sure that we understand a bit about regularity; this will illuminate the assumptions needed.

3.1. **Regularity.** While regularity of moving boundary-value problems is an incredibly challenging area (see [CS05]), we can make some headway. Namely, if we assume enough regularity for the boundary, we can get better control of the sense in which the boundary behavior holds.

The following representation result will help us in carrying out this analysis. Define

(3)
$$p_\circ(t,x) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{x^2}{4t}\right] \qquad t>0,\ x\in\mathbb{R}$$
$$p_\pm(t,x,y) \overset{\text{def}}{=} \{p_\circ(t,x-y) \pm p_\circ(t,x+y)\}\, e^{\alpha t} = \{p_\circ(t,x-y) \pm p_\circ(t,-x-y)\}\, e^{\alpha t} \qquad t>0,\ x,y\in\mathbb{R}$$
$$\hat{p}_\pm(t,x,y) \overset{\text{def}}{=} \{p_\circ(t,x-y) \pm p_\circ(t,x+y)\}\, e^{\hat{\alpha} t} = \{p_\circ(t,x-y) \pm p_\circ(t,-x-y)\}\, e^{\hat{\alpha} t}; \qquad t>0,\ x,y\in\mathbb{R}$$

the second representations of $p_\pm$ and $\hat{p}_\pm$ stem from the fact that $p_\circ$ is even in its second argument. The distinction between $p_\pm$ and $\hat{p}_\pm$ naturally lies in the distinction between Ito and Stratonovich calculations. We then have that

$$\begin{aligned}
&\frac{\partial p_\pm}{\partial t}(t,x,y) = \frac{\partial^2 p_\pm}{\partial y^2}(t,x,y) + \alpha p_\pm(t,x,y) \qquad t>0,\ x,y\in\mathbb{R}\\
&\frac{\partial \hat{p}_\pm}{\partial t}(t,x,y) = \frac{\partial^2 \hat{p}_\pm}{\partial y^2}(t,x,y) + \hat{\alpha} \hat{p}_\pm(t,x,y) \qquad t>0,\ x,y\in\mathbb{R}\\
&\lim_{t\searrow 0} p_\pm(t,x,\cdot) = \lim_{t\searrow 0}\hat{p}_\pm(t,x,\cdot) = \delta_x; \qquad x\in\mathbb{R}\setminus\{0\}
\end{aligned} \tag{4}$$

the relevant distinction between p_+ and p_- is their behavior at $x=0$. This will come up in the arguments of Lemma 3.2.

Lemma 3.1. *Let* u *be a weak solution of* (1) *and assume that* β *is continuous. If* $0<t<\tau$ *and* $x>\beta(t)$*, then*

(5)
$$u(t,x) = -\int_{s=0}^t e^{B_t-B_s} p_\pm(t-s, x-\beta(t), \beta(s)-\beta(t))ds + e^{B_t}\int_{y\in\mathbb{R}} p_\pm(t,x-\beta(t),y-\beta(t))u_\circ(y)dy.$$

Furthermore, $u(t,\cdot)$ *is* C^∞ *on* $(\beta(t),\infty)$*.*

Proof. Fix $t>0$, $x\in\mathbb{R}$, $\delta>0$ and $c\in\mathbb{R}$ and define

$$\varphi_{t,x,\delta,c}(s,y) \overset{\text{def}}{=} \hat{p}_\pm(t+\delta-s, x-c, y-c) \qquad s\in[0,t],\ y\in\mathbb{R}$$

$$U_{t,x,\delta,c}(s) \stackrel{\text{def}}{=} \int_{y\in\mathbb{R}} \varphi_{t,x,\delta,c}(s,y)u(s,y)dy. \qquad s<\tau$$

Fix next a finite stopping time $\tau'<\tau$. For $s\in[0,t]$, we have that

(6)

$$\begin{aligned} U_{t,x,\delta,c}(s\wedge\tau') &= U_{t,x,\delta,c}(0) + \int_{r=0}^{s\wedge\tau'} U_{t,x,\delta,c}(r)dB_r - \int_{r=0}^{s\wedge\tau'} \varphi_{t,x,\delta,c}(r,\beta(r))dr \\ &= U_{t,x,\delta,c}(0) + \int_{r=0}^{s} U_{t,x,\delta,c}(r\wedge\tau')\chi_{[0,\tau']}(r)dB_r - \int_{r=0}^{s} \varphi_{t,x,\delta,c}(r,\beta(r))\chi_{[0,\tau']}(r)dr. \end{aligned}$$

Thus by Ito's formula and some simple calculations, we have that

$$\begin{aligned} U_{t,x,\delta,c}(s\wedge\tau')\exp\left[-B_s+\frac{1}{2}s\right] &= U_{t,x,\delta,c}(0) - \int_{r=0}^{s} U_{t,x,\delta,c}(r\wedge\tau')\exp\left[-B_r+\frac{1}{2}r\right]\chi_{(\tau',\infty)}(r)dB_r \\ &\quad + \frac{1}{2}\int_{r=0}^{s} U_{t,x,\delta,c}(r\wedge\tau')\exp\left[-B_r+\frac{1}{2}r\right]\chi_{(\tau',\infty)}(r)dr \\ &\quad - \int_{r=0}^{s} \varphi_{t,x,\delta,c}(r,\beta(r))\exp\left[-B_r+\frac{1}{2}r\right]\chi_{[0,\tau']}(r)dr \end{aligned}$$

for all $s\in[0,t]$. Taking $s=t\wedge\tau'$, we have that

$$U_{t,x,\delta,c}(t\wedge\tau')\exp\left[-B_{t\wedge\tau'}+\frac{1}{2}t\wedge\tau'\right] = U_{t,x,\delta,c}(0) - \int_{r=0}^{t\wedge\tau'} \varphi_{t,x,\delta,c}(r,\beta(r))\exp\left[-B_r+\frac{1}{2}r\right]dr$$

and thus (using the fact that the integral is against ds)

$$\begin{aligned} U_{t,x,\delta,c}(t\wedge\tau') &= U_{t,x,\delta,c}(0)\exp\left[B_{t\wedge\tau'}-\frac{1}{2}t\wedge\tau'\right] \\ &\quad - \int_{r=0}^{t\wedge\tau'} \varphi_{t,x,\delta,c}(r,\beta(r))\exp\left[B_{t\wedge\tau'}-B_r-\frac{1}{2}(t\wedge\tau'-r)\right]dr. \end{aligned}$$

Next taking $\tau'\nearrow\tau$, we have that

$$U_{t,x,\delta,c}(t\wedge\tau) = U_{t,x,\delta,c}(0)\exp\left[B_{t\wedge\tau}-\frac{1}{2}t\wedge\tau\right] - \int_{r=0}^{t\wedge\tau} \varphi_{t,x,\delta,c}(r,\beta(r))\exp\left[B_{t\wedge\tau}-B_r-\frac{1}{2}(t\wedge\tau-r)\right]dr.$$

Again using the fact that this is an integral against ds, we can take $c=\beta(t)$. If $t<\tau$, then

$$U_{t,x,\delta,\beta(t)}(t) = U_{t,x,\delta,\beta(t)}(0)\exp\left[B_t-\frac{1}{2}t\right] - \int_{r=0}^{t} \varphi_{t,x,\delta,\beta(t)}(r,\beta(r))\exp\left[B_t-B_r-\frac{1}{2}(t-r)\right]dr.$$

If $t<\tau$, $x>\beta(t)$, and β is continuous,

$$\begin{aligned} &\inf_{0\le s\le t}\{|t-s|+|(x-\beta(t))-(\beta(t)-\beta(s))|\} = \min_{0\le s\le t}\{|t-s|+|(x-\beta(t))-(\beta(t)-\beta(s))|\} > 0 \\ &\inf_{0\le s\le t}\{|t-s|+|(x-\beta(t))+(\beta(t)-\beta(s))|\} = \min_{0\le s\le t}\{|t-s|+|(x-\beta(t))+(\beta(t)-\beta(s))|\} > 0. \end{aligned}$$

Thus

$$\lim_{\delta\searrow 0}\sup_{0\le s\le t}\left|\varphi_{t,x,\delta,\beta(t)}(s,\beta(s)) - \hat{p}_\pm(t-s,x-\beta(t),\beta(s)-\beta(t))\right| = 0$$

$$\lim_{\delta\searrow 0}\sup_{y\in\mathbb{R}}\left|\varphi_{t,x,\delta,\beta(t)}(0,y) - \hat{p}_\pm(t,x-\beta(t),y-\beta(t))\right| = 0.$$

This gives us the claimed representation result. We can then differentiate to get the claimed smoothness. □

Note that (5) is *not* an explicit formula for u since the right-hand side of (5) depends on u through β. We also note that the proof effectively converts the Ito integral of (6) into a Stratonovich one, implying that $\hat{p}_\pm$ is converted back into $p_\pm$.

Next, let's see what happens if we in fact assume that β is continuously differentiable. It turns out that not only does the boundary behavior of (1) hold pointwise, but we can find an evolution equation for β (which depends on u). To get the general idea of this latter fact, let's return to our deterministic PDE (2). By definition $v(t,\beta_\circ(t)) = 0$, so differentating (and using an approximation just to the right of $\beta_\circ$) we get that

$$\frac{\partial v}{\partial t}(t,\beta_\circ(t)) + \frac{\partial v}{\partial x}(t,\beta_\circ(t))\dot{\beta}_\circ(t) = 0.$$

Using the PDE for v and the boundary conditions (again, a rigorous proof would require pushing the calculation just a bit to the right of $\beta_\circ$), we get that in fact

$$\dot{\beta}_\circ(t) = -\left\{\frac{\partial^2 v}{\partial x^2}(t,\beta_\circ(t)) + \alpha v(t,\beta_\circ(t))\right\} = -\frac{\partial^2 v}{\partial x^2}(t,\beta_\circ(t)). \tag{7}$$

For the SPDE (1) we should have the same result (since the noise term vanishes at the boundary).

To proceed, let's rewrite (5) in a slightly more convenient way. If $\{u(t,\cdot) \mid 0 \le t < \tau\}$ is a weak solution of (1) and $0 < t < \tau$, set

$$\begin{aligned} A_1(t,\varepsilon) &\overset{\text{def}}{=} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] p_\circ(s,\varepsilon + \beta(t) - \beta(t-s))ds \qquad \varepsilon \in \mathbb{R}\setminus\{0\} \\ A_2^\pm(t,\varepsilon) &\overset{\text{def}}{=} e^{B_t}\int_{y\in\mathbb{R}} p_\pm(t,\varepsilon,y-\beta(t))u_\circ(y)dy. \qquad \varepsilon \in \mathbb{R} \end{aligned}$$

Then some simple manipulation (which reflects the second representation of $p_\pm$ in (3) and the fact that $p_\circ$ is even in its second argument) shows that

$$u(t,\beta(t)+\varepsilon) = -A_1(t,\varepsilon) - A_1(t,-\varepsilon) + A_2^+(t,\varepsilon) = -A_1(t,\varepsilon) + A_1(t,-\varepsilon) + A_2^-(t,\varepsilon). \tag{8}$$

We in fact have

Lemma 3.2. *Let $\{u(t,\cdot) \mid 0 \le t < \tau\}$ be a solution of (1). If β is continuously differentiable, then*

$$\lim_{x\searrow\beta(t)} \frac{\partial u}{\partial x}(t,x) = 1 \qquad \textit{and} \qquad \lim_{x\searrow\beta(t)} \frac{\partial^2 u}{\partial x^2}(t,x) = -\dot{\beta}(t) \tag{9}$$

for all $t \in [0,\tau)$.

Proof. From (8), we have that

$$\begin{aligned} \frac{\partial u}{\partial x}(t,\beta(t)+\varepsilon) &= -\frac{\partial A_1}{\partial \varepsilon}(t,\varepsilon) + \frac{\partial A_1}{\partial \varepsilon}(t,-\varepsilon) + \frac{\partial A_2^+}{\partial \varepsilon}(t,\varepsilon) \\ \frac{\partial^2 u}{\partial x^2}(t,\beta(t)+\varepsilon) &= -\frac{\partial^2 A_1}{\partial \varepsilon^2}(t,\varepsilon) + \frac{\partial^2 A_1}{\partial \varepsilon^2}(t,-\varepsilon) + \frac{\partial^2 A_2^-}{\partial \varepsilon^2}(t,\varepsilon). \end{aligned}$$

Note that

$$\frac{\partial p_\circ}{\partial x}(t,x) = -\frac{1}{2\sqrt{4\pi}}\frac{x}{t^{3/2}}\exp\left[-\frac{x^2}{4t}\right] \qquad \text{and} \qquad \frac{\partial^2 p_\circ}{\partial x^2}(t,x) = \frac{1}{2\sqrt{4\pi}t^{3/2}}\left\{\frac{x^2}{2t} - 1\right\}\exp\left[-\frac{x^2}{4t}\right]. \tag{10}$$

Thus

$$\begin{aligned}
\frac{\partial A_1}{\partial \varepsilon}(t,\varepsilon) &= \int_{s=0}^t \exp\left[B_t - B_{t-s} - \alpha s\right] \frac{\partial p_\circ}{\partial x}(s, \varepsilon + \beta(t) - \beta(t-s))ds \\
&= -\frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\varepsilon + \beta(t) - \beta(t-s)}{s^{3/2}} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] ds \\
&= -\tilde{A}_{1,1}(t,\varepsilon) - \tilde{A}_{1,2}(t,\varepsilon) \\
\frac{\partial^2 A_1}{\partial \varepsilon^2}(t,\varepsilon) &= \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\partial^2 p_\circ}{\partial x^2}(s, \varepsilon + \beta(t) - \beta(t-s))ds \\
&= \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \left\{ \frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{2s} - 1 \right\} \\
&\qquad \times \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] \frac{1}{s^{3/2}} ds \\
&= \tilde{A}_{2,1}(t,\varepsilon) + \tilde{A}_{2,2}(t,\varepsilon) + \tilde{A}_{2,3}(t,\varepsilon)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{A}_{1,1}(t,\varepsilon) &= \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\varepsilon}{s^{3/2}} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] ds \\
\tilde{A}_{1,2}(t,\varepsilon) &= \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\beta(t) - \beta(t-s)}{s^{3/2}} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] ds \\
\tilde{A}_{2,1}(t,\varepsilon) &= \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \left\{ \frac{\varepsilon^2}{2s} - 1 \right\} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] \frac{1}{s^{3/2}} ds \\
\tilde{A}_{2,2}(t,\varepsilon) &= \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\varepsilon}{s^{3/2}} \frac{\beta(t) - \beta(t-s)}{s} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] ds \\
\tilde{A}_{2,3}(t,\varepsilon) &= \frac{1}{4\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{(\beta(t) - \beta(t-s))^2}{s^{5/2}} \exp\left[-\frac{(\varepsilon + \beta(t) - \beta(t-s))^2}{4s}\right] ds.
\end{aligned}$$

Since β is by assumption continuously differentiable,

$$K \overset{\text{def}}{=} \sup_{0<\delta\le t} \frac{|\beta(t) - \beta(t-\delta)|}{\delta}$$

is finite. Thus

$$\begin{aligned}
\left| \frac{\beta(t) - \beta(t-s)}{s^{3/2}} \right| &\le \frac{K}{s^{1/2}} \\
\left| \frac{(\beta(t) - \beta(t-s))^2}{s^{5/2}} \right| &\le \frac{K^2}{s^{1/2}}
\end{aligned}$$

for all $s \in (0,t]$. Since $s \mapsto \frac{1}{\sqrt{s}}$ is integrable on $(0,t]$, we can use dominated convergence to see that

$$\lim_{\varepsilon \to 0} \tilde{A}_{1,2}(t,\varepsilon) = \frac{1}{2\sqrt{4\pi}} \int_{s=0}^t \exp\left[B_t - B_{t-s} + \alpha s\right] \frac{\beta(t) - \beta(t-s)}{s^{3/2}} \exp\left[-\frac{(\beta(t) - \beta(t-s))^2}{4s}\right] ds$$

$$\lim_{\varepsilon\to 0}\tilde{A}_{2,3}(t,\varepsilon)=\frac{1}{4\sqrt{4\pi}}\int_{s=0}^{t}\exp\left[B_t-B_{t-s}+\alpha s\right]\frac{(\beta(t)-\beta(t-s))^2}{s^{5/2}}\exp\left[-\frac{(\beta(t)-\beta(t-s))^2}{4s}\right]ds.$$

To understand $\tilde{A}_{1,1}$, $\tilde{A}_{2,2}$ and $\tilde{A}_{2,3}$ we make the change of variables $u=|\varepsilon|/\sqrt{s}$ and rearranging things to get that

$$\tilde{A}_{1,1}(t,\varepsilon)=\operatorname{sgn}(\varepsilon)\frac{1}{\sqrt{4\pi}}\int_{u=|\varepsilon|/\sqrt{t}}^{\infty}\exp\left[B_t-B_{t-\varepsilon^2/u^2}+\alpha\varepsilon^2/u^2\right]\exp\left[-\frac{u^2}{4}\left(1+\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right)^2\right]du$$

$$\begin{aligned}\tilde{A}_{2,2}(t,\varepsilon)=\frac{\operatorname{sgn}(\varepsilon)}{\sqrt{4\pi}}&\int_{u=|\varepsilon|/\sqrt{t}}^{\infty}\exp\left[B_t-B_{t-\varepsilon^2/u^2}+\alpha\varepsilon^2/u^2\right]\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon^2/u^2}\\&\times\exp\left[-\frac{u^2}{4}\left(1+\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right)^2\right]du.\end{aligned}$$

Suppose that $\varepsilon<1/(2K)$. If $u\geq 1$, then

$$\left|\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right|\leq K\frac{\varepsilon}{u^2}\leq K\varepsilon\leq\frac{1}{2}$$

in which case

$$\exp\left[-\frac{u^2}{4}\left(1+\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right)^2\right]\leq\exp\left[-\frac{u^2}{16}\right].\tag{11}$$

On the other hand, if $u<1$, we obviously have that

$$\exp\left[-\frac{u^2}{4}\left(1+\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right)^2\right]\leq 1.$$

Dominated convergence here ensures that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\operatorname{sgn}(\varepsilon)}\tilde{A}_{1,1}(t,\varepsilon)&=\frac{1}{\sqrt{4\pi}}\int_{u=0}^{\infty}\exp\left[-\frac{u^2}{4}\right]du=\frac{1}{2}\\\lim_{\varepsilon\to 0}\frac{1}{\operatorname{sgn}(\varepsilon)}\tilde{A}_{2,2}(t,\varepsilon)&=\frac{\dot{\beta}(t)}{\sqrt{4\pi}}\int_{u=0}^{\infty}\exp\left[-\frac{u^2}{4}\right]du=\frac{\dot{\beta}(t)}{2}.\end{aligned}$$

We next consider $\tilde{A}_{2,1}(t,\varepsilon)$. In fact, we should jointly consider $\tilde{A}_{2,1}(t,\varepsilon)$ and $\tilde{A}_{2,1}(t,-\varepsilon)$. We have that

$$\tilde{A}_{2,1}(t,\varepsilon)-\tilde{A}_{2,1}(t,-\varepsilon)=\frac{1}{2}\int_{s=0}^{t}\exp\left[B_t-B_{t-s}\right]\left(\frac{\varepsilon^2}{2s}-1\right)p_-(s,\varepsilon,\beta(t)-\beta(t-s))\frac{1}{s}ds.$$

The value of this is that $p_-(s,\varepsilon,0)=0$ for all $s>0$ and $\varepsilon\in\mathbb{R}$. We also note that

$$\frac{\partial p_-}{\partial y}(t,x,0)=-2\frac{\partial p_\circ}{\partial x}(t,x)e^{\alpha t}\qquad t>0,x\in\mathbb{R}$$

and

$$\begin{aligned}\frac{\partial^2 p_-}{\partial y^2}(t,x,y)&=\left\{\frac{\partial^2 p_\circ}{\partial x^2}(t,x-y)-\frac{\partial^2 p_\circ}{\partial x^2}(t,x+y)\right\}e^{\alpha t}\\&=\left\{\frac{\partial^2 p_\circ}{\partial x^2}(t,x-y)-\frac{\partial^2 p_\circ}{\partial x^2}(t,-x-y)\right\}e^{\alpha t}\end{aligned}\qquad t>0,x\in\mathbb{R},y\in\mathbb{R}$$

where the last representation uses the fact that $p_\circ$ is even in its second argument. Thus

$$\begin{aligned}p_-(s,\varepsilon,\beta(t)-\beta(t-s)) &= -2\left(\beta(t)-\beta(t-s)\right)\frac{\partial p_\circ}{\partial x}(s,\varepsilon)e^{\alpha s}\\ &\quad+(\beta(t)-\beta(t-s))^2e^{\alpha s}\int_{r=0}^1(1-r)\frac{\partial^2 p_\circ}{\partial x^2}\left(s,\varepsilon-r(\beta(t)-\beta(t-s))\right)dr\\ &\quad-(\beta(t)-\beta(t-s))^2e^{\alpha s}\int_{r=0}^1(1-r)\frac{\partial^2 p_\circ}{\partial x^2}\left(s,-\varepsilon-r(\beta(t)-\beta(t-s))\right)dr.\end{aligned}$$

Thus

$$\tilde{A}_{2,1}(t,\varepsilon)-\tilde{A}_{2,1}(t,-\varepsilon)=\tilde{A}^a_{2,1}(t,\varepsilon)+\tilde{A}^b_{2,1}(t,\varepsilon)-\tilde{A}^b_{2,1}(t,-\varepsilon)+\tilde{A}^c_{2,1}(t,\varepsilon)-\tilde{A}^c_{2,1}(t,\varepsilon)$$

where

$$\begin{aligned}\tilde{A}^a_{2,1}(t,\varepsilon)&=-\int_{s=0}^t\exp\left[B_t-B_{t-s}+\alpha s\right]\left(\frac{\varepsilon^2}{2s}-1\right)\frac{\beta(t)-\beta(t-s)}{s}\frac{\partial p_\circ}{\partial x}(s,\varepsilon)ds\\ \tilde{A}^b_{2,1}(t,\varepsilon)&=\frac{1}{2}\int_{s=0}^t\int_{r=0}^1\exp\left[B_t-B_{t-s}+\alpha s\right]\frac{\varepsilon^2}{2s}\frac{(\beta(t)-\beta(t-s))^2}{s}\frac{\partial^2 p_\circ}{\partial x^2}\left(s,\varepsilon-r(\beta(t)-\beta(t-s))\right)dr\,ds\\ &=\frac{\varepsilon^2}{4}\int_{s=0}^t\int_{r=0}^1\exp\left[B_t-B_{t-s}+\alpha s\right]\left(\frac{\beta(t)-\beta(t-s)}{s}\right)^2\frac{\partial^2 p_\circ}{\partial x^2}\left(s,\varepsilon-r(\beta(t)-\beta(t-s))\right)dr\,ds\\ \tilde{A}^c_{2,1}(t,\varepsilon)&=-\frac{1}{2}\int_{s=0}^t\int_{r=0}^1\exp\left[B_t-B_{t-s}+\alpha s\right]\frac{(\beta(t)-\beta(t-s))^2}{s}\frac{\partial^2 p_\circ}{\partial x^2}\left(s,\varepsilon-r(\beta(t)-\beta(t-s))\right)dr\,ds\\ &=-\frac{1}{2}\int_{s=0}^t\int_{r=0}^1\exp\left[B_t-B_{t-s}+\alpha s\right]\left(\frac{\beta(t)-\beta(t-s)}{s}\right)^2 s\frac{\partial^2 p_\circ}{\partial x^2}\left(s,\varepsilon-r(\beta(t)-\beta(t-s))\right)dr\,ds.\end{aligned}$$

We will again use the transformation $u=|\varepsilon|/\sqrt{s}$. We compute that for $\varepsilon>0$

$$\begin{aligned}\tilde{A}^a_{2,1}(t,\varepsilon)&=-\frac{1}{2\sqrt{4\pi}}\int_{s=0}^t\exp\left[B_t-B_{t-s}+\alpha s\right]\left(\frac{\varepsilon^2}{2s}-1\right)\frac{\beta(t)-\beta(t-s)}{s}\frac{\varepsilon}{s^{3/2}}\exp\left[-\frac{\varepsilon^2}{4s}\right]ds\\ &=-\operatorname{sgn}(\varepsilon)\int_{u=\varepsilon/\sqrt{t}}^\infty\exp\left[B_t-B_{t-\varepsilon^2/u^2}+\alpha\varepsilon^2/u^2\right]\left(\frac{u^2}{2}-1\right)\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon^2/u^2}\\ &\quad\times\exp\left[-\frac{u^2}{4}\right]\frac{1}{\sqrt{4\pi}}du.\end{aligned}$$

Thus by dominated convergence,

$$\lim_{\varepsilon\searrow 0}\tilde{A}^a_{2,1}(t,\varepsilon)=\dot{\beta}(t)\int_{u=0}^\infty\left(\frac{u^2}{2}-1\right)\exp\left[-\frac{u^2}{4}\right]\frac{1}{\sqrt{4\pi}}du=0.$$

Similarly,

$$\begin{aligned}\tilde{A}^b_{2,1}(t,\varepsilon)&=\frac{\varepsilon}{2}\int_{u=|\varepsilon|/\sqrt{t}}^\infty\int_{r=0}^1\exp\left[B_t-B_{t-\varepsilon^2/u^2}+\alpha\varepsilon^2/u^2\right]\left(\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon^2/u^2}\right)^2\\ &\quad\times\left(\varepsilon^3/u^3\right)\frac{\partial^2 p_\circ}{\partial x^2}\left(\varepsilon^2/u^2,\varepsilon-r(\beta(t)-\beta(t-\varepsilon^2/u^2))\right)dr\,du.\end{aligned}$$

From the second equality of (10), we see that there is a $K>0$ such that

$$\left|\frac{\partial^2 p_\circ}{\partial x^2}(s,x)\right|\le\frac{K}{s^{3/2}}\exp\left[-\frac{x^2}{8s}\right]\le\frac{K}{s^{3/2}}\tag{12}$$

for all $s \in (0,t]$ and $x \in \mathbb{R}$. Assume again that $\varepsilon < 1/(2K)$. If $u \le 1$, then

$$\left|(\varepsilon^3/u^3)\frac{\partial^2 p_\circ}{\partial x^2}\left(\varepsilon^2/u^2, \varepsilon - r(\beta(t)-\beta(t-\varepsilon^2/u^2))\right)\right| \le K\frac{(\varepsilon^3/u^3)}{(\varepsilon^2/u^2)^{3/2}} = K.$$

On the other hand, if $u \ge 1$, we have that

$$\begin{aligned}&\left|(\varepsilon^3/u^3)\frac{\partial^2 p_\circ}{\partial x^2}\left(\varepsilon^2/u^2, \varepsilon - r(\beta(t)-\beta(t-\varepsilon^2/u^2))\right)\right| \\ &\qquad\le K\frac{(\varepsilon^3/u^3)}{(\varepsilon^2/u^2)^{3/2}}\exp\left[-\frac{\left(\varepsilon - r(\beta(t)-\beta(t-\varepsilon^2/u^2))\right)^2}{4\varepsilon^2/u^2}\right] \\ &\qquad\le K\exp\left[-\frac{u^2}{4}\left(1-r\frac{\beta(t)-\beta(t-\varepsilon^2/u^2)}{\varepsilon}\right)^2\right] \le K\exp\left[-\frac{u^2}{8}\right]\end{aligned}$$

by again using (11). Combining things together, we see that there is a $K>0$ such that

$$\left|\tilde{A}^b_{2.1}(t,\varepsilon)\right| \le K\varepsilon\int_{u=0}^\infty \exp\left[-\frac{u^2}{8}\right]du = K\varepsilon\sqrt{2\pi};$$

thus indeed

$$\lim_{\varepsilon\to 0}\tilde{A}^b_{2.1}(t,\varepsilon) = 0.$$

We next turn to $\tilde{A}^c_{2.1}$. We here use the last bound of (12). Since $s \mapsto 1/\sqrt{s}$ is integrable on $(0,1]$, we can use dominated convergence to see that

$$\lim_{\varepsilon\to 0}\tilde{A}^c_{2,\varepsilon}(t,\varepsilon) = -\frac{1}{2}\int_{s=0}^t\int_{r=0}^1 \exp\left[B_t - B_{t-s} + \alpha s\right]\left(\frac{\beta(t)-\beta(t-s)}{s}\right)^2 s\frac{\partial^2 p_\circ}{\partial x^2}\left(s, r(\beta(t)-\beta(t-s))\right)dr\,ds,$$

this integral being finite. We have again availed ourselves of the fact that $p_\circ$ is even in its second argument.

Finally, let's understand the relevant behavior of $A_2^\pm$. We have that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\frac{\partial A_2^+}{\partial\varepsilon}(t,\varepsilon) &= e^{B_t}\int_{y\in\mathbb{R}}\frac{\partial p_+}{\partial x}(t,0,y-\beta(t))u_\circ(y)dy \\ \lim_{\varepsilon\searrow 0}\frac{\partial^2 A_2^-}{\partial\varepsilon^2}(t,\varepsilon) &= e^{B_t}\int_{y\in\mathbb{R}}\frac{\partial^2 p_-}{\partial x^2}(t,0,y-\beta(t))u_\circ(y)dy.\end{aligned}$$

From the second expression for $p_\pm$ in (3), we have that

$$\frac{\partial p_+}{\partial x}(t,0,y) = e^{\alpha t}\left\{\frac{\partial p_\circ}{\partial x}(t,-y) - \frac{\partial p_\circ}{\partial x}(t,-y)\right\} = 0.$$

We also have that $p_-(t,0,y)=0$ for all $t>0$, so

$$\frac{\partial^2 p_-}{\partial x^2}(t,0,y) = \frac{\partial p_-}{\partial t}(t,0,y) - \alpha p_-(t,0,y) = 0.$$

Thus in fact

$$\lim_{\varepsilon\searrow 0}\frac{\partial A_2^+}{\partial\varepsilon}(t,\varepsilon) = 0 \qquad\text{and}\qquad \lim_{\varepsilon\searrow 0}\frac{\partial^2 A_2^-}{\partial\varepsilon^2}(t,\varepsilon) = 0.$$

Combining things together, we indeed get (9). □

3.2. **A Transformation.** The characterization of β given in (9) allows us to rewrite the moving boundary-value problem in a more convenient way. The calculation which gives us some analytical traction is found in [Lun04] (see also [Fri64, Ch. 8]). Again, let's return to our deterministic PDE (2). For all $t \geq 0$ and $x \in \mathbb{R}$, define $\tilde{v}(t,x) = v(t, x + \beta_\circ(t)) + e^{-x}$. Then $v(t,x) = \tilde{v}(t, x - \beta_\circ(t)) - \exp\left[-x + \beta_\circ(t)\right]$. Assuming that $\beta_\circ$ is differentiable, we have that for $x > 0$ and $t > 0$,

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t,x) &= \frac{\partial v}{\partial t}(t, x+\beta_\circ(t)) + \frac{\partial v}{\partial x}(t, x+\beta_\circ(t))\dot{\beta}_\circ(t) \\ \frac{\partial \tilde{v}}{\partial x}(t,x) &= \frac{\partial v}{\partial x}(t, x+\beta_\circ(t)) - e^{-x} \\ \frac{\partial^2 \tilde{v}}{\partial x^2}(t,x) &= \frac{\partial^2 v}{\partial x^2}(t, x+\beta_\circ(t)) + e^{-x}. \end{aligned} \tag{13}$$

We can combine these equations and use the PDE for v to rewrite the evolution of $\tilde{v}$ as

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t,x) &= \frac{\partial^2 v}{\partial x^2}(t, x+\beta_\circ(t)) + \alpha v(t, x+\beta_\circ(t)) + v(t, x+\beta_\circ(t)) b(t) \\ &\qquad + \frac{\partial v}{\partial x}(t, x+\beta_\circ(t))\dot{\beta}_\circ(t) \\ &= \frac{\partial^2 \tilde{v}}{\partial x^2}(t,x) - e^{-x} + \alpha\left(\tilde{v}(t,x) - e^{-x}\right) + \left(\frac{\partial \tilde{v}}{\partial x}(t,x) + e^{-x}\right)\dot{\beta}_\circ(t) \\ &\qquad + \left(\tilde{v}(t,x) - e^{-x}\right) b(t). \end{aligned} \tag{14}$$

Note also that

$$\frac{\partial \tilde{v}}{\partial x}(t,0) = 1 - 1 = 0.$$

Furthermore, $\tilde{v}(t,0) = 1$ for all $t > 0$, so evaluating (14) at $x = 0$ (or more accurately, as $x \searrow 0$), we get that

$$0 = \frac{\partial \tilde{v}}{\partial t}(t,0) = \frac{\partial^2 \tilde{v}}{\partial x^2}(t,0) - 1 + \dot{\beta}_\circ(t).$$

Thus in fact

$$\dot{\beta}_\circ(t) = 1 - \frac{\partial^2 \tilde{v}}{\partial x^2}(t,0); \tag{15}$$

alternately by combining (7) and the last line of (13), we have that

$$\dot{\beta}_\circ(t) = -\frac{\partial^2 v}{\partial x^2}(t, \beta_\circ(t)) = -\left\{\frac{\partial^2 \tilde{v}}{\partial x^2}(0,x) - 1\right\}.$$

Inserting the dynamics of $\beta_\circ$ back into (14) we can collect things and get a PDE for $\tilde{v}$; we have that

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t,x) &= \frac{\partial^2 \tilde{v}}{\partial x^2}(t,x) - e^{-x} + \alpha\left(\tilde{v}(t,x) - e^{-x}\right) + \left(\frac{\partial \tilde{v}}{\partial x}(t,x) + e^{-x}\right)\left(1 - \frac{\partial^2 \tilde{v}}{\partial x^2}(t,0)\right) \\ &\qquad + \left(\tilde{v}(t,x) - e^{-x}\right) b(t) \qquad t > 0,\ x > 0 \\ \frac{\partial \tilde{v}}{\partial x}(t,0) &= 0 \qquad t > 0 \\ \tilde{v}(0,x) &= \tilde{u}_\circ(x) \stackrel{\text{def}}{=} u_\circ(x) + e^{-x} \qquad x > 0 \end{aligned}$$

Replacing b by our Brownian motion B and α by $\hat{\alpha}$, we now get the following.

Lemma 3.3. *Suppose that $\{u(t,\cdot) \mid 0 \le t < \tau\} \subset C(\mathbb{R}) \cap L(\mathbb{R})$ is a weak solution of* (1). *Suppose also that β is continuously differentiable and $\{\mathscr{F}_t\}_{t\ge 0}$-adapted. Then $\tilde{u}(t,x) = u(t,x+\beta(t)) + e^{-x}$ satisfies the integral equation*

(16)

$$\begin{aligned}\tilde{u}(t,x) &= \int_{y=0}^{\infty} \hat{p}_+(t,x,y)\left(u_\circ(y) + e^{-y}\right) dy \\ &\quad + \int_{s=0}^{t}\int_{y=0}^{\infty} \hat{p}_+(t-s,x,y)\left\{\left(\frac{\partial \tilde{u}}{\partial x}(s,y) + e^{-y}\right)\left(1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\right) - (\hat{\alpha}+1)e^{-y}\right\} dy\, ds \\ &\quad + \int_{s=0}^{t}\int_{y=0}^{\infty} \hat{p}_+(t-s,x,y)\left\{\tilde{u}(s,y) - e^{-y}\right\} dy\, dB_s\end{aligned}$$

for all $t > 0$ and $x > 0$.

Thanks to Lemmas 3.1 and 3.2, the assumption that β is continuously differentiable ensures that the spatial derivatives of $\tilde{u}$ on the right-hand side of (16) are well-defined.

Proof of Lemma 3.3. Fix $x > 0$ and $T > 0$. For $t \in [0,\tau)$, define

$$U^T(t) \overset{\text{def}}{=} \int_{y=0}^{\infty} \tilde{u}(t,y)\hat{p}_+(T-t,x,y)dy = A_1^T(t) + A_2^T(t)$$

where

$$\begin{aligned}A_1^T(t) &= \int_{y=0}^{\infty} u(t,y+\beta(t))\hat{p}_+(T-t,x,y)dy = \int_{y=\beta(t)}^{\infty} u(t,y)\hat{p}_+(T-t,x,y-\beta(t))dy \\ &= \int_{y\in\mathbb{R}} u(t,y)\hat{p}_+(T-t,x,y-\beta(t))dy \\ A_2^T(t) &= \int_{y=0}^{\infty} e^{-y}\hat{p}_+(T-t,x,y)dy.\end{aligned}$$

Using Definition 2.2 and (4), we get that

$$\begin{aligned}dA_1^T(t) &= \int_{y\in\mathbb{R}} u(t,y)\left\{-\frac{\partial \hat{p}_+}{\partial t}(T-t,x,y-\beta(t)) + \frac{\partial^2 \hat{p}_+}{\partial y^2}(T-t,x,y-\beta(t))\right. \\ &\qquad\qquad \left. + \hat{\alpha}\hat{p}_+(T-t,x,y-\beta(t)) - \frac{\partial \hat{p}_+}{\partial y}(T-t,x,y-\beta(t))\dot{\beta}(t)\right\} dxdt \\ &\qquad + \left\{\int_{y\in\mathbb{R}} u(t,y)\hat{p}_+(T-t,x,y-\beta(t))dy\right\} dB_t - \hat{p}_+(T-t,x,0)dt \\ &= -\left\{\int_{y=\beta(t)}^{\infty} u(t,y)\frac{\partial \hat{p}_+}{\partial y}(T-t,x,y-\beta(t))dx\right\}\dot{\beta}(t)dt \\ &\qquad + \left\{\int_{y=\beta(t)}^{\infty} u(t,y)\hat{p}_+(T-t,x,y-\beta(t))dy\right\} dB_t - \hat{p}_+(T-t,x,0)dt \\ &= \left\{\int_{y=\beta(t)}^{\infty} \frac{\partial u}{\partial y}(t,y)\hat{p}_+(T-t,x,y-\beta(t))dx\right\}\dot{\beta}(t)dt \\ &\qquad + \left\{\int_{y=\beta(t)}^{\infty} u(t,y)\hat{p}_+(T-t,x,y-\beta(t))dy\right\} dB_t - \hat{p}_+(T-t,x,0)dt\end{aligned}$$

$$
\begin{aligned}
&= \left\{\int_{y=0}^{\infty} \frac{\partial u}{\partial y}(t, y+\beta(t))\hat{p}_+(T-t,x,y)dx\right\}\dot{\beta}(t)dt \\
&\qquad + \left\{\int_{y=0}^{\infty} u(t,y+\beta(t))\hat{p}_+(T-t,x,y)dy\right\}dB_t - \hat{p}_+(T-t,x,0)dt \\
&= \left\{\int_{y=0}^{\infty}\left(\frac{\partial \tilde{u}}{\partial y}(t,y)+e^{-y}\right)\hat{p}_+(T-t,x,y)dx\right\}\dot{\beta}(t)dt \\
&\qquad + \left\{\int_{y=0}^{\infty}\left(\tilde{u}(t,y)-e^{-y}\right)\hat{p}_+(T-t,x,y)dy\right\}dB_t - \hat{p}_+(T-t,x,0)dt.
\end{aligned}
$$

We have here used the fact that $u(t,\beta(t)) = 0$. We have also employed a fairly straightforward generalization of the integral equality in Definition 2.2 to predictable integrands; the continuous differentiability and adaptedness of β allow us to apply this. Combining the characterization of $\dot{\beta}$ as in Lemma 3.2 and a calculation as in (15), we get that

$$\dot{\beta}(t) = 1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(t,0).$$

Thus

$$
\begin{aligned}
A_1^T(t) = &\int_{s=0}^{t}\left\{\int_{y=0}^{\infty}\left(\frac{\partial \tilde{u}}{\partial y}(s,y)+e^{-y}\right)\left(1-\frac{\partial^2\tilde{u}}{\partial x^2}(s,0)\right)\hat{p}_+(T-s,x,y)dy\right\}ds \\
&+\int_{s=0}^{t}\left\{\int_{y=0}^{\infty}\left(\tilde{u}(s,y)-e^{-y}\right)\hat{p}_+(T-s,x,y)dy\right\}dB_s - \int_{s=0}^{t}\hat{p}_+(T-s,x,0)ds.
\end{aligned}
$$

A straightforward differentiation, on the other hand, shows that

$$
\begin{aligned}
\dot{A}_2^T(t) &= -\int_{y=0}^{\infty} e^{-y}\frac{\partial \hat{p}_+}{\partial t}(T-t,x,y)dy \\
&= -\int_{y=0}^{\infty} e^{-y}\frac{\partial^2 \hat{p}_+}{\partial y^2}(T-t,x,y)dy - \int_{y=0}^{\infty} e^{-y}\hat{\alpha}\hat{p}_+(T-t,x,y)dy \\
&= \frac{\partial \hat{p}_+}{\partial y}(T-t,x,0) + \hat{p}_+(T-t,x,0) - (\hat{\alpha}+1)\int_{y=0}^{\infty} e^{-y}\hat{p}_+(T-t,x,y)dy.
\end{aligned}
$$

Note that $\frac{\partial \hat{p}_+}{\partial y}(T-t,x,0) = 0$. Combine things to get that

$$
\begin{aligned}
U^T(t) = U^T(0) &+ \int_{s=0}^{t}\left\{\int_{y=0}^{\infty}\left(\frac{\partial \tilde{u}}{\partial y}(s,y)+e^{-y}\right)\left(1-\frac{\partial^2\tilde{u}}{\partial x^2}(s,0)\right)\hat{p}_+(T-s,x,y)dy\right\}ds \\
&- (\hat{\alpha}+1)\int_{s=0}^{t}\left\{\int_{y=0}^{\infty} e^{-y}\hat{p}_+(T-s,x,y)dy\right\}ds \\
&+ \int_{s=0}^{t}\left\{\int_{y=0}^{\infty}\left(\tilde{u}(s,y)-e^{-y}\right)\hat{p}_+(T-s,x,y)dy\right\}dB_s.
\end{aligned}
$$

Now let $T \searrow t$ to get the claimed result. □

Of course (16) is equivalent to the SPDE

$$
\begin{aligned}
d\tilde{u}(t,x) = &\left\{\frac{\partial^2\tilde{u}}{\partial x^2}(t,x) + \hat{\alpha}\left(\tilde{u}(t,x)-e^{-x}\right) - e^{-x} + \left(\frac{\partial \tilde{u}}{\partial x}(t,x)+e^{-x}\right)\left(1-\frac{\partial^2\tilde{u}}{\partial x^2}(t,0)\right)\right\}dt \\
&+ \left(\tilde{u}(t,x)-e^{-x}\right)dB_t \qquad t>0,\ x>0
\end{aligned}
$$

$$\frac{\partial \tilde{u}}{\partial x}(t,0) = 0 \qquad t > 0$$

$$\tilde{u}(0,x) = \tilde{u}_\circ(x) = u_\circ(x) + e^{-x}. \qquad x > 0$$

We can also find a converse to Lemma 3.3. First of all note the following.

Lemma 3.4. *Suppose that* $\{\tilde{u}(t,\cdot) \mid 0 \le t < \tau\} \subset C^2(\mathbb{R}_+)$ *satisfies* (16). *Then* $\tilde{u}(t,0) = 1$ *for all* $0 \le t < \tau$. *Furthermore,* $\tilde{u}(t,x) > 0$ *for all* $t \ge 0$ *and* $x \ge 0$.

Proof. Let's first smooth things out. Fix $\delta > 0$ and define

$$\begin{aligned}
\tilde{u}_\delta(t,x) &\overset{\text{def}}{=} \int_{y=0}^\infty \hat{p}_+(\delta,x,y)\tilde{u}(t,y)dy = \int_{y=0}^\infty \hat{p}_+(t+\delta,x,y)\left(u_\circ(y) + e^{-y}\right)dy \\
&\quad + \int_{s=0}^t \int_{y=0}^\infty \hat{p}_+(t+\delta-s,x,y)\left\{\left(\frac{\partial \tilde{u}}{\partial x}(s,y) + e^{-y}\right)\left(1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\right) - (\hat{\alpha}+1)e^{-y}\right\} dy\,ds \\
&\quad + \int_{s=0}^t \int_{y=0}^\infty \hat{p}_+(t+\delta-s,x,y)\left\{\tilde{u}(s,y) - e^{-y}\right\} dy\,dB_s;
\end{aligned}$$

we have of course used the fact that $\hat{p}_+$ is a semigroup of integral kernels. For each $x > 0$, some straightforward computations show that

$$\begin{aligned}
d\tilde{u}_\delta(t,x) &\overset{\text{def}}{=} \left\{\frac{\partial^2 \tilde{u}_\delta}{\partial x^2}(t,x) + \hat{\alpha}\tilde{u}_\delta(t,x)\right\} dt \\
&\quad + \left(\int_{y=0}^\infty \hat{p}_+(\delta,x,y)\left\{\left(\frac{\partial \tilde{u}}{\partial x}(s,y) + e^{-y}\right)\left(1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\right) - (\hat{\alpha}+1)e^{-y}\right\} dy\right) dt \\
&\quad + \left(\int_{y=0}^\infty \hat{p}_+(\delta-s,x,y)\left\{\tilde{u}(s,y) - e^{-y}\right\} dy\right) dB_t.
\end{aligned}$$

We now let $\delta \searrow 0$ and use the assumed continuity of $\tilde{u}$. We also fix $\varepsilon > 0$ and evaluate the result at $x = \varepsilon$. We get that

$$\begin{aligned}
\tilde{u}(t,\varepsilon) &= \tilde{u}_\circ(\varepsilon) + \int_{s=0}^t \frac{\partial^2 \tilde{u}}{\partial x^2}(s,\varepsilon)ds + \hat{\alpha}\int_{s=0}^t \tilde{u}(s,\varepsilon)ds \\
&\quad + \int_{s=0}^t \left(\frac{\partial \tilde{u}}{\partial x}(s,\varepsilon) + e^{-s}\right)\left(1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\right) ds - (\hat{\alpha}+1)\int_{s=0}^t e^{-\varepsilon}ds \\
&\quad + \int_{s=0}^t \left(\tilde{u}(s,\varepsilon) - e^{-\varepsilon}\right) dB_s \\
&= \tilde{u}_\circ(\varepsilon) + \int_{s=0}^t \left\{\frac{\partial^2 \tilde{u}}{\partial x^2}(s,\varepsilon) - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\left(\frac{\partial \tilde{u}}{\partial x}(s,\varepsilon) + e^{-s}\right)\right\} ds \\
&\quad + \int_{s=0}^t \frac{\partial \tilde{u}}{\partial x}(s,\varepsilon)ds + \int_{s=0}^t \left(\tilde{u}(s,\varepsilon) - e^{-\varepsilon}\right) dB_s.
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we see that

$$\tilde{u}(t,0) = 1 + \int_{s=0}^1 \left(\tilde{u}(s,0) - 1\right) dB_s$$

or alternately

$$\tilde{u}(t,0) - 1 = \int_{s=0}^1 \left(\tilde{u}(s,0) - 1\right) dB_s$$

which indeed implies that $\tilde{u}(t,0) = 1$ for all $t \in [0,\tau)$.

To see the positivity, we define

$$u^*(t,x) \stackrel{\text{def}}{=} \left(\tilde{u}(t,x) - e^{-x}\right) e^{-B_t}. \qquad t \ge 0,\, x \ge 0 \tag{17}$$

Some straightforward calculations show that u^* satisfies the random PDE

(18)

$$\begin{aligned} \frac{\partial u^*}{\partial x}(t,x) &= \frac{\partial^2 u^*}{\partial x^2}(t,x) + \alpha u^*(t,x) - \left(\frac{\partial^2 u^*}{\partial x^2}(t,0)e^{B_t}\right)\frac{\partial u^*}{\partial x}(t,x) \qquad t>0,\, x>0 \\ u^*(t,0) &= e^{-B_t} \qquad t>0 \\ u^*(0,x) &= u_\circ(x). \qquad x>0 \end{aligned}$$

Note that $e^{-B_t} > 0$ for all $t > 0$ and $u_\circ(x) > 0$ for all $x > 0$. Standard calculations for the heat equation then ensure that indeed $u^*(t,x) > 0$ for all $t > 0$ and $x^* > 0$. □

We then have

Lemma 3.5. *Suppose that* $\{\tilde{u}(t,\cdot) \mid 0 \le t < \tau\} \subset C^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ *satisfies* (16). *Set*

$$\beta(t) = \int_{s=0}^t \left\{1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0)\right\} ds \qquad 0 \le t < \tau \tag{19}$$

and define

$$u(t,x) \stackrel{def}{=} \begin{cases} \tilde{u}(t, x-\beta(t)) - \exp\left[-\left(x-\beta(t)\right)\right] & x \ge \beta(t),\, 0 \le t < \tau \\ 0 & x < \beta(t),\, 0 \le t < \tau \end{cases} \tag{20}$$

Then $\{u(t,\cdot) \mid 0 \le t < \tau\}$ *is a weak solution of* (1).

Proof. Fix $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ and define

$$U(t) \stackrel{\text{def}}{=} \int_{x\in\mathbb{R}} \varphi(t,x)u(t,x)dx = A_1(t) - A_2(t) \qquad 0 \le t < \tau$$

where

$$\begin{aligned} A_1(t) &= \int_{x=\beta(t)}^\infty \varphi(t,x)\tilde{u}(t,x-\beta(t))dx = \int_{x=0}^\infty \varphi(t,x+\beta(t))\tilde{u}(t,x)dx \\ A_2(t) &= \int_{x=\beta(t)}^\infty \varphi(t,x)\exp\left[-(x-\beta(t))\right]dx = \int_{x=0}^\infty \varphi(t,x+\beta(t))e^{-x}dx \end{aligned}$$

To see the evolution of A_1, we repeat some of the regularization we used in Lemma 3.4. Fix $\delta > 0$ and define

$$\begin{aligned} \tilde{u}_\delta(t,x) &\stackrel{\text{def}}{=} \int_{y=0}^\infty \hat{p}_+(\delta,x,y)\tilde{u}(t,y)dy \qquad x \ge 0 \\ u_\delta(t,x) &\stackrel{\text{def}}{=} \begin{cases} \tilde{u}_\delta(t,x-\beta(t)) - \exp\left[-\left(x-\beta(t)\right)\right] & x \ge \beta(t),\, 0 \le t < \tau \\ 0 & x < \beta(t),\, 0 \le t < \tau \end{cases} \end{aligned}$$

Then define

$$A_1^\delta(t) \stackrel{\text{def}}{=} \int_{x=0}^\infty \varphi(t,x+\beta(t))\tilde{u}_\delta(t,x)dx = A_1^{\delta,a}(t) + A_1^{\delta,b}(t) + A_1^{\delta,c}(t)$$

where

$$A_1^{\delta,a}(t) = \int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\xi_1(s,y)dx\,dy\,ds$$

$$A_1^{\delta,b}(t) = \int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\xi_2(s,y)dx\,dy\,dB_s$$
$$A_1^{\delta,c}(t) = \int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(t+\delta,x,y)\tilde{u}_\circ(y)dx\,dy$$

where finally

$$\begin{aligned}\xi_1(t,x) &= \left(\frac{\partial \tilde{u}}{\partial x}(t,x)+e^{-y}\right)\left(1-\frac{\partial^2 \tilde{u}}{\partial x^2}(t,0)\right)-(\hat{\alpha}+1)e^{-x}\\ &= -\frac{\partial u}{\partial x}(t,x+\beta(t))\frac{\partial^2 u}{\partial x^2}(t,\beta(t))-(\hat{\alpha}+1)e^{-x}\\ \xi_2(t,x) &= \tilde{u}(t,x)-e^{-x} = u(t,x+\beta(t)).\end{aligned}$$

We also note that we can rewrite the evolution of β as

$$\dot{\beta}(t) = -\frac{\partial^2 u}{\partial x^2}(t,\beta(t)). \qquad t\in[0,\tau)$$

Thus

$$\begin{aligned}dA_1^{\delta,a}(t) &= \left(\int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \left\{\frac{\partial \varphi}{\partial t}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)+\frac{\partial \varphi}{\partial x}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\dot{\beta}(t)\right.\right.\\ &\qquad\left.\left.+\varphi(t,x+\beta(t))\frac{\partial \hat{p}_+}{\partial t}(t+\delta-s,x,y)\right\}\xi_1(s,y)dx\,dy\,ds\right)dt\\ &\quad+\left(\int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(\delta,x,y)\xi_1(t,y)dx\,dy\right)dt\\ &= \left(\int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \left\{\frac{\partial \varphi}{\partial t}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\right.\right.\\ &\qquad\left.+\varphi(t,x+\beta(t))\frac{\partial^2 \hat{p}_+}{\partial x^2}(t+\delta-s,x,y)+\hat{\alpha}\varphi(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\right\}\xi_1(s,y)dx\,dy\,ds\\ &\quad\left.+\int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \frac{\partial \varphi}{\partial x}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\xi_1(s,y)ds\,dx\,dy\,\dot{\beta}(t)\right)dt\\ &\quad+\left(\int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(\delta,x,y)\xi_1(t,y)dx\,dy\right)dt.\end{aligned}$$

Similar calculations show that

$$\begin{aligned}dA_1^{\delta,b}(t) &= \left(\int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \left\{\frac{\partial \varphi}{\partial t}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\right.\right.\\ &\qquad\left.+\varphi(t,x+\beta(t))\frac{\partial^2 \hat{p}_+}{\partial x^2}(t+\delta-s,x,y)+\hat{\alpha}\varphi(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\right\}\xi_2(s,y)dx\,dy\,dB_s\\ &\quad\left.+\int_{s=0}^t \int_{x=0}^\infty \int_{y=0}^\infty \frac{\partial \varphi}{\partial x}(t,x+\beta(t))\hat{p}_+(t+\delta-s,x,y)\xi_2(s,y)dB_s\,dx\,dy\,\dot{\beta}(t)\right)dt\\ &\quad+\left(\int_{x=0}^\infty \int_{y=0}^\infty \varphi(t,x+\beta(t))\hat{p}_+(\delta,x,y)\xi_2(t,y)dx\,dy\right)dB_t\end{aligned}$$

and finally

$$dA_1^{\delta,c}(t) = \left(\int_{x=0}^\infty \int_{y=0}^\infty \left\{\frac{\partial \varphi}{\partial t}(t,x+\beta(t))\hat{p}_+(t+\delta,x,y)\right.\right.$$

$$+\varphi(t,x+\beta(t))\frac{\partial^2 \hat{p}_+}{\partial x^2}(t+\delta,x,y)+\hat{\alpha}\varphi(t,x+\beta(t))\hat{p}_+(t+\delta,x,y)\Big\}\,\tilde{u}_\circ(y)dx\,dy$$
$$+\int_{x=0}^{\infty}\int_{y=0}^{\infty}\frac{\partial\varphi}{\partial x}(t,x+\beta(t))\hat{p}_+(t+\delta,x,y)\hat{u}_\circ(y)dx\,dy\,\dot{\beta}(t)\Bigg)\,dt.$$

Adding these expressions together, we get that, we get that

$$\begin{aligned}
A_1^\delta(t)-A_1^\delta(0)=\int_{s=0}^{t}&\left(\int_{x=0}^{\infty}\left(\frac{\partial\varphi}{\partial t}+\hat{\alpha}\varphi\right)(s,x+\beta(s))\tilde{u}_\delta(s,x)dx\right.\\
&\left.+\int_{x=0}^{\infty}\varphi(s,x+\beta(s))\frac{\partial^2\tilde{u}_\delta}{\partial x^2}(s,x)dx+\int_{x=0}^{\infty}\frac{\partial\varphi}{\partial x}(t,x+\beta(t))\tilde{u}_\delta(s,x)dx\dot{\beta}(s)\right)ds\\
&+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat{p}_+(\delta,x,y)\xi_1(s,y)dx\,dy\right)ds\\
&+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat{p}_+(\delta,x,y)\xi_2(s,y)dx\,dy\right)dB_s.
\end{aligned}$$

We can force the evolution of A_2 into a similar expression. We have

$$\begin{aligned}
\dot{A}_2(t)&=\int_{x=0}^{\infty}\left\{\frac{\partial\varphi}{\partial t}(t,x+\beta(t))+\frac{\partial\varphi}{\partial x}(t,x+\beta(t))\dot{\beta}(t)\right\}e^{-x}dx\\
&=\int_{x=0}^{\infty}\left(\frac{\partial\varphi}{\partial t}+\hat{\alpha}\varphi\right)(t,x+\beta(t))e^{-x}dx\\
&\quad+\int_{x=0}^{\infty}\varphi(t,x+\beta(t))e^{-x}dx+\int_{x=0}^{\infty}\frac{\partial\varphi}{\partial x}(t,x+\beta(t))e^{-x}dx\,\dot{\beta}(t)\\
&\quad-(\hat{\alpha}+1)\int_{x=0}^{\infty}\varphi(t,x+\beta(t))e^{-x}dx.
\end{aligned}$$

Again combining things we get that

$$\begin{aligned}
&\big(A_1^\delta(t)-A_2(t)\big)-\big(A_1^\delta(0)-A_2(0)\big)\\
&\quad=\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\left(\frac{\partial\varphi}{\partial t}+\hat{\alpha}\varphi\right)(s,x+\beta(s))u_\delta(s,x+\beta(s))dx\right.\\
&\qquad+\int_{x=0}^{\infty}\varphi(s,x+\beta(s))\frac{\partial^2 u_\delta}{\partial x^2}(s,x+\beta(s))dx+\int_{x=0}^{\infty}\frac{\partial\varphi}{\partial x}(t,x+\beta(s))u_\delta(s,x+\beta(s))dx\dot{\beta}(s)\\
&\qquad\left.+(\hat{\alpha}+1)\int_{x=0}^{\infty}\varphi(t,x+\beta(t))e^{-x}dx\right)ds\\
&\qquad+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat{p}_+(\delta,x,y)\xi_1(s,y)dx\,dy\right)ds\\
&\qquad+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat{p}_+(\delta,x,y)\xi_2(s,y)dx\,dy\right)dB_s\\
&\quad=\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\left(\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}+\hat{\alpha}\varphi\right)(s,x+\beta(s))u_\delta(s,x+\beta(s))dx\right.\\
&\qquad-\int_{x=0}^{\infty}\varphi(t,x+\beta(s))\frac{\partial u_\delta}{\partial x}(s,x+\beta(s))dx\dot{\beta}(s)\\
&\qquad-\varphi(s,\beta(s))\frac{\partial u_\delta}{\partial x}(s,\beta(s))+\frac{\partial\varphi}{\partial x}(s,\beta(s))u_\delta(s,\beta(s))-\varphi(s,\beta(s))u_\delta(s,\beta(s))\dot{\beta}(s)dx
\end{aligned}$$

$$
\begin{aligned}
&+(\hat\alpha+1)\int_{x=0}^{\infty}\varphi(t,x+\beta(t))e^{-x}dx\Bigg)\,ds\\
&+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat p_+(\delta,x,y)\xi_1(s,y)dx\,dy\right)ds\\
&+\int_{s=0}^{t}\left(\int_{x=0}^{\infty}\int_{y=0}^{\infty}\varphi(s,x+\beta(s))\hat p_+(\delta,x,y)\xi_2(s,y)dx\,dy\right)dB_s.
\end{aligned}
$$

By definition of $\hat p_+$, we conclude that $\frac{\partial u_\delta}{\partial x}(s,\beta(s))=0$. We also have by Lemma 3.4 that $\lim_{\delta\searrow 0}u_\delta(s,\beta(s))=0$. Upon letting $\delta\searrow 0$ and rearranging things, we indeed get a weak solution of (1). □

4. A Picard Iteration

Our main task now is to show that we can indeed solve (16). The main complication is that (16) is fully *nonlinear* due to the presence of the $\frac{\partial^2\tilde u}{\partial x^2}(t,0)$ term in the drift. If we turn off the noise, we can do this via semigroup theory as in [Lun04]. The noise, however, complicates things, as we need to respect the rules of Ito integration and (unless we want to use more advanced theories of stochastic integrals) integrate against predictable functions.

Our approach will be to set up a functional framework in which we can use Picard-type iterations to show existence and uniqueness. As usual, $C_0^\infty(\mathbb{R}_+)$ is the collection of infinitely smooth functions on $[0,\infty)$ whose support is bounded. Define next

$$C^\infty_{0,\text{even}}(\mathbb{R}_+)\overset{\text{def}}{=}\left\{\varphi\in C_0^\infty(\mathbb{R}_+)\mid \varphi^{(n)}(0)=0 \text{ for all odd } n\in\mathbb{N}\right\};$$

in other words, $C^\infty_{0,\text{even}}(\mathbb{R}_+)$ are those elements of $C_0^\infty(\mathbb{R}_+)$ which can be extended to an even element of $C^\infty(\mathbb{R})$ (namely, consider the map $y\mapsto\varphi(|y|)$). For all $\varphi\in C_0^\infty(\mathbb{R}_+)$, define

$$\|\varphi\|_H\overset{\text{def}}{=}\sqrt{\sum_{i=0}^{3}\int_{x\in(0,\infty)}\left|\varphi^{(i)}(x)\right|^2dx}.$$

Let H be the closure of $C_0^\infty(\mathbb{R}_+)$ with respect to $\|\cdot\|_H$ and let H_{even} be the closure of $C^\infty_{0,\text{even}}(\mathbb{R}_+)$ with respect to $\|\cdot\|_H$. We also define

$$\|\varphi\|_L\overset{\text{def}}{=}\sqrt{\int_{x\in(0,\infty)}|\varphi(x)|^2\,dx}$$

for all square-integrable functions on $\mathbb{R}_+$. Of course H and H_{even} are Hilbert spaces (H is more commonly written as H^3; i.e., it is the collection of functions on $\mathbb{R}_+$ which possess three weak square-integrable derivatives). The important aspect of H is the following fairly standard result.

Lemma 4.1. *We have that $H\subset C^2$. More precisely, for any $\varphi\in H$, we have that*

$$\sup_{\substack{x\in\mathbb{R}_+\\ i\in\{0,1,2\}}}\left|\varphi^{(i)}(x)\right|\le 2\|\varphi\|_H.$$

Finally, for $i\in\{0,1,2\}$, $\varphi^{(i)}(0)\overset{def}{=}\lim_{x\searrow 0}\varphi^{(i)}(x)$ is well-defined.

The proof is in Subsection 4.1.

Fix $L>0$ and $\Psi_L \in C^\infty(\mathbb{R};[0,1])$ such that $\Psi_L(x)=1$ if $|x|\le L$ and $\Psi_L(x)=0$ if $|x|\ge L+1$. Set

$$\tilde{u}_1^L(t,x)=\int_{y=0}^\infty \hat{p}_+(t,x,y)\tilde{u}_\circ(y)dy$$

for all $t>0$ and $x\in\mathbb{R}$ and recursively define

$$(21)\quad \tilde{u}_{n+1}^L(t,x)=\int_{y=0}^\infty \hat{p}_+(t,x,y)\tilde{u}_\circ(y)dy$$
$$+\int_{s=0}^t\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\left\{\left(\frac{\partial \tilde{u}_n^L}{\partial x}(t,y)+e^{-y}\right)\left(1-\frac{\partial^2\tilde{u}_n^L}{\partial x^2}(t,0)\right)\Psi_L\left(\|\tilde{u}_n^L(t,\cdot)\|_H\right)\right.$$
$$\left.-(\hat{\alpha}+1)e^{-y}\right\}dy\,ds+\int_{s=0}^t\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\left\{\tilde{u}_n^L(s,y)-e^{-y}\right\}dy\,dB_s.\qquad t>0,\ x>0$$

For each $n\in\mathbb{N}$, $\{\tilde{u}_n^L(t,\cdot);\, t\ge 0\}$ is a well-defined, adapted, and continuous path in H_{even}.

To study (21), we will use the Neumann heat semigroup. For $\varphi\in C_0^\infty(\mathbb{R}_+)$, $t>0$, and $x>0$, define

$$(T_t\varphi)(x)\overset{\text{def}}{=}\int_{y=0}^\infty p_+(t,x,y)\varphi(y)dy.$$

Lemma 4.2. *For each $t>0$, T_t has a unique extension from $C_0^\infty(\mathbb{R}_+)$ to H such that $T_tH\subset H_{even}$ and such that $\|T_tf\|_H\le\|f\|_H$ for all $f\in H$. Secondly, there is a $K_A>0$ such that*

$$\|T_t\dot{f}\|_H\le\frac{K_A}{t^{3/4}}\|f\|_H$$

for all $f\in H_{even}\cap C^4(\mathbb{R}_+)$.

Again, we delay the proof until Subsection 4.1.

Another convenience will be to rewrite the ds part of (21). Define

$$\tilde{\Psi}_L(\psi)\overset{\text{def}}{=}(1-\ddot{\psi}(0))\Psi_L\left(\|\psi\|_H\right)$$

for all $\psi\in H$. Then

$$\left\{\frac{\partial\tilde{u}_n^L}{\partial x}(t,x)+e^{-x}\right\}\left(1-\frac{\partial^2\tilde{u}_n^L}{\partial x^2}(t,0)\right)\Psi_L\left(\|\tilde{u}_n^L(t,\cdot)\|_H\right)=\left\{\frac{\partial\tilde{u}_n^L}{\partial x}(t,x)+e^{-x}\right\}\tilde{\Psi}_L(\tilde{u}_N^L(t,\cdot))$$

for all $n\in\mathbb{N}$. For ψ and η in H, let's also define

$$(D\tilde{\Psi}_L)(\psi;\eta)\overset{\text{def}}{=}-\ddot{\eta}(0)\Psi_L(\|\psi\|_H)+(1-\ddot{\psi}(0))\dot{\Psi}_L(\|\psi\|_H)\frac{\langle\psi,\eta\rangle_H}{\|\psi\|_H}.$$

Lemma 4.3. *For each ψ and η in H, $(D\tilde{\Psi}_L)(\psi,\eta)$ is the Gâteaux derivative of $\tilde{\Psi}_L$ at ψ in the direction of η. Furthermore, there is a $K_B>0$ such that*

$$\left|(D\tilde{\Psi}_L)(\psi,\eta)\right|\le K_B\chi_{[0,L]}(\|\psi\|_H)\|\eta\|_H$$

for all ψ and η in H.

Proof. The claim is straightforward. $\square$

For each $n \in \mathbb{N}$, we now define $\tilde{w}_n^L(t,x) \overset{\text{def}}{=} \tilde{u}_{n+1}^L(t,x) - \tilde{u}_n^L(t,x)$ for all $x \geq 0$ and $t \geq 0$. Clearly $\sup_{0\leq t\leq T} \mathbb{E}\left[\|\tilde{w}_1^L\|_H^2\right] < \infty$ for all $T > 0$. We then write that

$$\tilde{w}_{n+1}^L(t,x) = \sum_{j=1}^4 A_j^{(n)}(t,x)$$

where

$$\begin{aligned}
A_1^{(n)}(t,x) &= \int_{\lambda=0}^1 \int_{s=0}^t \left(\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\frac{\partial \tilde{w}_n^L}{\partial x}(s,y)dy\right) \tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
&= \int_{\lambda=0}^1 \int_{s=0}^t \left(T_{t-s}\frac{\partial \tilde{w}_n^L}{\partial x}(s,\cdot)\right)(x)\tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
A_2^{(n)}(t,x) &= \int_{\lambda=0}^1 \int_{s=0}^t \left(\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\left\{\frac{\partial u_n^L}{\partial x}(s,y) + \lambda \frac{\partial \tilde{w}_L^n}{\partial x}(s,y)\right\} dy\right) \\
&\qquad \times D\tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot), \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
&= \int_{\lambda=0}^1 \int_{s=0}^t \left(T_{t-s}\left\{\frac{\partial u_n^L}{\partial x}(s,\cdot) + \lambda \frac{\partial \tilde{w}_L^n}{\partial x}(s,\cdot)\right\}\right)(x) D\tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot), \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
A_3^{(n)}(t,x) &= \int_{\lambda=0}^1 \int_{s=0}^t \int_{y=0}^\infty \hat{p}_+(t-s,x,y)e^{-y}dy D\tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot), \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
&= \int_{\lambda=0}^1 \int_{s=0}^t (T_{t-s}E)(x) D\tilde{\Psi}_L\left(\tilde{u}_n^L(s,\cdot) + \lambda \tilde{w}_n^L(s,\cdot), \tilde{w}_n^L(s,\cdot)\right) ds\, d\lambda \\
A_4^{(n)}(t,x) &= \int_{s=0}^t \left(\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\tilde{w}_n^L(s,y)dy\right) dB_s \\
&= \int_{s=0}^t \left(T_{t-s}\tilde{w}_n^L(s,\cdot)\right)(x) dB_s
\end{aligned}$$

where for convenience we have set $E(x) \overset{\text{def}}{=} e^{-x}$ for all $x \geq 0$. Note that the $\tilde{u}_n^L$'s and $\tilde{w}_n^L$'s are all in H_{even}.

An easy calculation gives us that

$$\mathbb{E}\left[\|A_4^{(n)}(t,\cdot)\|_H^2\right] = \int_{s=0}^t \mathbb{E}\left[\|T_{t-s}\tilde{w}_n^L(s,\cdot)\|^2\right] ds \leq \int_{s=0}^t \mathbb{E}\left[\|\tilde{w}_n^L(s,\cdot)\|_H^2\right] ds.$$

We similarly have (using Jensen's inequality) that

$$\mathbb{E}\left[\|A_3^{(n)}(t,\cdot)\|_H^2\right] \leq tK_B^2 \int_{s=0}^t \|T_{t-s}E\|_H^2 \mathbb{E}\left[\|\tilde{w}_n^L(s,\cdot)\|_H^2\right] ds \leq tK_B^2\|E\|_H^2 \int_{s=0}^t \mathbb{E}\left[\|\tilde{w}_n^L(s,\cdot)\|_H^2\right] ds.$$

To bound $A_1^{(n)}$ and $A_2^{(n)}$, we use the fact that $t^{-3/4}$ is locally integrable. More precisely,

$$\int_{s=0}^t \frac{1}{(t-s)^{3/4}} ds = 4t^{1/4}$$

for all $t > 0$. Thus

$$\mathbb{E}\left[\|A_1^{(n)}(t,\cdot)\|_H^2\right] \leq K_B^2 \mathbb{E}\left[\left|\int_{s=0}^t \left\|T_{t-s}\frac{\partial \tilde{w}_n^L}{\partial x}(s,\cdot)\right\|_H ds\right|^2\right]$$

$$\leq K_B^2 \mathbb{E}\left[\left|\int_{s=0}^t \frac{\left\|\tilde{w}_n^L(s,\cdot)\right\|_H^2}{(t-s)^{3/4}} ds\right|^2\right] \leq 4K_B^2 t^{1/4} \int_{s=0}^t \frac{\mathbb{E}\left[\left\|\tilde{w}_n^L(s,\cdot)\right\|_H^2\right]}{(t-s)^{3/4}} ds.$$

Finally, we have that

$$\begin{aligned}
&\mathbb{E}\left[\|A_2^{(n)}(t,\cdot)\|_H^2\right] \\
&\leq K_B^2 \mathbb{E}\left[\left|\int_{s=0}^t \left\|T_{t-s}\left(\frac{\partial \tilde{u}_n^L}{\partial x}(s,\cdot)+\lambda\frac{\partial \tilde{u}_n^L}{\partial x}(s,\cdot)\right)\right\|_H \chi_{[0,L]}\left(\left\|\tilde{u}_n^L(s,\cdot)+\lambda\tilde{w}_n^L(s,\cdot)\right\|_H\right)\|\tilde{w}_n^L(s,\cdot)\|_H ds\right|^2\right] \\
&\leq K_B^2 \mathbb{E}\left[\left|\int_{s=0}^t \frac{L}{(t-s)^{3/4}}\|\tilde{w}_n^L(s,\cdot)\|_H ds\right|^2\right] \leq 4K_B^2 L^2 t^{1/4}\int_{s=0}^t \frac{\mathbb{E}[\|\tilde{w}_n^L(s,\cdot)\|_H^2]}{(t-s)^{3/4}} ds.
\end{aligned}$$

Lemma 4.4. *For each $T>0$, we have that $\sum_{n=1}^\infty \sup_{0\leq t\leq T}\mathbb{E}\left[\|\tilde{u}_{n+1}^L-\tilde{u}_n^L\|_H\right]<\infty$. Thus $\mathbb{P}$-a.s., $u^L(t,\cdot)\overset{def}{=}\lim_{n\to\infty}u_n^L(t,\cdot)$ exists as a limit in $C([0,T];H)$ and u^L satisfies the integral equation*

(22)

$$\begin{aligned}
\tilde{u}^L(t,x) &= \int_{y=0}^\infty \hat{p}_+(t,x,y)\tilde{u}_\circ(y)dy \\
&\quad+\int_{s=0}^t\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\left\{\left(\frac{\partial\tilde{u}^L}{\partial x}(t,y)+e^{-y}\right)\left(1-\frac{\partial^2\tilde{u}^L}{\partial x^2}(t,0)\right)\Psi_L\left(\|\tilde{u}^L(t,\cdot)\|_H\right)\right. \\
&\qquad\left.-(\hat{\alpha}+1)e^{-y}\right\}dy\,ds \\
&\quad+\int_{s=0}^t\int_{y=0}^\infty \hat{p}_+(t-s,x,y)\left\{\tilde{u}^L(s,y)-e^{-y}\right\}dB_s. \qquad t>0,\ x>0
\end{aligned}$$

Proof. See also [Wal86, Lemma 3.3]. Fixing $T>0$ we collect the above calculations to see that there is a $K_T>0$ such that

$$\mathbb{E}[\|\tilde{w}_{n+1}^L(t,\cdot)\|_H^2]\leq K_T\int_{s=0}^t\frac{\mathbb{E}[\|\tilde{w}_n^L(t,\cdot)\|_H^2]}{(t-s)^{3/4}}ds$$

for all $t\in[0,T]$. Iterating this, we get that

$$\mathbb{E}[\|\tilde{w}_n^L(t,\cdot)\|_H^2]\leq K_T^{n-1}t^{(n-1)/4}\left\{\prod_{j=1}^{n-2}B(1+j/4,1/4)\right\}\sup_{0\leq t\leq T}\mathbb{E}[\|\tilde{w}_1^L\|_H^2]$$

where B is the standard Beta function and thus that

$$\sqrt{\mathbb{E}[\|\tilde{w}_n^L(t,\cdot)\|_H^2]}\leq K_T^{(n-1)/2}t^{(n-1)/8}\left\{\prod_{j=1}^{n-2}B(1+j/4,1/4)\right\}^{1/2}\sup_{0\leq t\leq T}\sqrt{\mathbb{E}[\|\tilde{w}_1^L\|_H^2]}.$$

To show that the terms on the right are summable, we use the ratio test. It suffices to show that

$$\lim_{n\to\infty}K_T^{1/2}t^{1/8}\left(B\left(1+\frac{n-2}{4},1/4\right)\right)^{1/2}=0. \tag{23}$$

We calculate that

$$B(1+n/4,1/4)=\int_{s=0}^1 s^{n/4}(1-s)^{-3/4}ds=\int_{s=0}^{1/2}s^{n/4}(1-s)^{-3/4}ds+\int_{s=1/2}^1 s^{n/4}(1-s)^{-3/4}ds$$

$$\leq \left(\frac{1}{2}\right)^{n/4} \int_{s=0}^{1/2} (1-s)^{-3/4} ds + \left(\frac{1}{2}\right)^{-3/4} \int_{s=1/2}^{1} s^{n/4} ds$$
$$= \left(\frac{1}{2}\right)^{(n+1)/4} + \left(\frac{1}{2}\right)^{-3/4} \frac{1-(1/2)^{n/4+1}}{n/4+1}.$$

This implies (23). The rest of the proof follows by standard calculations. □

We can finally show uniqueness.

Lemma 4.5. *The solution of* (22) *is unique.*

Proof. Let u_1 and u_2 be two solutions. Define $\tilde{w} \overset{\text{def}}{=} u_1 - u_2$. By calculations as above we get that

$$\mathbb{E}[\|\tilde{w}(t,\cdot)\|_H^2] \leq K_T \int_{s=0}^{t} (t-s)^{-3/4} \mathbb{E}[\|\tilde{w}(s,\cdot)\|_H^2] ds.$$

We can iterate this inequality several times to get (cf. [Wal86, Theorem 3.2])

$$\begin{aligned}
\mathbb{E}[\|\tilde{w}(t,\cdot)\|_H^2] &\leq K_T^2 \int_{s=0}^{t} (t-s)^{-3/4} \int_{r=0}^{s} (s-r)^{-3/4} \mathbb{E}[\|\tilde{w}(r,\cdot)\|_H^2] dr\, ds \\
&= K_T^2 B(1/4,1/4) \int_{r=0}^{t} (t-r)^{-2/4} \mathbb{E}[\|\tilde{w}(r,\cdot)\|_H^2] dr \\
&\leq K_T^3 B(1/4,1/4) \int_{r=0}^{t} (t-r)^{-2/4} \int_{s=0}^{r} (r-s)^{-3/4} \mathbb{E}[\|\tilde{w}(s,\cdot)\|_H^2] ds\, dr \\
&= K_T^3 B(1/4,1/4) B(1/2,1/4) \int_{r=0}^{t} (t-s)^{-1/4} \mathbb{E}[\|\tilde{w}(s,\cdot)\|_H^2] ds \\
&\leq K_T^4 B(1/4,1/4) B(1/2,1/4) \int_{s=0}^{t} (t-s)^{-1/4} \int_{r=0}^{s} (s-r)^{-3/4} \mathbb{E}[\|\tilde{w}(r,\cdot)\|_H^2] dr\, ds \\
&= K_T^4 B(1/4,1/4) B(1/2,1/4) B(3/4,1/4) \int_{r=0}^{t} \mathbb{E}[\|\tilde{w}(r,\cdot)\|_H^2] dr
\end{aligned}$$

We can now use Gronwall's inequality. □

Let's now see what happens as $L \nearrow \infty$. Define the random times

$$\tau_L \overset{\text{def}}{=} \inf\{t \geq 0 : \|\tilde{u}^L(t,\cdot)\|_H \geq L\} \qquad L > 0$$
$$\tau \overset{\text{def}}{=} \varlimsup_{L\to\infty} (\tau_L \wedge L).$$

Let's also define

$$\tilde{u}(t,x) \overset{\text{def}}{=} \varlimsup_{L\to\infty} \tilde{u}^L(t \wedge \tau_L, x). \qquad t \geq 0,\, x \geq 0$$

Lemma 4.6. *We have that*

$$\varlimsup_{t\nearrow\tau} \|\tilde{u}(t,\cdot)\|_H = \infty.$$

Define u as in (19)–(20). *Then $\{u(t,\cdot) \mid 0 \leq t < \tau\}$ is a weak solution of* (1).

Proof. Fixing $L' > L$ we have from the uniqueness claim of Lemma 4.5 that $\tilde{u}^{L'}(t,\cdot) = \tilde{u}^L(t,\cdot)$ for $0 \leq t \leq \tau_L$. Thus $\tau_{L'} \geq \tau_L$ for all $L' > L$, and so $\tau = \lim_{L\to\infty} \tau_L = \lim_{L\to\infty}(\tau_L \wedge L)$ and τ is predictable. We also have that $\tilde{u}(t,\cdot) = \lim_{L\to\infty} \tilde{u}^L(t,\cdot)$ for $0 \leq t < \tau$. From this and Lemma 3.5, we conclude that $\{u(t,\cdot) \mid 0 \le t < \tau\}$ as defined by (19)–(20) indeed is a weak solution of (1). The characterization of $\|\tilde{u}(t,\cdot)\|_H$ at $\tau-$ is obvious. $\square$

In fact, we have a more explicit characterization of τ.

Lemma 4.7. *We have that*

$$\varlimsup_{t\nearrow\tau} \left|\frac{\partial^2 \tilde{u}}{\partial x^2}(t,0)\right| = \infty.$$

Proof. For each $L>0$, define

$$\tau'_L \stackrel{\text{def}}{=} \inf\left\{t\in[0,\tau] \mid \left|\frac{\partial^2 \tilde{u}}{\partial x^2}(t-,0)\right| \ge L\right\}. \qquad (\inf\emptyset = \tau)$$

By standard SPDE calculations like we used in Lemma 4.4, we know that (22) has a solution on $[0,\tau'_L]$. Thus in fact $\tau > \tau'_L$ and hence

$$\left|\frac{\partial^2 \tilde{u}}{\partial x^2}(\tau'_L,0)\right| = L.$$

Consequently

$$\lim_{L\to\infty} \left|\frac{\partial^2 \tilde{u}}{\partial x^2}(\tau'_L,0)\right| = \infty.$$

Since $\tau'_L \le \tau$, we of course also have that $\lim_{L\to\infty}\tau'_L \le \tau$. On the other hand, $\|u(t,\cdot)\|_H$ may become large for many reasons other than $\left|\frac{\partial^2 \tilde{u}}{\partial x^2}(\tau'_L,0)\right|$ becoming large, so necessarily $\tau \le \lim_{L\to\infty}\tau'_L$. Putting things together, we get that $\lim_{L\to\infty}\tau_L = \tau$. The claimed result now follows. $\square$

To finish things off, we prove uniqueness.

Lemma 4.8 (Uniqueness)**.** *If* $\{u(t,\cdot) \mid 0\le t<\tau\} \subset H$ *and* $\{u'(t,\cdot) \mid 0 \le t < \tau'\} \subset H$ *are two solutions of* (16)*, then* $u(t,\cdot) = u'(t,\cdot)$ *for* $0 \le t < \min\{\tau,\tau'\}$.

Proof. For each $L>0$, define

$$\sigma_L \stackrel{\text{def}}{=} \inf\left\{t\in[0,\tau\wedge\tau'): \left|\frac{\partial^2 \tilde{u}}{\partial x^2}(t,0)\right| \ge L \text{ or } \left|\frac{\partial^2 \tilde{u}'}{\partial x^2}(t,0)\right| \ge L\right\}. \qquad \inf\emptyset = \tau\wedge\tau'$$

Then $\tau\wedge\tau' \le \lim_{L\to\infty}\sigma_L$. We can use standard uniqueness theory to conclude that u and u' coincide on $[0,\sigma_L]$, and we then let $L\nearrow\infty$. $\square$

4.1. **Proofs.** We here give the delayed proofs. We start with the structural claims about H.

Proof of Lemma 4.1. The fact that $H\subset C^2$ is well-known; [Eva98]. Fix $\varphi\in C_0^\infty(\mathbb{R}_+)$, $x\in(0,\infty)$, and $i\in\{0,1,2\}$. We then have that

$$\frac{\partial^i\varphi}{\partial x^i}(x) = \int_{s=x}^{x+1}\frac{\partial^i\varphi}{\partial x^i}(s)ds - \int_{s=x}^{x+1}\left\{\frac{\partial^i\varphi}{\partial x^i}(s) - \frac{\partial^i\varphi}{\partial x^i}(x)\right\}ds$$

$$= \int_{s=x}^{x+1}\frac{\partial^i\varphi}{\partial x^i}(s)ds - \int_{s=x}^{x+1}\int_{r=x}^{s}\frac{\partial^{i+1}\varphi}{\partial x^{i+1}}(r)dr\,ds = \int_{s=x}^{x+1}\frac{\partial^i\varphi}{\partial x^i}(s)ds - \int_{r=x}^{x+1}(x+1-r)\frac{\partial^{i+1}\varphi}{\partial x^{i+1}}(r)dr$$

Thus

$$\left|\frac{\partial^i\varphi}{\partial x^i}(x)\right| \le \sqrt{\int_{s=x}^{x+1}\left|\frac{\partial^i\varphi}{\partial x^i}(s)\right|^2 ds} + \sqrt{\int_{r=x}^{x+1}\left|\frac{\partial^{i+1}\varphi}{\partial x^{i+1}}(r)\right|^2 dr} \le 2\|\varphi\|_H.$$

Of course we also have that

$$\left|\varphi^{(i)}(x)-\varphi^{(i)}(y)\right|\le\sqrt{\|\varphi\|_H}\sqrt{|x-y|}$$

so the stated limits at $x=0$ exist. □

We next study $\{T_t\}_{t>0}$.

Proof of Lemma 4.2. The proof relies upon a combination of fairly standard calculations.

To begin, fix $\varphi\in C_0^\infty(\mathbb{R}_+)$ and define

$$u(t,x)\overset{\text{def}}{=}\int_{y\in\mathbb{R}}p_\circ(t,x-y)\varphi(|y|)dy$$
$$=\int_{y=0}^\infty p_\circ(t,x-y)\varphi(y)dy+\int_{y=-\infty}^0 p_\circ(t,x-y)\varphi(-y)dy=\int_{y=0}^\infty\left\{p_\circ(t,x-y)+p_\circ(t,x+y)\right\}\varphi(y)dy.$$

Thus $u(t,x)=(T_t\varphi)(x)$ for $x>0$, and since $p_\circ$ is even in its second argument,

$$u(t,-x)=\int_{y\in\mathbb{R}}p_\circ(t,-x+y)\varphi(|y|)dy=\int_{y\in\mathbb{R}}p_\circ(t,-x-y)\varphi(|y|)dy=u(t,x)$$

so in fact $u(t,\cdot)$ is even. Thus we indeed have that $\frac{\partial^n u}{\partial x^n}(t,0)=0$ for all odd $n\in\mathbb{N}$; thus $T_t\varphi\in H_{\text{even}}$.

A standard calculation shows that T_t is a contraction on H. Indeed, for each nonnegative integer n,

$$\frac{d}{dt}\int_{x\in\mathbb{R}}\left|\frac{\partial^n u}{\partial x^n}(t,x)\right|^2dx=2\int_{x\in\mathbb{R}}\frac{\partial^{n+2}u}{\partial x^{n+2}}(t,x)\frac{\partial^n u}{\partial x^n}(t,x)dx=-2\int_{x\in\mathbb{R}}\left|\frac{\partial^{n+1}u}{\partial x^{n+1}}(t,x)\right|^2dx\le 0$$

and thus

$$\begin{aligned}\int_{x=0}^\infty\left|\frac{\partial^n u}{\partial x^n}(t,x)\right|^2dx&=\frac12\int_{x\in\mathbb{R}}\left|\frac{\partial^n u}{\partial x^n}(t,x)\right|^2dx\le\frac12\int_{x\in\mathbb{R}}\left|\frac{\partial^n u}{\partial x^n}(0,x)\right|^2dx\\&=\int_{x=0}^\infty\left|\varphi^{(n)}(x)\right|^2dx.\end{aligned}\tag{24}$$

Summing these inequalities up for $n\in\{0,1,2,3\}$, we see that $\|T_t\varphi\|_H^2\le\|\varphi\|_H^2$ for all $\varphi\in C_0^\infty(\mathbb{R}_+)$. This implies that T_t is a contraction on $C_0^\infty(\mathbb{R}_+)$ and has the claimed extension.

To proceed, fix $\varphi\in C_{0,\text{even}}^\infty(\mathbb{R}_+)$ and define

$$v(t,x)=\int_{y=0}^\infty\hat{p}_+(t,x,y)\varphi^{(1)}(y)dy=\int_{y\in\mathbb{R}}p_\circ(t,x-y)\varphi^{(1)}(|y|)dy;$$

note for future reference that since $\varphi^{(1)}(0)=0$, $y\mapsto\varphi^{(1)}(|y|)$ is continuous at $y=0$. Differentiating, we get that

$$\frac{\partial v}{\partial x}(t,x)=\int_{y\in\mathbb{R}}\frac{\partial p_\circ}{\partial x}(t,x-y)\varphi^{(1)}(|y|)dy=\int_{y\in\mathbb{R}}p_\circ(t,x-y)\varphi^{(2)}(|y|)\operatorname{sgn}(y)dy.\tag{25}$$

Thus in particular

$$\int_{x=0}^\infty|v(t,x)|^2dx\le\int_{x=0}^\infty\left|\varphi^{(1)}(x)\right|^2dx$$
$$\int_{x=0}^\infty\left|\frac{\partial v}{\partial x}(t,x)\right|^2dx\le\int_{x=0}^\infty\left|\varphi^{(2)}(x)\right|^2dx.$$

Differentiating (25) again, we get that

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}(t,x) &= \int_{y\in\mathbb{R}} \frac{\partial p_\circ}{\partial x}(t,x-y)\varphi^{(2)}(|y|)\,\mathrm{sgn}(y)dy\\ \frac{\partial^3 v}{\partial x^3}(t,x) &= \int_{y\in\mathbb{R}} \frac{\partial^2 p_\circ}{\partial x^2}(t,x-y)\varphi^{(2)}(|y|)\,\mathrm{sgn}(y)dy\\ &= 2\frac{\partial p_\circ}{\partial x}(t,x)\varphi^{(2)}(0) + \int_{y\in\mathbb{R}} \frac{\partial p_\circ}{\partial x}(t,x-y)\varphi^{(3)}(|y|)dy.\end{aligned}$$

We now note that there is a $K>0$ such that

$$\left|\frac{\partial p_\circ}{\partial x}(t,x)\right| \le \frac{K}{\sqrt{t}} p_\circ(2t,x)$$

for all $t>0$ and $x\in\mathbb{R}$. Thus

$$\begin{aligned}\left|\frac{\partial^2 v}{\partial x^2}(t,x)\right| &\le \frac{K}{\sqrt{t}}\int_{y\in\mathbb{R}} p_\circ(2t,x-y)\left|\varphi^{(2)}(|y|)\right| dy\\ \left|\frac{\partial^3 v}{\partial x^3}(t,x)\right| &\le \frac{2K}{\sqrt{t}} p_\circ(2t,x)\left|\varphi^{(2)}(0)\right| + \frac{K}{\sqrt{t}}\int_{y\in\mathbb{R}} p_\circ(2t,x-y)\left|\varphi^{(3)}(|y|)\right| dy.\end{aligned}$$

We can now fairly easily conclude from (24) with $n=0$ that

$$\sqrt{\int_{x=0}^\infty \left|\frac{\partial^2 v}{\partial x^2}(t,x)\right|^2 dx} \le \frac{K}{\sqrt{t}}\sqrt{\int_{x=0}^\infty \left|\varphi^{(2)}(x)\right|^2 dx}.$$

We also note that

$$\sqrt{\int_{x=0}^\infty p_\circ^2(2t,x)dx} = \sqrt{\frac{1}{\sqrt{2\pi t}}\int_{x=0}^\infty \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}\right]dx} \le \frac{1}{(2\pi t)^{1/4}}.$$

Thus

$$\sqrt{\int_{x=0}^\infty \left|\frac{\partial^3 v}{\partial x^3}(t,x)\right|^2 dx} \le \frac{K}{(2\pi)^{1/4}t^{3/4}}|\varphi^{(2)}(0)| + \frac{K}{\sqrt{t}}\sqrt{\int_{x=0}^\infty \left|\varphi^{(3)}(x)\right|^2 dx}.$$

Combine things together to get the last claim. □

5. Numerical Simulation

In this section, we will see from numerical simulations where the moving boundary is. In general, it is difficult to simulate the SPDE (1) directly since we need to find a solution of a stochastic heat equation and at the same time we need to trace the position of the moving boundary. Here we can avoid this difficulty since we have the explicit formula for the solution u in Lemma 3.5. That is,

$$u(t,x) \overset{\text{def}}{=} \begin{cases} \tilde{u}(t,x-\beta(t)) - \exp\left[-\left(x-\beta(t)\right)\right] & x\ge\beta(t),\ 0\le t<\tau\\ 0 & x<\beta(t),\ 0\le t<\tau,\end{cases} \tag{26}$$

where $\beta(t)$ is defined as

$$\beta(t) = \int_{s=0}^t \left\{1 - \frac{\partial^2\tilde{u}}{\partial x^2}(s,0)\right\}ds \qquad 0\le t<\tau \tag{27}$$

and $\tilde{u}$ is a solution of the SPDE

(28)

$$\begin{aligned} d\tilde{u}(t,x) &= \left\{ \frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) + \hat{\alpha}\left(\tilde{u}(t,x) - e^{-x}\right) - e^{-x} + \left(\frac{\partial \tilde{u}}{\partial x}(t,x) + e^{-x}\right)\left(1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(t,0)\right)\right\} dt \\ &\quad + \left(\tilde{u}(t,x) - e^{-x}\right) dB_t \qquad t > 0,\ x > 0 \\ \frac{\partial \tilde{u}}{\partial x}(t,0) &= 0 \qquad t > 0 \\ \tilde{u}(0,x) &= \tilde{u}_\circ(x) = u_\circ(x) + e^{-x}. \qquad x > 0 \end{aligned}$$

Therefore we first need to solve the SPDE (28) numerically in order to obtain the moving boundary $\beta(t)$ and then the weak solution $u(t,x)$. Here we first discretize space by using the explicit finite difference scheme, then we can obtain SDE's. Now we use the Euler-Maruyama Method to find numerical solutions of SDE's (see [Gai96, Hig02]). Since there is a stability issue for parabolic PDE we note that $\Delta t/(\Delta x)^2 < 1/2$, where Δt is a time step and Δx is a space step. Figure 1 is a simulation with initial condition

$$u_\circ(x) = \begin{cases} \frac{x+x^2}{1+x^4} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

and $\alpha = .5$. We can clearly see that there are two phases separated by the black line, which is the moving boundary, and how u is changing on the colored region where $u > 0$.

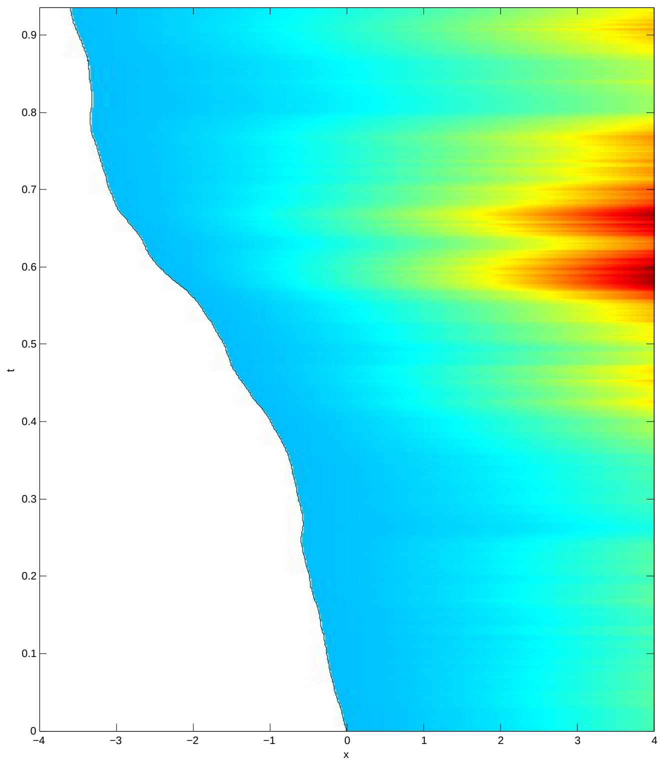

FIGURE 1. Weak solution $u(t, x)$

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