Intermittency properties in a hyperbolic Anderson problem

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Abstract

We study the asymptotics of the even moments of solutions to a stochastic wave equation with linear multiplicative noise. Our main theorem states that these moments grow more quickly than one might expect. This phenomenon is well-known for parabolic stochastic partial differential equations, under the name of intermittency. Our results seem to be the first example of this phenomenon for hyperbolic equations.

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1 Introduction

This paper studies intermittency properties of the solution to the following (semi-)linear stochastic wave equation in spatial dimension $d = 3$, with random potential $\dot{F}$:

$$\frac{\partial^2}{\partial t^2}u(t,x) = \Delta u(t,x) + u(t,x)\dot{F}(t,x), \tag{1.1}$$
$$u(0,x) \equiv v_0, \qquad \frac{\partial}{\partial t}u(0,x) \equiv \tilde{v}_0,$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^3$, Δ denotes the Laplacian on $\mathbb{R}^3$ and $v_0, \tilde{v}_0 \in \mathbb{R}$ and $v_0, \tilde{v}_0 > 0$. The process $\dot{F}$ is the *formal* derivative of a Gaussian random field, white in time and correlated in space, whose covariance function formally satisfies

$$E\left[\dot{F}(t,x)\dot{F}(s,y)\right] = \delta_0(t-s)f(x-y).$$

In this equation, $\delta(\cdot)$ denotes the Dirac delta function, and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d$, satisfying certain standard conditions that are specified in Section 2.

Next we summarize the concept of intermittency. This idea arose in physics, and different authors give it different meanings. Physicists say that a system is intermittent if its solution is dominated by a few large peaks. On the mathematical side Zeldovich, Molchanov, and coauthors [13], [16], [17], [11] formulated the following definition and developed the idea in the context of linear parabolic s.p.d.e's. For $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we define the upper and lower (moment) Lyapunov exponents of $u(t,x)$ to be

$$\begin{aligned}
\bar{\lambda}_n &= \limsup_{t\to\infty} \frac{\log E[|u(t,x)|^n]}{t}, \\
\underline{\lambda}_n &= \liminf_{t\to\infty} \frac{\log E[|u(t,x)|^n]}{t}.
\end{aligned}$$

In principle, the upper and lower Lyapunov exponents depend on x, but because our initial functions v_0 and $\tilde{v}_0$ are constant and the random potential $\dot{F}$ is spatially homogeneous, it turns out that there is no dependence on x. In case the upper and lower Lyapunov exponents agree, we write the common value as λ_n and call it the n-th (moment) Lyapunov exponent.

The reader can easily deduce from Jensen's or Hölder's inequalities that if the Lyapunov exponents exist, then

$$\lambda_1 \leq \frac{\lambda_2}{2} \leq \frac{\lambda_3}{3} \leq \cdots \tag{1.2}$$

We know that equality holds in Jensen's or Hölder's inequalities if and only if the random variable involved is constant. Intermittency should mean that as a function of x, $u(t,x)$ is far from constant and consists of "high peaks and low valleys." From this informal reasoning, we are led to say that the solution $u(t,x)$ is *intermittent* if

$$\lambda_1 < \frac{\lambda_2}{2} < \frac{\lambda_3}{3} < \cdots \tag{1.3}$$

The fact that these inequalities do imply the existence of high peaks is established by Cranston and Molchanov [3] in the case of the stochastic heat equation.

We do not establish the existence of Lyapunov exponents and therefore intermittency for $u(t,x)$ in the sense of (1.3). However, we prove strict inequalities in the sense of (1.3) for the upper and lower Lyapunov exponents of even order. This suggests that some form of the intermittency phenomenon is present in our hyperbolic s.p.d.e. (1.1).

One motivation for studying (1.1) is its similarity to the parabolic Anderson model studied in [1] and in many subsequent papers (see for instance [3, 4, 9, 10, 14]). Another comes from the following idea. The right hand side of the wave equation usually represents elastic forces (the Laplacian term) plus a forcing term, according to Newton's law which states that the acceleration $\partial^2 u/\partial t^2$ equals the force. We can easily imagine that the force might be random, and the strength of the randomness could depend on the solution u. This would lead to a term of the form $h(u(t,x))\dot{F}(t,x)$ for some function h. If we use a linear approximation, $h(u) \approx h_0 u$, we are left with the equation (1.1).

For the hyperbolic equation (1.1), one would expect the intermittency property (1.3) to translate into a different sample path behavior than the "high-peak" picture that is valid for the stochastic heat equation. Indeed, the heat equation has monotonicity properties of solutions that are not present in the wave equation. For the wave equation, one would rather expect intermittency to translate into very large oscillations of the sample paths. Making this picture precise is a research project.

The main results of this paper are stated in Theorems 3.2 and 4.1, and the reader can look ahead to see the assumptions. Here, we give the implications of those theorems for the upper and lower Lyapunov exponents.

Theorem 1.1. *There exist constants $C_1, C_2 > 0$ such that the following holds. Firstly, for $n \in \mathbb{N}$, under the assumptions of Theorem 3.2,*

$$\frac{\bar{\lambda}_n}{n} \leq C_1 n^{1/3}.$$

Secondly, for $n \in \mathbb{N}$ even, under the assumptions of Theorem 4.1,

$$\frac{\underline{\lambda}_n}{n} \geq C_2 n^{1/3}.$$

In other words, a kind of intermittency holds for the even Lyapunov exponents, in the sense that when divided by n, the even upper and lower exponents grow like $n^{1/3}$, and equality must fail infinitely many times in (1.2).

In order to prove this theorem, it is first necessary to give a rigorous meaning to the s.p.d.e. (1.1). For this, we use the extension of Walsh's martingale measure stochastic integral developed by the first author in [5], and the associated integral formulation of (1.1). The second key ingredient is a formula for the moments of the solution to (1.1), analogous to the Feynman-Kac formula. Indeed, for the parabolic s.p.d.e. considered in [1], this formula plays a central role. For the stochastic wave equation (1.1), the authors, together with R. Tribe, have developed a more general Feynman-Kac-type formula that leads to an expression for the moments of $u(t, x)$ (see [7]). These formulas are recalled in Section 2 and used in Sections 3 and 4.

Remark 1.2. Our methods should also apply to the one- and two-dimensional wave equations, for which the fundamental solutions are nonnegative functions $S(t, x)$ with $\int S(t, x)dx = t$. However, they will not apply directly to the stochastic wave equation in dimensions $d \geq 4$, in which the fundamental solution is a Schwarz distribution which is not a signed measure. Some results on moments of the solution to the stochastic wave equation in high dimensions are contained in [2].

2 Existence, uniqueness and moments of the solution

We begin by giving a formal definition of the Gaussian noise $\dot{F}$. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions (see [12]). On a given probability space $(\Omega, \mathcal{F}, P)$, we define a Gaussian process $F = (F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1}))$ with mean zero and covariance functional

$$E\big[F(\varphi)F(\psi)\big] = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \varphi(t,x)\, f(x-y)\, \psi(t,y).$$

Since this is a covariance, it is well-known [12, Schwartz, Chap. VII, Théorème XVII] that f must be symmetric and be the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$, termed the *spectral measure* : $f = \mathcal{F}\mu$. In this case, F extends to a worthy martingale measure $M = (M_t(B),\ t \geq 0,\ B \in \mathcal{B}_b(\mathbb{R}^d))$ in the sense of [15], with covariation measure Q defined by

$$Q([0,t] \times A \times B) = \langle M(A), M(B) \rangle_t = t \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ 1_A(x) f(x-y) 1_B(y),$$

and dominating measure $K = Q$ (see [6, 5]). By construction, $t \mapsto M_t(B)$ is a continuous martingale and

$$F(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t,x) M(dt,dx),$$

where the stochastic integral is as defined in [15].

For $d = 3$, the fundamental solution of the wave equation is the measure defined by

$$S(t) = \frac{1}{4\pi t} \sigma_t, \tag{2.1}$$

for any $t > 0$, where σ_t denotes the uniform surface measure (with total mass $4\pi t^2$) on the sphere $B(0,t)$ of radius t. In particular,

$$S(t, \mathbb{R}^3) = t. \tag{2.2}$$

Hence, in the mild formulation of equation (1.1), Walsh's classical stochastic integration theory developed in [15] does not apply. In this paper, we use the extension of the stochastic integral developed in Dalang [5].

2.1 Existence and uniqueness

The following assumption is needed (see [5, Theorem 11 and Example 6]) for equation (1.1) to have a solution.

Assumption A. The spectral measure μ of the Gaussian process F satisfies

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+\|\xi\|^2} < \infty.$$

We term a *solution to* (1.1) a jointly measurable and adapted process $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3)$ that satisfies the stochastic integral equation

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u(s,y) F(ds,dy), \tag{2.3}$$

where $w(t,x)$ is the solution to the homogeneous (and deterministic) wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) w(t,x) = 0, \qquad w(0,x) \equiv v_0, \qquad \frac{\partial}{\partial t} w(0,x) \equiv \tilde{v}_0. \tag{2.4}$$

In particular,

$$w(t,x) = v_0 + t\tilde{v}_0 \tag{2.5}$$

so w does not depend on x.

The following proposition is proved in [5].

Proposition 2.1. *Fix $T > 0$. If Assumption A holds, then (2.3) has a unique square-integrable solution $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3)$. Moreover, this solution is L^2-continuous and for all $T > 0$ and $p \geq 0$,*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^3} E[|u(t,x)|^p] < \infty.$$

Hölder continuity of $(u(t,x))$ is studied in [8].

2.2 The probabilistic representation of second moments

Following [7], we give a kind of Feynman-Kac formula for the second moments of our solutions. Instead of Brownian motion, our underlying process moves with speed 1 and changes directions at random times.

Let $\tilde{X}_t = t\,\Theta_0$, where Θ_0 is chosen according to the uniform probability measure on $\partial B(0,1)$. In particular, $t \mapsto \tilde{X}_t$ is uniform motion in the randomly chosen direction Θ_0, with starting point X_0 to be specified.

Let $\tilde{\Theta}_i$, $i = 1, 2, \ldots$ be i.i.d. copies of Θ_0, and let $\tilde{X}^{(i)} = (\tilde{X}^{(i)}_t,\ t \geq 0),\ i \geq 1$, be defined by $\tilde{X}^{(i)}_t = t\tilde{\Theta}_i$, so that they are i.i.d. copies of $(\tilde{X}_t,\ t \geq 0)$. Let $(N(t),\ t \geq 0)$ be a rate 1 Poisson process independent of the $(\tilde{X}^{(i)})$. Let $0 < \tau_1 < \tau_2 < \cdots$ be the jump times of $(N(t))$ and set $\tau_0 \equiv 0$. Define a process $X = (X_t,\ t \geq 0)$ as follows :

$$X_t = X_0 + \tilde{X}^{(1)}_t, \qquad \text{for } 0 \leq t \leq \tau_1,$$

and for $i \geq 1$,

$$X_t = X_{\tau_i} + \tilde{X}^{(i+1)}_{t-\tau_i}, \qquad \text{for } \tau_i < t \leq \tau_{i+1}.$$

We use P_x to denote a probability under which, in addition, $X_0 = x$ with probability one. Informally, the process X follows $\tilde{X}^{(1)}$ during the interval $[0, \tau_1]$, then follows $\tilde{X}^{(2)}$ started at X_{τ_1} during $[\tau_1, \tau_2]$, then $\tilde{X}^{(3)}$ started at X_{τ_2} during $[\tau_2, \tau_3]$, etc.

Using two independent i.i.d. families $(\tilde{X}^{(i,1)}_{\cdot}, i \geq 1)$ and $(\tilde{X}^{(i,2)}_{\cdot}, i \geq 1)$, construct, as for X above, two processes $X^1 = (X^1_t,\ t \geq 0)$ and $X^2 = (X^2_t,\ t \geq 0)$ which renew themselves at the same set of jump times τ_i of the process N, and which start, under P_{x_1,x_2}, at x_1 and x_2 respectively. Expectation relative to P_{x_1,x_2} is denoted $E_{x_1,x_2}[\cdot]$.

Taking (2.2) into account, the following result is proved in [7, Theorem 4.3].

Theorem 2.2. *Let $u(t,x)$ be the solution of (2.3) given in Proposition 2.1. Then*

$$\begin{aligned} E[u(t,x)u(t,y)] &= e^t E_{x,y}\Big[w\left(t - \tau_{N(t)}, X^1_{\tau_{N(t)}}\right) w\left(t - \tau_{N(t)}, X^2_{\tau_{N(t)}}\right) \\ &\qquad \times \prod_{i=1}^{N(t)} \left((\tau_i - \tau_{i-1})^2 f\left(X^1_{\tau_i} - X^2_{\tau_i}\right)\right) \Big] \end{aligned}$$

(where, on $\{N(t) = 0\}$, the product is defined to take the value 1).

2.3 Moments of order n

Theorem 2.2 extends to higher moments as follows. Let $\mathcal{P}_n$ denote the set of unordered pairs from $\mathcal{L}_n = \{1, \ldots, n\}$ and for $\rho \in \mathcal{P}_n$, we write $\rho = \{\rho_1, \rho_2\}$, with $\rho_1 < \rho_2$. Note that card $(\mathcal{P}_n) = n(n-1)/2$. Let $(N_{\cdot}(\rho),\ \rho \in \mathcal{P}_n)$ be independent rate 1 Poisson processes. For $A \subseteq \mathcal{P}_n$, let $N_t(A) = \sum_{\rho \in A} N_t(\rho)$. This defines a Poisson random measure such that for fixed A, $(N_t(A),\ t \geq 0)$ is a Poisson process with intensity card(A). Let $\sigma_1 < \sigma_2 < \cdots$ be the jump times of $(N_t(\mathcal{P}_n),\ t \geq 0)$, and let $R^i = \{R_1^i, R_2^i\}$ be the pair corresponding to time σ_i.

For $\ell \in \mathcal{L}_n$, let $\mathcal{P}^{(\ell)} \subseteq \mathcal{P}_n$ be the set of pairs that contain ℓ, so that card$(\mathcal{P}^{(\ell)}) = n-1$. Let $\tau_1^\ell < \tau_2^\ell < \cdots$ be the jump times of $(N_t(\mathcal{P}^{(\ell)}),\ t \geq 0)$. We write $N_t(\ell)$ instead of $N_t(\mathcal{P}^{(\ell)})$. Note that

$$\sum_{\rho \in \mathcal{P}_n} N_t(\rho) = N_t(\mathcal{P}_n) = \frac{1}{2} \sum_{\ell \in \mathcal{L}_n} N_t(\ell). \tag{2.6}$$

We now define the motion process needed. For $\ell \in \mathcal{L}_n$ and $i \geq 0$, let $(\tilde{X}_t^{\ell,(i)},\ t \geq 0)$ be i.i.d. copies of the uniform motion process $(\tilde{X}_t)$ defined in Section 2.2. Set

$$X_t^\ell = \begin{cases} X_0^\ell + \tilde{X}_t^{\ell,(1)}, & 0 \leq t \leq \tau_1^\ell, \\ \\ X_{\tau_i^\ell}^\ell + \tilde{X}_{t-\tau_i^\ell}^{\ell,(i+1)}, & \tau_i^\ell < t < \tau_{i+1}^\ell. \end{cases}$$

In particular, at time σ_i, the two processes $X_{\cdot}^{R_1^i}$ and $X_{\cdot}^{R_2^i}$ change directions, while the other motions do not. For an illustration of these motions, see [7, Section 5].

It will be useful to define X_t^ℓ for certain $t < 0$. For given $(t_1, x_1), \ldots, (t_n, x_n)$, under the measure $P_{(t_1,x_1),\ldots,(t_n,x_n)}$, we set

$$X_t^\ell = \tilde{X}_{t+t_\ell}^{\ell,(0)} \quad \text{for } -t_\ell \leq t \leq 0.$$

Finally we set $\tau_0^\ell = -t_\ell$. Taking (2.2) into account, the following theorem is established in [7, Theorem 5.1].

Theorem 2.3. *The n-th product moments are given by*

$$
\begin{aligned}
&E[u(t,x_1)\cdots u(t,x_n)] \qquad (2.7)\\
&= e^{tn(n-1)/2} E_{(0,x_1),\ldots,(0,x_n)}\left[\prod_{i=1}^{N_t(\mathcal{P}_n)} f(X_{\sigma_i}^{R_1^i} - X_{\sigma_i}^{R_2^i})\right.\\
&\qquad \left.\times \prod_{\ell\in\mathcal{L}_n}\prod_{i=1}^{N_t(\ell)} (\tau_i^\ell - \tau_{i-1}^\ell)\cdot \prod_{\ell\in\mathcal{L}_n} w(t-\tau_{N_t(\ell)}^\ell, X_{\tau_{N_t(\ell)}}^\ell)\right].
\end{aligned}
$$

3 Upper bounds on the moments

In this section, we shall work under the following assumption.

Assumption B. The covariance function f is bounded (hence uniformly continuous and attains its maximum at 0). We let $\alpha = f(0)$.

Note that under Assumption B, the spectral measure μ satisfies $f(0) = \mu(\mathbb{R}^3) < \infty$, and so Assumption A is also satisfied.

3.1 Second moments

In this subsection, we show that $t \mapsto E[(u(t,x))^2]$ grows at most at an exponential rate. The method is specific to the second moment, and is much simpler that what will be needed for higher moments, which are dealt with in the next section.

Proposition 3.1. *Under Assumption B, there is $C < \infty$ such that for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^3$,*

$$|E[u(t,x)u(t,y)]| \leq C(v_0 + t\tilde{v}_0)^2 \exp(t(2\alpha)^{1/3}). \qquad (3.1)$$

Proof. By Theorem 2.2, (2.5) and Assumption B,

$$|E[u(t,x)u(t,y)]| \leq (v_0 + t\tilde{v}_0)^2\, e^t\, h(t),$$

where

$$h(t) = E_{x,y}\left[\prod_{i=1}^{N(t)} (\alpha(\tau_i - \tau_{i-1})^2)\right].$$

Using the strong Markov property at the first jump time τ_1 of $N(t)$ and letting $\mathcal{F}_1 = \sigma(\tau_1, X^1_{\tau_1}, X^2_{\tau_1})$, we see that

$$\begin{aligned}
h(t) &= E\left[1_{\{N(t)=0\}}\right] + E\left[1_{\{N(t)>0\}}\,(\alpha\,\tau_1^2)\,E\left[\prod_{i=2}^{N(t)}(\alpha(\tau_i-\tau_{i-1})^2)\middle|\mathcal{F}_1\right]\right]\\
&= e^{-t} + \alpha\int_0^t s^2\,h(t-s)\,e^{-s}\,ds\\
&= e^{-t} + \alpha\int_0^t (t-s)^2\,h(s)\,e^{-(t-s)}\,ds.
\end{aligned}$$

Letting $g(t) = e^t\,h(t)$, we see that

$$g(t) = 1 + \alpha\int_0^t (t-s)^2\,g(s)\,ds.$$

Therefore, $g(0) = 1$ and

$$g'(t) = 2\alpha\int_0^t (t-s)\,g(s)\,ds.$$

It follows that $g'(0) = 0$ and

$$g''(t) = 2\alpha\int_0^t g(s)\,ds.$$

Therefore, $g''(0) = 0$ and

$$g'''(t) = 2\alpha\,g(t).$$

The general solution of this ordinary differential equation is

$$\begin{aligned}
g(t) = c_1 e^{t(2\alpha)^{1/3}} &+ c_2 e^{-t(2\alpha)^{1/3}/2}\sin\left((2\alpha)^{1/3}\frac{\sqrt{3}}{2}\,t\right)\\
&+ c_3 e^{-t(2\alpha)^{1/3}/2}\cos\left((2\alpha)^{1/3}\frac{\sqrt{3}}{2}\,t\right),
\end{aligned}$$

and c_1, c_2 and C_3 are determined by the initial conditions $g(0) = 1$, $g'(0) = 0$ and $g''(0) = 0$. Therefore, (3.1) holds. $\square$

3.2 Higher moments

In this section, we obtain upper bounds on higher moments and, in particular, establish the following theorem.

Theorem 3.2. *Under Assumption B, there exists a universal constant $C < \infty$ such that for all $n \geq 2$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$,*

$$|E[u^n(t,x)]| \leq C(|v_0| + t|\tilde{v}_0|)^n \exp(C\alpha^{1/3} n^{4/3} t).$$

The main technical effort is contained in the following lemma, which uses the notation of Section 2.3.

Lemma 3.3. *There is a universal constant $C < \infty$ such that for all $n \geq 2$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$,*

$$e^{tn(n-1)/2} E_{(0,x),\dots,(0,x)} \left[\prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_t(\ell)} \left(\alpha^{\frac{1}{2}} (\tau_i^\ell - \tau_{i-1}^\ell) \right) \right] \leq C \exp(C\alpha^{1/3} n^{4/3} t).$$

Assuming this lemma for the moment, we prove Theorem 3.2.

Proof of Theorem 3.2. We use Theorem 2.3 with $x_1 = \cdots = x_n = x$. $S(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}^3) = \tau_i^\ell - \tau_{i-1}^\ell$. From (2.7), Assumption B and (2.5), we find that

$$|E[u^n(t,x)]| \leq (|v_0| + t|\tilde{v}_0|)^n \, e^{tn(n-1)/2} \, E_{(0,x),\dots,(0,x)} \left[\alpha^{N_t(\mathcal{P}_n)} \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_t(\ell)} (\tau_i^\ell - \tau_{i-1}^\ell) \right].$$

Use (2.6) to rewrite this as

$$(|v_0| + t|\tilde{v}_0|)^n e^{tn(n-1)/2} E_{(0,x),\dots,(0,x)} \left[Z_{n,t} \right],$$

where

$$Z_{n,t} = \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_t(\ell)} \left(\alpha^{\frac{1}{2}} (\tau_i^\ell - \tau_{i-1}^\ell) \right), \tag{3.2}$$

then apply Lemma 3.3 to conclude the proof of Theorem 3.2. $\square$

Proof of Lemma 3.3. First, we recall the arithmetic-geometric inequality, namely, for positive numbers $a_1, \ldots, a_k$,

$$\left(\prod_{\ell=1}^{k} a_\ell\right)^{1/k} \leq \frac{\sum_{\ell=1}^{k} a_\ell}{k}. \tag{3.3}$$

Let $Z_{n,t}$ be as in (3.2) and set

$$\nu_n = \binom{n}{2} = \frac{n(n-1)}{2}, \tag{3.4}$$

so that $N_t(\mathcal{P}_n)$ is a Poisson random variable with parameter $t\nu_n$. For $k \in \mathbb{N}$, given that $N_t(\mathcal{P}_n) = k$, the number of factors in the product that defines $Z_{n,t}$ is $2k$, by (2.6), so by (3.3),

$$\begin{aligned} E_{(0,x),\ldots,(0,x)}\left[Z_{n,t} \mid N_t(\mathcal{P}_n) = k\right] &\leq \left(\frac{1}{2k}\sum_{\ell\in\mathcal{L}_n}\sum_{i=1}^{N_t(\ell)}\left(\alpha^{\frac{1}{2}}(\tau_i^\ell - \tau_{i-1}^\ell)\right)\right)^{2k} \\ &\leq \left(\frac{\alpha^{\frac{1}{2}}nt}{2k}\right)^{2k}. \end{aligned}$$

Therefore,

$$\begin{aligned} E_{(0,x),\ldots,(0,x)}\left[Z_{n,t}1_{\{N_t(\mathcal{P}_n)=k\}}\right] &\leq \left(\frac{\alpha^{\frac{1}{2}}nt}{2k}\right)^{2k} P\{N_t(\mathcal{P}_n) = k\} \\ &= \left(\frac{\alpha^{\frac{1}{2}}nt}{2k}\right)^{2k} e^{-\nu_n t}\,\frac{(\nu_n t)^k}{k!}. \end{aligned}$$

Using Stirling's approximation $k! \simeq \sqrt{2\pi}k^k e^{-k}\sqrt{k}$, we get, for $k \geq k_0$, where

$k_0 > 1$ is a universal constant, that

$$\begin{aligned}
k!\,(2k)^{2k} &\geq \frac{\sqrt{2\pi}}{2}\, k^k e^{-k}\sqrt{k}\,(2k)^{2k} \\
&\geq \frac{\sqrt{2\pi}}{2}\, k^{3k} 2^{2k} e^{-k}\sqrt{k} \\
&= \frac{\sqrt{2\pi}}{2\sqrt{3}}\, \left(e^{2k}3^{-3k}2^{2k}\right)\left((3k)^{3k}e^{-3k}\sqrt{3k}\right) \\
&\geq c_0\left(\frac{e^{2k}2^{2k}}{3^{3k}}\right)\,(3k)! \\
&= c_0\zeta^{3k}\,(3k)!,
\end{aligned}$$

where $c_0 = \frac{1}{4\sqrt{3}}$ and ζ is a universal positive constant. It follows that for $k \geq k_0$,

$$E_{(0,x),\ldots,(0,x)}\left[Z_{n,t}1_{\{N_t(\mathcal{P}_n)=k\}}\right] \leq \frac{e^{-\nu_n t}}{c_0\,(3k)!}\left(\zeta^{-1}2^{-1/3}\alpha^{1/3}n^{4/3}t\right)^{3k}$$

and

$$\begin{aligned}
e^{tn(n-1)/2}E_{(0,x),\ldots,(0,x)}[Z_{n,t}] &\leq 1+\sum_{k=1}^{k_0-1}\frac{(\alpha^{1/3}n^{4/3}t)^{3k}}{k!\,(2k)^{2k}} \\
&\quad +\frac{1}{c_0}\sum_{k=0}^{\infty}\frac{1}{(3k)!}\left(\zeta^{-1}2^{-1/3}\alpha^{1/3}n^{4/3}t\right)^{3k} \\
&\leq C\exp\left(C\alpha^{1/3}n^{4/3}t\right),
\end{aligned}$$

provided the universal constant C is chosen large enough. This proves Lemma 3.3. $\square$

4 Lower bounds on the moments

In this section, we will work under the following assumption.

Assumption C. The covariance function f has the following property: there exist $\delta > 0$ and $\alpha_0 > 0$ such that for $\|x\| < 2\delta$, $f(x) \geq \alpha_0$.

Figure 1: A projection of the cone $C(x,y)$.

Theorem 4.1. *Under Assumptions A and C, there exists a universal constant $c > 0$ such that for all even $n \in \mathbb{N}$, $x \in \mathbb{R}^3$ and $t > 0$,*

$$E(u^n(t,x)) \geq (v_0 + t\tilde{v}_0)^n \exp(c\alpha_0^{1/3} n^{4/3} t).$$

Remark 4.2. Without Assumption C, the inequality $E[u^n(t,x)] \geq (v_0+t\tilde{v}_0)^n$ holds for all $t \geq 0$. Indeed, by (2.7),

$$E[u^n(t,x)] \geq e^{tn(n-1)/2} E_{(0,x),\dots,(0,x)}[1_{\{N_t(\mathcal{P}_n)=0\}}\,(v_0 + t\tilde{v}_0)^n] = (v_0 + t\tilde{v}_0)^n.$$

Proof of Theorem 4.1. Fix $x \in \mathbb{R}^3$. Given $y \in \mathbb{R}^3$, let $C(x,y)$ denote the solid cone with vertex at y whose axis passes through x and y and consisting of those $z \in \mathbb{R}^3$ such that $(y-z)\cdot(y-x) \geq \cos(\frac{\pi}{4})\|y-z\|\,\|y-x\|$. Let $\delta > 0$ be as in Assumption C. An elementary geometric argument (see Figure 1) shows that if $\|y-x\| \leq \delta$, $z \in C(x,y)$ and $\|y-z\| \leq \delta$, then $\|z-x\| \leq \delta$.

Let $t > 0$. Consider the event

$$D(t) = \bigcap_{\ell=1}^{n} \{X^{\ell}_{\tau_i^\ell} + \tilde{\Theta}^{\ell,(i)} \in C(x, X^{\ell}_{\tau_i^\ell}),\ i = 1,\dots,N_t(\ell)\}.$$

Informally, on the event $D(t)$ and under $P_{(0,x),\dots,(0,x)}$, each motion process $X^\ell_\cdot$ starts at x, moves away from x to $X^\ell_{\tau_1^\ell}$, but then "comes back in the general direction of x" repeatedly, since the variable $\Theta^{\ell,(i)}$ falls in the cone $C(x, X^\ell_{\tau_i^\ell})$. By the observation above, we note that if $\tau_{i+1}^\ell - \tau_i^\ell \leq \delta$, for $i = 1,\dots,N_t(\ell)$, and $\ell = 1,\dots,n$, then $\|X^{R_j^i}_{\sigma_i} - x\| \leq \delta$, $i = 1,\dots,N_t(\mathcal{P}_n)$, $j = 1,2$, and, in particular,

$$\|X^{R_1^i}_{\sigma_i} - X^{R_2^i}_{\sigma_i}\| \leq 2\delta, \qquad P_{(0,x),\dots,(0,x)}\text{-a.s.}. \tag{4.1}$$

Let $m = m(t) \in \mathbb{N}$, and set $k = \frac{m}{\delta}\frac{n}{2}$. Let $\ell = \frac{\delta t}{2(m+1)}$, and, for $j = 1,\dots,\frac{m}{\delta}$, let $t_j = \frac{jt\delta}{2(m+1)}$ and $I_j = [a_j, b_j]$, where $a_j = t_j - \ell/4$ and $b_j = t_j + \ell/4$, so that the length of I_j is $\frac{\ell}{2} = \frac{\delta t}{4(m+1)}$ and I_j and I_{j+1} are separated by an interval of length $a_{j+1} - b_j = \ell/2$.

Let

$$C(k,n,t) = \bigcap_{j=1}^{m/\delta} G_j(n,k),$$

where

$$G_j(n,k) = \left\{ N_{b_j}(\mathcal{P}_n) - N_{a_j}(\mathcal{P}_n) = \frac{n}{2} \right\} \cap \left\{ N_{b_j}(\ell) - N_{a_j}(\ell) = 1,\ \ell = 1,\dots,n \right\}.$$

Notice that on $C(k,n,\ell)$, $N_t(\mathcal{P}_n) = \frac{m}{\delta}\frac{n}{2} = k$, and during each time interval I_j, each process $X^\ell_\cdot$ changes direction exactly once. In particular, on $C(k,n,t)$,

$$\frac{\delta t}{4(m+1)} \leq \tau^\ell_{i+1} - \tau^\ell_i \leq \frac{\delta t}{m+1}, \qquad i = 0,\dots,m, \tag{4.2}$$

so $\tau^\ell_{i+1} - \tau^\ell_i \leq \delta$ if m is large enough.

Let ν_n be defined as in (3.4). Then, by the fact that $w(s,y) = v_0 + t\tilde{v}_0$ and the observation (4.1) above, for m (or k) large, Theorem 2.3 implies that

$$E(u^n(t,x)) \geq (v_0 + t\tilde{v}_0)^n\, e^{\nu_n t} E_{(0,x),\dots,(0,x)}\left[1_{D(t)} 1_{C(k,n,t)} \alpha_0^{N_t(\mathcal{P}_n)} \tilde{Z}_{n,t} \right], \tag{4.3}$$

where

$$\tilde{Z}_{n,t} = \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_t(\ell)} (\tau^\ell_i - \tau^\ell_{i-1}).$$

Let $\gamma = P\{y + \Theta_0 \in C(0,y)\} > 0$ (which does not depend on y). The right hand side above is bounded below by

$$\begin{aligned}
&e^{\nu_n t} \alpha_0^k \gamma^{2k} E[\tilde{Z}_{n,t} 1_{C(k,n,t)} \mid N_t(\mathcal{P}_n) = k] P\{N_t(\mathcal{P}_n) = k\} \\
&\quad = (v_0 + t\tilde{v}_0)^n \frac{(\alpha_0 \gamma^2 \nu_n t)^k}{k!} E[\tilde{Z}_{n,t} 1_{C(k,n,t)} \mid N_t(\mathcal{P}_n) = k].
\end{aligned}$$

By (4.2),

$$\begin{aligned}
E[\tilde{Z}_{n,t} 1_{C(k,n,t)} \mid N_t(\mathcal{P}_n) = k] \quad &\geq \quad \left(\frac{\delta t}{4(m+1)} \right)^{2k} P(C(k,n,t) \mid N_t(\mathcal{P}_n) = k) \\
&\geq \quad \left(\frac{ctn}{k} \right)^{2k} P(C(k,n,t) \mid N_t(\mathcal{P}_n) = k),
\end{aligned} \tag{4.4}$$

where $c = \frac{1}{8}$.

We now estimate the conditional probability $P(C(k,n,t) \mid N_t(\mathcal{P}_n))$. Given $N_t(\mathcal{P}_n) = k$, the jump times $(\sigma_1, \dots, \sigma_k)$ have the same distribution as the order statistics of a sequence of k uniform random variables with values in $[0,t]$, and the pairs $(R^1, \dots, R^k)$ form a uniform random vector with values in $(\mathcal{P}_n)^k$, which is independent of $(\sigma_1, \dots, \sigma_k)$. Therefore, the (mixed discrete/continuous) probability density function of the random vector

$$(\sigma_1, \dots, \sigma_k, R^1, \dots, R^k)$$

is

$$P\{\sigma_1 \in dx_1, \dots, \sigma_k \in dx_k,\ R^1 = \rho^1, \dots, R^k = \rho^k\} = \frac{k!}{t^k}\, dx_1 \cdots dx_k\, \frac{1}{\nu_n^k},$$

if $\rho^1, \dots, \rho^k \in \mathcal{P}_n$ and $x_1 < \cdots < x_k$, and equals 0 otherwise.

The event $G_1(n,k)$ occurs if and only if $a_1 \le \sigma_1 < \cdots < \sigma_{\frac{n}{2}} \le b_1$ and the $\frac{n}{2}$ pairs $R^1, \dots, R^{\frac{n}{2}}$ form an ordered partition of $\{1, \dots, n\}$. Notice that there are

$$\binom{n}{2, 2, \dots, 2} = \frac{n!}{2^{\frac{n}{2}}}$$

such partitions, and a similar characterisation holds for the other $G_j(n,k)$. Therefore,

$$\begin{aligned} P(C(k,n,t) \mid N_t(\mathcal{P}_n)) &= \frac{k!}{t^k \nu_n^k} \int_{a_1}^{b_1} dx_1 \int_{x_1}^{b_1} dx_2 \cdots \int_{x_{\frac{n}{2}-1}}^{b_1} dx_{\frac{n}{2}} \\ &\quad \cdots \int_{a_{\frac{m}{\delta}}}^{b_{\frac{m}{\delta}}} dx_{k-\frac{n}{2}+1} \int_{x_{k-\frac{n}{2}+1}}^{b_{\frac{m}{\delta}}} dx_{k-\frac{n}{2}+2} \cdots \int_{x_{k-1}}^{b_{\frac{m}{\delta}}} dx_k \binom{n}{2, 2, \dots, 2}^{\frac{m}{\delta}}. \end{aligned}$$

Each group of $\frac{n}{2}$ integrals is equal to the volume of a simplex in $\mathbb{R}^{\frac{n}{2}}$, which is $\frac{1}{(n/2)!}(\ell/2)^{\frac{n}{2}}$. Therefore,

$$P(C(k,n,t) \mid N_t(\mathcal{P}_n)) = \frac{k!}{t^k \nu_n^k} \left(\frac{1}{(n/2)!} \left(\frac{\ell}{2}\right)^{\frac{n}{2}} \right)^{\frac{m}{\delta}} \binom{n}{2, 2, \dots, 2}^{\frac{m}{\delta}}. \tag{4.5}$$

We note that $\frac{m}{\delta} = \frac{2k}{n}$, and

$$\frac{1}{(n/2)!} \binom{n}{2, 2, \dots, 2} = 2^{-n/2} \frac{n!}{(n/2)!}.$$

According to Stirling's approximation, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$n! \geq n^n e^{-n}\sqrt{n} \qquad \text{and} \qquad (n/2)! \leq 6(n/2)^{n/2}e^{-n/2}\sqrt{n/2}.$$

Let

$$\tilde{c} = \inf_{2\leq n\leq n_0} \frac{n!}{n^n e^{-n}\sqrt{n}}, \qquad \tilde{C} = \sup_{2\leq n\leq n_0} \frac{(n/2)!}{6(n/2)^{n/2}e^{-n/2}\sqrt{n/2}}.$$

Letting c denote the universal constant $c = \frac{1}{6} \wedge \frac{\tilde{c}}{\tilde{C}}$, we see that

$$\frac{1}{(n/2)!}\binom{n}{2,2,\ldots,2} \geq c2^{-n/2}\frac{n^n e^{-n}\sqrt{n}}{(n/2)^{n/2}e^{-n/2}\sqrt{n/2}} \geq \sqrt{2}ce^{-n/2}n^{n/2}.$$

Now observe from the definition of ℓ, ν_n and k that

$$\frac{1}{t^k\nu_n^k}\left(\frac{\ell}{2}\right)^{\frac{nm}{2\delta}} = \left(\frac{\delta t}{4(m+1)}\,\frac{2}{tn(n-1)}\right)^k \geq \left(\frac{n}{8kn^2}\right)^k = \left(\frac{1}{8kn}\right)^k.$$

Therefore, we see from (4.5) that

$$\begin{aligned} P(C(k,n,t) \mid N_t(\mathcal{P}_n)) &\geq k!\left(\sqrt{2}ce^{-n/2}n^{n/2}\right)^{m/\delta}\left(\frac{1}{8kn}\right)^k \\ &= k!\,(\sqrt{2}c)^{2k/n}e^{-k}n^k\left(\frac{1}{8kn}\right)^k, \end{aligned}$$

since $m/\delta = 2k/n$. Looking back at (4.3) and (4.4), we conclude that

$$\begin{aligned} E(u^n(t,x)) &\geq (v_0+t\tilde{v}_0)^n\frac{(\alpha_0\gamma^2\nu_n t)^k}{k!}\left(\frac{ctn}{k}\right)^{2k}k!\left(\sqrt{2}c\right)^{2k/n}e^{-k}n^k\left(\frac{1}{8kn}\right)^k \\ &= (v_0+t\tilde{v}_0)^n\left(\frac{\alpha_0\gamma^2n^3(n-1)t^3c^2\left(\sqrt{2}c\right)^{2/n}e^{-1}}{k^3}\right)^k. \end{aligned}$$

There is again a universal positive constant, which we denote again by c, such that

$$E(u^n(t,x)) \geq (v_0+t\tilde{v}_0)^n\left(\frac{\alpha_0\gamma^2c\,n^4t^3}{k^3}\right)^k.$$

Let

$$k = e^{-1/3}c^{1/3}\alpha_0^{1/3}\gamma^{2/3}n^{4/3}t,$$

to conclude that for t sufficiently large,

$$E(u^n(t,x)) \geq (v_0 + t\tilde{v}_0)^n \exp\left(e^{-1/3} c^{1/3} \alpha_0^{1/3} \gamma^{2/3} n^{4/3} t\right).$$

This concludes the proof. $\square$

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