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Markov Processes and **Related Fields**

Small Noise Asymptotics of Traveling Waves

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Abstract. We consider the randomly perturbed Fisher – KPP equation

$$\partial_t u = \partial_x^2 u + u(1-u) + \varepsilon\sqrt{u(1-u)}\dot{W}.$$

where $\dot{W} = \dot{W}(t,x)$ is a space-time white noise. We discuss our proof of the Brunet – Derrida conjecture that the speed of traveling fronts for small ε is

$$2 - \pi^2 |\log \varepsilon^2|^{-2}$$

with an error of order $(\log|\log\varepsilon|)|\log\varepsilon|^{-3}$.

Keywords: heat equation, white noise, stochastic partial differential equations

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1. The stochastic Fisher – KPP equation

Traveling waves are some of the most widely studied phenomena in the mathematical sciences. One of the simplest examples involves the Fisher equation, also called the Kolmogorov – Petrovskii – Piscounov (KPP) equation. A function $u(t,x) : t \geq 0, x \in \mathbf{R}$ satisfies the KPP equation if

$$\begin{aligned} &\frac{\partial u}{\partial t} = \partial_x^2 u + u(1-u), \\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

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and we often take the initial data $u_0(x)$ to be the Heaviside function

$$\mathbf{1}_{(-\infty,0]}(x). \tag{1.2}$$

A traveling wave, or front, for (1.1) is a solution $u(t,x) = h(x - vt)$ which keeps a fixed shape h and moves at velocity v. For (1.1), there is a family of traveling waves with different velocities, but there is a distinguished traveling wave h_0 such that the Heaviside initial data $u_0(x) = \mathbf{1}_{(-\infty,0]}(x)$ gives a solution $u(t,x)$ with

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} \big|u(t,x) - h_0(x - m(t))\big| = 0$$

and $m(t)/t \to 2$ as $t \to \infty$. In other words, the solution converges in shape to the traveling wave with shape h_0 and speed $\mathrm{v}_0 = 2$.

Our goal is to study how traveling wave speeds change under the influence of random noise. The main result deals with randomly perturbed KPP equations,

$$\partial_t u = \partial_x^2 u + u(1-u) + \varepsilon\sqrt{u(1-u)}\dot{W}, \qquad t \geq 0, \quad x \in \mathbf{R}, \quad u \geq 0. \tag{1.3}$$

The two parameter white noise $\dot{W}$ is defined by specifying, for square integrable deterministic functions $\varphi(s,y)$, that $W(\varphi) := \iint_0^\infty \varphi(s,y)\, W(ds,dy)$ are a Gaussian family with mean zero and covariance

$$E[W(\varphi)W(\psi)] = \iint_0^\infty \varphi(s,y)\psi(s,y)\, ds\, dy. \tag{1.4}$$

See [20] and [16] for background.

The main justification for the form of the noise in (1.3) is that this is the way it arises in approximations by microscopic particle models. For example, (1.3) appears as an appropriate limit of the long range voter model [13].

When $\varepsilon > 0$ with Heaviside initial data, one has non-negative, continuous solutions, with a finite upper bound on the support

$$r(t) = \sup\{x \in \mathbf{R} : u(t,x) > 0\}. \tag{1.5}$$

In this case, there is also a finite lower bound on the support of $1 - u$, but it is not so relevant for our discussion. The process viewed from $r(t)$,

$$\tilde{u}(t,x) = u(t, x + r(t)) \tag{1.6}$$

should have a unique nondegenerate stationary solution, and this is the *random traveling wave*. One expects $t^{-1}r(t)$ to have a nonrandom limit,

$$\mathrm{v}_\varepsilon = \lim_{t\to\infty} t^{-1} r(t), \tag{1.7}$$

which was proved in [12] for (1.3).

In this work we are primarily concerned with the asymptotics of v_ε as $\varepsilon \to 0$ in (1.3). Taking the expectation of (1.3) gives

$$\partial_t E[u] = \partial_x^2 E[u] + E[u] - E[u^2]. \tag{1.8}$$

By Jensen's inequality, $E[u^2] \geq E[u]^2$ and hence $E[u]$ is a subsolution of (1.1). From this it is not hard to convince oneself that if v_ε exists we must have

$$\mathrm{v}_\varepsilon \leq \mathrm{v}_0. \tag{1.9}$$

Brunet and Derrida [2] and [3] (see also [10, 17]) conjectured that as $\varepsilon \to 0$,

$$\mathrm{v}_0 - \mathrm{v}_\varepsilon \simeq \frac{\pi^2}{|\log \varepsilon^2|^2}. \tag{1.10}$$

The effect is remarkable; for example, a naive Taylor expansion might suggest, since symmetry implies $\varepsilon = 0$ is a local maximum, that $\mathrm{v}_0 - \mathrm{v}_\varepsilon \simeq \mathcal{O}(\varepsilon^2)$. In fact, it is enormously larger. The phenomenon was not expected. It was first observed in computer simulations of particle systems modeled by (1.3). It was quickly understood at the physical level as a consequence of the pulled nature of the fronts. Recall that in an evolution equation with traveling fronts between an unstable and a stable state, the front is said to be *pulled* if its asymptotic speed is the same as that in the linearization of the equation about the unstable state, and *pushed* if the speed is larger than that of the linearization (see [19]). The KPP equation has a pulled front. Because in pulled fronts the front speed is determined in the region where the density u is very small, fluctuations can have a dramatic effect on the front speed.

The asymptotics (1.10) is based on a comparison with the cutoff KPP equation

$$\partial_t u = \partial_x^2 u + u(1-u)\,\mathbf{1}(u \geq \varepsilon^2). \tag{1.11}$$

The idea is that when $u < \varepsilon^2$, $u(1-u) < \varepsilon\sqrt{u(1-u)}$ and the noise term wins. Brunet and Derrida [2] used matched asymptotics to show that the traveling front velocity in (1.11) is

$$\mathrm{v}_{\mathrm{cutoff}} \simeq \mathrm{v}_0 - \frac{\pi^2}{|\log \varepsilon^2|^2}. \tag{1.12}$$

Panja [14, 15] gives a very comprehensive review of the physical aspects of the Brunet – Derrida theory.

The asymptotics (1.12) for the wave speed in (1.11) are made rigorous in [7] and [1]. It is considerably more difficult to justify the heuristic connection between (1.3) and (1.11). The only previous work is by Conlon and Doering [5] who proved that for sufficiently small ε,

$$\mathrm{v}_\varepsilon \geq \mathrm{v}_0 - \frac{C \log|\log \varepsilon|}{|\log \varepsilon|^2}, \tag{1.13}$$

by coupling (1.3) to a contact process (see Liggett [11]).

Our main result is a proof of both the upper bound and a lower bound for the Brunet – Derrida conjecture.

Theorem 1.1. *Let $u(t,x)$ be the solution of* (1.3) *with initial data* (1.2) *and* v_ε *be as in* (1.7). *There exists $C<\infty$ and ε_0 such that for $\varepsilon\le\varepsilon_0$,*

$$\Big|\mathrm{v}_\varepsilon-\mathrm{v}_0+\frac{\pi^2}{|\log\varepsilon^2|^2}\Big|\le C\frac{\log|\log\varepsilon|}{|\log\varepsilon^2|^3}. \tag{1.14}$$

Remark 1.1. Our proof works for general KPP type equations

$$\partial_t u=\partial_x^2 u+f(u)+\varepsilon\sigma(u)\dot W \tag{1.15}$$

where f and σ are Lipschitz with

$$\begin{aligned} &f(0)=f(1)=0;\qquad 0<f(u)\le uf'(0),\quad u\in(0,1),\\ &\sigma^2(u)\le u. \end{aligned} \tag{1.16}$$

We must also assume that

$$f(u)\le 2-u,\quad \text{for } u\ge 1,$$

and there exist constants $\gamma>0$ and $0<u^*<1$ such that

$$\frac{\sigma^2(u_2)-\sigma^2(u_1)}{u_2-u_1}\ge\gamma,\quad \text{for } 0\le u_1\le u_2\le u^*.$$

The key assumption is that *weak uniqueness* holds for the equation. This is not known in general, but it is known for (1.3) using duality, and also using duality for the case

$$\partial_t u=\partial_x^2 u+u(1-u)+\varepsilon\sqrt{u}\dot W, \tag{1.17}$$

which appears as the limit of the long range contact process [13]. Tribe [18] studies traveling waves in (1.17) but it is left open whether the asymptotic wave speed is unique and deterministic.

Remark 1.2. We have assumed that the initial data $u_0(x)$ is a Heaviside function. But it can be much more general. All one needs is $\int_0^\infty u_0(x)\,dx<\infty$ and enough mass to the left of the origin; for example, $u_0(x)\ge\theta>0$ for $x<0$ suffices. The speed is always given by (1.14), in sharp contrast to the deterministic case (1.1) where $u_0(x)\simeq\exp(-\gamma x)$, $x\to\infty$ produces a wave speed $\mathrm{v}=\gamma+\gamma^{-1}$ when $\gamma\in(0,1]$, a fact that irked Fisher (see the original [8]).

Remark 1.3. Recently, Brunet et. al [4] have conjectured, based on a phenomenological argument, that

$$\mathrm{v}_\varepsilon-\mathrm{v}_0+\frac{\pi^2}{|\log\varepsilon^2|^2}\simeq\frac{3\pi^2\log|\log\varepsilon|}{|\log\varepsilon^2|^3}. \tag{1.18}$$

Note that the error term in Theorem 1.1 is of the same order.

2. Outline of the proof

The general idea behind our proof of Theorem 1.1 is to compare the stochastic KPP evolution (1.3) to a deterministic traveling wave whose speed we can control. Since a lower bound (1.13) of some form was already obtained by Conlon and Doering, we concentrate here on the new ideas for the upper bound.

We use the traveling wave which solves (1.3), but without the noise and with an extra Dirichlet condition at a point which moves forward at a fixed speed. Let $v(t,x)$ satisfy

$$\begin{cases} \partial_t v = \partial_x^2 v + v(1-v), & x < \kappa t, \\ v(t,x) = 0, & x \geq \kappa t. \end{cases} \tag{2.1}$$

We search for the κ for which there exists a traveling front solution

$$v(t,x) = F(x - \kappa t) \tag{2.2}$$

with $\lim_{x\to-\infty} F(x) = 1$ and

$$F'(0) = -\nu. \tag{2.3}$$

If $\nu = \varepsilon^2$, the solution will have a mass of $\mathcal{O}(\varepsilon^2)$ within a distance $\mathcal{O}(1)$ of $x = \kappa t$. This is the critical mass which can survive in the stochastic equation when u is small and hence this provides a consistent strategy for a stochastic traveling front in (1.3) to propagate.

To determine the resulting κ, let $\mathbf{x}(t) = F(-t)$ and note that the problem is equivalent to that of finding the $\kappa = \kappa(\nu)$ such that the solution of the ordinary differential equation

$$\mathbf{x}'' = \kappa \mathbf{x}' - \mathbf{x}(1-\mathbf{x}), \qquad \mathbf{x}(0) = 0, \quad \mathbf{x}'(0) = \nu \tag{2.4}$$

has $\mathbf{x}(\infty) = 1$. Through phase plane analysis of (2.4) it is not too hard to derive the following relationship between κ and ν.

Proposition 2.1. *Let $\kappa(\nu)$ be the solution of (2.1)–(2.3). There exists $C_0 < \infty$ and $\nu_0 > 0$ such that for $\nu \leq \nu_0$,*

$$\left|\kappa(\nu) - 2 + \pi^2 |\log \nu|^{-2}\right| \leq C_0 \log|\log\nu| \, |\log\nu|^{-3}. \tag{2.5}$$

Here is a brief outline of how we implement our comparison scheme. The reader should be suspicious here, because the speed of F is arbitrary, so in principle we could slow down F and obtain an even better upper bound. We will explain later why this is not possible.

In our comparison argument, we work with time intervals of length T and space intervals of length L. Here $T = T(\varepsilon)$ and $L = L(\varepsilon)$ are large, chosen so that $L, T \to \infty$ as $\varepsilon \to 0$.

Now we describe the first time stage. Let F be as in (2.2), and let $u(t,x)$ satisfy (1.3) with $u(0,x) = F(x)$. Let τ be the first time $t \le T$ such that

$$u(t,x) \ge F(x - \kappa t - L);$$

if there is no such time, let $\tau = T$. In other words, we start the traveling wave $F(x - \kappa t - L)$ shifted L to the right of $u(t,x)$, and stop the time stage when u overtakes F, or at time T if u does not overtake F. In either case, we then start a new solution $u_1(\tau + t, x)$ with

$$u_1(\tau, x) = F(x - \kappa\tau - L).$$

In other words, we start u_1 at the position of the old comparison function. For the second time interval, we compare $u_1(t,x)$ with $F(x - \kappa t - 2L)$, which is the old comparison function shifted $2L$ to the right. We repeat this process over and over again.

If the solution u always stayed to the left of F, then we would have an upper bound on the speed of u given by the speed of F, namely κ, plus the effect of moving F a distance L units to the right at the end of each time stage. Therefore, an upper bound on the speed would be $\kappa + L/T$, which is very close to κ if T is large.

We also need to consider the possibility that u overtakes F before the end of the time interval, that is, that $\tau < T$. If this event occurs, we must start over again and move F an extra unit to the right. To compute the effect of this event on the speed, we reason as follows. A time interval is of length $E\tau$ on average. The distance moved over that time interval is $\kappa E\tau$ plus L for the readjustment which occurs at the end of the interval. To get the average speed, we must divide this quantity by the average length of the time interval, namely $E\tau$. Dividing, we obtain the following upper bound for the speed:

$$\frac{\kappa E\tau + L}{E\tau} = \kappa + \frac{L}{E\tau}.$$

Thus, our upper bound on the speed will be about κ provided $L/E\tau$ is small. It is sufficient to assume that L/T is small and $P(\tau < T)$ is bounded away from 1.

Now we indicate how to control $P(\tau < T)$. Roughly speaking, there are two ways in which u could overtake F.

1. u could rise high enough to bump into F away from the boundary $x = \kappa t$.
2. Since u does not satisfy a Dirichlet condition but F does, u could spill over beyond the point where $F(x - \kappa t + L) = 0$.

1. The probability of u rising higher than average can be controlled using large deviation theory. In particular, one can show that for $M, \delta > 0$ there are

constants $C_1, C_2 > 0$ such that for all $\lambda > 0$

$$P\Big(\sup_{0\le t\le T} \sup_{\kappa t - M \le x \le \kappa t} |u(t,x) - F(x-\kappa t)| > \lambda \Big) \le C_1 \exp\Big(- \frac{C_2 \lambda^2}{T^{1/2-\delta}} \Big).$$

Since $F(x - \kappa t + L) \ge F(x - \kappa t)$, the above bound is enough to control the probability of bumping F away from the boundary $x = \kappa t + L$.

2. The probability that $u(t,x)$ spills over the moving boundary $x = \kappa t + L$ can be controlled using ideas from the theory of superprocesses. We give a brief introduction; the reader can consult [6] for more details.

For simplicity, we consider the case (1.17). Note that it looks different than (1.3). However the behavior of both solutions near the points where u is small is similar; and only this behavior is relevant for controlling the mass that may spill over the boundary $x = \kappa t + L$. Except for the drift $u(1-u)$, (1.17) resembles

$$\partial_t v = \partial_x^2 v + \sqrt{v}\dot{W}, \tag{2.6}$$

which is the stochastic partial differential equation for the density of super Brownian motion or Dawson – Watanabe process. Consider the measure $X_t(dx) = v(t,x)\,dx$. This measure represents the limit of critical branching Brownian motions, as follows. Fix a parameter $m > 0$, and for $t > 0$ let $\{B_i(t)\}_{i=1}^{N(t)}$ be independent Brownian motions, and let $N(t)$ represent the number of Brownian particles present at time t. At each time $k/m : k \ge 1$, each Brownian particle splits in two or dies with probability $1/2$ each, independently. We consider the normalized empirical measure for this system of particles:

$$X_t^{(m)}(dx) = \frac{1}{m} \sum_{i=1}^{N(t)} \delta_{B_i(t)}(dx)$$

where δ_x is the delta measure centered at x. Let $\mu_m = X_0^{(m)}$, and suppose that as $m \to \infty$, μ_m converges weakly to $\mu(dx) = v(0,x)dx$. Then the main existence theorem for superprocesses asserts that in the appropriate topology, $X_t^{(m)}$ converges in distribution to a measure-valued process X_t with initial value $X_0 = \mu$. Furthermore, in one dimension, the measure $X_t(dx)$ has a density $v(t,x)$ with respect to Lebesgue measure, and $v(t,x)$ satisfies (2.6) for some white noise $\dot{W}$.

Since we can think of superprocesses as systems of infinitesimal branching Brownian motions which do not interact, there are many powerful tools available to study them. Although (1.17) is different than (2.6), we can adapt certain tools from superprocesses to our situation. In particular, we use the following two ideas.

1. We can split solutions of (2.6) and (1.17) as $u(t,x) = u_1(t,x) + u_2(t,x)$. In the superprocess case (2.6), u_1, u_2 would not interact. This will no longer

be true for (1.17), but the interactions can be controlled reasonably well. We can also generalize this argument to deal with other coefficients of the noise.

2. The total mass of a superprocess represents the limiting size of a critical branching process. It is well known that if the initial mass is small, the branching process will die quickly, with high probability. Indeed, the total mass $M(t) = X_t(\mathbf{R})$ of the superprocess satisfies the following stochastic differential equation
$$dM = \sqrt{M}\, dB. \tag{2.7}$$
This process is called Feller's branching diffusion.

Using these ideas, we now explain how to control the mass from u which spills over beyond the point $x = \kappa t + L$, which represents the point at which we impose the Dirichlet condition on the traveling wave $F(x - \kappa t + L)$. Viewing $u(t, x)$ as a collection of infinitesimal particles, we first ask which particles are most likely to move to the right of the curve $x = \kappa t + L$ in our time interval $[0, T]$. It seems reasonable to look at the particles closest to the boundary, namely those within a fixed distance K from $x = 0$ at time $t = 0$. Indeed, these are the rightmost particles of $u(0, x)$, so we decompose $u(t, x) = u_1(t, x) + u_2(t, x)$, where $u_1(t, x)$ represents the particles descended from the mass $u_1(0, x) = u(0, x)\mathbf{1}(x \geq -K)$. Let $M(t)$ denote the mass of $u_1(t, x)$, namely
$$M(t) = \int_{\mathrm{R}} u_1(t, x)\, dx.$$
From what we have said, $M(t)$ resembles Feller's branching diffusion, so it tends to die quickly, provided the initial mass is small. However, the noise term in (1.17) is multiplied by ε, so the branching is slower than in (2.7). Thus $M(t)$ will die quickly only if the initial mass $M(0)$ is especially small. Here is where we use the relationship (2.5) between the slope $\nu = \varepsilon^2$ of $u(0, x) = F(x)$ at $x = 0$. The mass $M(0)$ is bounded by $CK\nu = CK\varepsilon^2$ for some constant C close to $1/2$. This bound is good enough to ensure that with high probability $u_1(t, x)$ dies quickly, i.e., $u_1(t, \cdot)$ is identically 0 after a short time. Thus, with high probability we can ignore the mass that spills over the boundary $x = \kappa t + L$. Note that a similar decomposition argument allows us to control the mass from u that spills over beyond the point $x = \kappa t + L$ works also for the equation (1.3).

Earlier we warned the reader to be suspicious about our choice of the speed κ of the traveling wave $F(x - \kappa t)$, since it seemed that κ could take any desired value. Now we see that κ is restricted by the requirement that the slope ν of $F(x)$ at $x = 0$ should be small, so that the initial mass $M(0)$ is small, and $u_1(t, x)$ dies out quickly with high probability.

We have now explained why the probability $P(\tau < T)$ is small, and our heuristic description of the proof of the upper bound in Theorem 1.1 is complete.

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