HITTING PROBABILITIES OF A BROWNIAN FLOW WITH RADIAL DRIFT

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ABSTRACT. We consider a stochastic flow $\phi_t(x, \omega)$ in $\mathbb{R}^n$ with initial point $\phi_0(x, \omega) = x$, driven by a single n-dimensional Brownian motion, and with an outward radial drift of magnitude $\frac{F(\|\phi_t(x)\|)}{\|\phi_t(x)\|}$, with F nonnegative, bounded and Lipschitz. We consider initial points x lying in a set of positive distance from the origin. We show that there exist constants $C^*, c^* > 0$ not depending on n, such that if $F > C^* n$ then the image of the initial set under the flow has probability 0 of hitting the origin. If $0 \le F \le c^* n^{3/4}$, and if the initial set has nonempty interior, then the image of the set has positive probability of hitting the origin.

1. INTRODUCTION AND MAIN RESULTS

In this paper we study a hitting problem of stochastic flows with radial drift. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $x \in \mathbb{R}^n \setminus \{0\}$ we define the unit radial vector in the x direction as

$$\mathbf{u}(x) = \frac{x}{\|x\|}$$

and we define

$$\mathbf{u}(0) = 0.$$

Here $\| \cdot \|$ denotes the Euclidean norm.

We consider a Brownian flow $\phi = \{\phi_t(x, \omega); t \ge 0, x \in \mathbb{R}^n, \omega \in \Omega\}$ starting from a set $\mathcal{A}_0 \subset \mathbb{R}^n$ such that

$$(1.1) \qquad d\phi_t(x) = \frac{F(\|\phi_t(x)\|)}{\|\phi_t(x)\|} \mathbf{u}(\phi_t(x)) dt + dW_t \quad \text{for } t \ge 0,$$

$$\phi_0(x) = x \in \mathcal{A}_0.$$

Here, W_t is a standard n-dimensional Brownian motion, not depending on x, and $F(\cdot)$ is a nonnegative, bounded Lipschitz function. Throughout the paper the differential d is taken with respect to t. We also set up the standard Brownian filtration

$$\mathcal{F}_t = \sigma(W_s : s \leq t).$$

Note that the flow might not be defined after ϕ reaches the origin, due to the singular drift there. In fact, we will use a stopping time which is intuitively defined as

(1.2) $$\tau = \tau_{\mathcal{A}_0} = \inf\{t : \phi_t(x) = 0 \text{ for some } x \in \mathcal{A}_0\}.$$

For additional details we refer the reader to Section 2.

We are interested in the question of whether the flow can hit the origin with positive probability, in the case where the initial set $\mathcal{A}_0$ is at positive distance from the origin. We say that ϕ hits 0 if $\tau < \infty$, and define the corresponding event

$$G_{\mathcal{A}_0,F,n} := \{\tau_{\mathcal{A}_0} < \infty\}.$$

Before stating our main results, we give some background on related hitting problems. It is well known that for a single Brownian motion W_t in $\mathbb{R}^n$, the radial distance $D_t = \|W_t\|$ is a Bessel process which satisfies the following SDE,

(1.3) $$dD_t = d\tilde{W}_t + \frac{n-1}{2D_t}dt,$$

where $\tilde{W}_t$ is a one-dimensional Brownian motion. Assuming $D_0 > 0$, it is a familiar fact that D_t can hit 0 with positive probability iff $n < 2$, where fractional values of n are allowed in (1.3). This question and many more can be answered using ideas from potential theory and harmonic functions; see [11] or most other books in Markov processes. In one dimension and with $F(y) = (n-1)/2$, we see that (1.1) is identical to the Bessel equation (1.3).

There is an intimate connection between Bessel processes and their associated flows, to stochastic Loewner evolution (SLE), also called Schramm-Loewner evolution (see Lawler [10] for some basic facts). Indeed, on page x of the preface of [10], Lawler states "With the Brownian input, the Loewner equation becomes an equation of Bessel type, and much of the analysis of SLE comes from studying such stochastic differential equations. For example, the different "phases" of SLE (simple/non-simple/space-filling) are deduced from properties of the Bessel equation". In his St. Flour notes [16], Werner states on page 131: "Then, we see that SLE_κ can be interpreted in terms of the flow

of a complex Bessel process". For a further explanation of this point of view, see Katori [8].

Here are a few details. SLE is usually thought of as a flow in a subset of the complex plane, with random parameters. The equation for chordal SLE $g_t(x)$, which takes values in the upper complex half-plane, is

$$\partial_t g_t(x) = \frac{2}{g_t(x) - \kappa W(t)},$$

where $W(t)$ is a one-dimensional Brownian motion and $\kappa > 0$ is a parameter. Setting $g_t(x) = \kappa h_t(x) + \kappa W(t)$, we find

$$dh_t(x) = \frac{2/\kappa^2}{h_t(x)} dt - dW(t),$$

which is an equation of Bessel type.

Since the Bessel process and its associated flow play such an important role in SLE, we feel that it is of interest to study Bessel flows in higher dimensions, and similar flows such as in (1.1). Perhaps the most basic property of the Bessel process is its probability of hitting the origin, and this is the question we investigate in this paper.

Although such hitting questions are classical and have been completely answered for a large class of Markov processes, the reader may be surprised to learn that for processes such as $\phi_t(\cdot)$ which take values in a function space, the potential theory is more difficult or even intractable. In such cases we must fall back on more basic ideas, such as covering arguments and comparison with a random walk. As usual, the critical case is the most difficult. The critical case is where the parameters of the process are at or near the boundary between hitting and not hitting.

Stochastic partial differential equations (SPDE) provide a source of such examples, and hitting questions for such equations have been studied in [3, 4, 5, 14, 15] among others. These papers deal with the stochastic heat and wave equations either with no drift or with well behaved drift. Since (1.1) has singular drift, the following result might be more relevant to our situation. Suppose that x lies in the unit circle $[0, 1]$ with endpoints identified, and that $u(t, x)$ satisfies

(1.4) $$\partial_t u = \partial_x^2 u + u^{-\alpha} + \dot{W},$$

where $\dot{W} = \dot{W}(t, x)$ is two-parameter white noise. Assume that $u(0, x)$ is continuous and strictly greater than 0. Then u hits 0 with positive probability if $\alpha < 3$, and hits 0 with probability 0 if $\alpha > 3$. We say that u hits 0 if there is a point (t, x) such that $u(t, x) = 0$. See [12] and [13] for details. A natural question about (1.4) is whether white noise $\dot{W}$

could be replaced by colored noise. We can regard (1.1) as a degenerate SPDE which does not have the Laplacian, and where the colored noise is independent of x, so the noise is at the opposite extreme from white noise.

Returning to our stochastic flow ϕ_t, we see that for a fixed point $x_0 \in \mathbb{R}^n$ and for $n \geq 2$ there is no chance that $\phi_t(x_0)$ can hit 0. In that situation, even with zero drift $F \equiv 0$, as already mentioned, $\|\phi_t(x_0)\|$ is a Bessel process with radial drift $(n-1)/(2\|\phi\|)$ and $n \geq 2$. Adding extra radial drift with $F \geq 0$ makes it even less likely for $\phi_t(x_0)$ to hit the origin. However, if we allow $x \in \mathcal{A}_0$ to vary, then we may have several chances for $\phi_t(x)$ to hit 0. Intuitively, the Brownian motion W_t which drives our flow has n independent components, so we may expect that ϕ has n independent chances to get closer to 0, and so the critical drift should be proportional to n. Our first guess might also be that the critical drift is proportional to $1/\|\phi\|$, as for the Bessel process. Here, the critical drift is that drift which lies on the boundary between a positive probability of hitting 0 and a zero probability of hitting 0. Unfortunately, there is no comparison principle for $\|\phi_t(x)\|$, so we cannot be sure that increasing the radial drift increases $\|\phi\|$. Thus we cannot conclude that larger radial drift leads to a smaller probability of ϕ hitting 0, and so we cannot give precise meaning to the term "critical drift".

By introducing new processes $\psi_t(x) = \phi_t(x) - W_t$ and $B_t = -W_t$, we may rewrite the problem in terms of ψ_t and B_t:

$$(1.5) \quad d\psi_t(x) = \frac{F(\|\psi_t(x) - B_t\|)}{\|\psi_t(x) - B_t\|} \mathbf{u}(\psi_t(x) - B_t)dt, \quad \text{for } 0 \leq t < \tau,$$

$$\psi_0(x) = x \in \mathcal{A}_0.$$

Note that B_t is a standard n-dimensional Brownian motion. We denote

$$\mathcal{A}_t = \{\psi_t(x) : \ x \in \mathcal{A}_0\},$$

which for any $t < \tau$ is a subset of $\mathbb{R}^n$.

Now we are ready to state our main results. Recall that $F : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative Lipschitz function. In our first theorem, we prove that for F bounded below by a large enough constant, the Brownian motion B_t does not hit the region $\mathcal{A}_t$. Moreover, the distance between $\mathcal{A}_t$ and B_t tends to ∞ as $t \to \infty$.

Theorem 1. *There exists $C^* > 0$ not depending on n such that for all $n \geq 1$ the following holds. Suppose the set $\mathcal{A}_0 \subset \mathbb{R}^n$ has a positive distance from the origin. If $F(x) \geq C^*n$ for all $x \in \mathbb{R}^n$, then we have*

$$P(G_{\mathcal{A}_0, F, n}) = 0.$$

In the next two theorems we prove that when F is bounded from above by a small enough constant, the Brownian motion B_t hits the region $\mathcal{A}_t$ with positive probability. Hitting the region with probability one requires that the region be rather large, or else the transience of Brownian motion in high dimensions works against hitting.

Theorem 2. *There exists $c^* > 0$ not depending on n such that for all $n \geq 1$ the following holds. Suppose*

$$\mathcal{A}_0 = \{(x_1, ..., x_n) : x_1 \geq 1 \ and \ x_2, ... x_n \in \mathbb{R}\}.$$

If $0 \leq F(x) \leq c^* n^{3/4}$ for all $x \in \mathbb{R}^n$, then

$$P(G_{\mathcal{A}_0, F, n}) = 1.$$

If we are willing to accept a positive probability of $\{\phi_t(x) : x \in \mathcal{A}_0\}$ hitting 0, we can extend Theorem 2 to a broad class of initial sets $\mathcal{A}_0$. If we make the set $\mathcal{A}_0$ smaller, then the set of ω for which $\{\phi_t(x) : x \in \mathcal{A}_0\}$ hits 0 will also become smaller. Theorem 3 shows that if $\mathcal{A}_0$ contains an open ball, then the probability of the above set hitting 0 is still positive.

Theorem 3. *There exists $c^* > 0$ not depending on n such that for all $n \geq 1$ the following holds. Suppose $\mathcal{A}_0 \subset \mathbb{R}^n$ has nonempty interior. If $0 \leq F(x) \leq c^* n^{3/4}$ for all $x \in \mathbb{R}^n$, then*

$$P(G_{\mathcal{A}_0, F, n}) > 0.$$

Roughly speaking, our theorems say that the critical drift (if there is one) for (1.1) has magnitude between $c^* n^{\frac{3}{4}}/\|\phi\|$ and $C^* n/\|\phi\|$.

We also note that in the case $n = 1$, our question reduces to finding the critical parameter for a Bessel process to hit the origin. We leave the details of this case to the reader.

2. Precise setup of the flow

A good reference for the general theory of stochastic flows is the book of Kunita [9]. However, since our flow is driven by a single Brownian motion, we can make use of properties of Brownian motion to simplify our setup. We also need to keep in mind that the flow should only exist up to time τ which was intuitively defined as the first time t that $\phi_t(x) = 0$ for some $x \in \mathcal{A}_0$.

Our strategy is to truncate the drift and take the limit as the truncation level tends to infinity. Let

$$(2.1) \qquad f_N(x) = \frac{F(\|x\|)}{\|x\| \vee (1/N)} \mathbf{u}(x), \qquad f_\infty(x) = \frac{F(\|x\|)}{\|x\|} \mathbf{u}(x),$$

where $\vee$ denotes the maximum. Observe that f_N is a globally Lipschitz function.

We start by defining the translated flow (1.5), with respect to the truncated drift f_N. Let $\psi_t^{(N)}(x)$ satisfy

(2.2) $\qquad d\psi_t^{(N)}(x) = f_N\big(\psi_t^{(N)}(x) - B_t\big)dt, \quad \text{for } t \geq 0,$

$$\psi_0^{(N)}(x) = x \in \mathbb{R}^n$$

with $B_t = -W_t$. Then (2.2) is an ordinary differential equation and $f_N\big(\psi_t^{(N)}(x) - B_t\big)$ is a (random) Lipschitz function of $\psi_t^{(N)}(x)$. It follows that (2.2) has a unique solution which is $\mathcal{F}_t$-adapted, where we recall that $\mathcal{F}_t$ is the Brownian filtration.

Given a set $\mathcal{A} \subset \mathbb{R}^n$, let

$$\tau^{(N)}(\mathcal{A}) = \inf\left\{t \geq 0 : \inf_{x \in \mathcal{A}} |\psi_t^{(N)}(x) - B_t| \leq 1/N\right\}$$

and let $\tau^{(N)}(\mathcal{A}) = \infty$ if the above set is empty. It follows that for $x \in \mathcal{A}$ and $0 \leq t \leq \tau^{(N)}(\mathcal{A})$, we have

(2.3) $\qquad d\psi_t^{(N)}(x) = f_N\big(\psi_t^{(N)}(x) - B_t\big)dt$

$$\psi_0^{(N)}(x) = x \in \mathbb{R}^n$$

and that $f_N(\psi_t^{(N)}(x)) = f_\infty(\psi_t^{(N)}(x))$ for $0 \leq t \leq \tau^{(N)}(\mathcal{A})$, almost surely. Again almost surely, we have $0 \leq \tau^{(1)}(\mathcal{A}) \leq \tau^{(2)}(\mathcal{A}) \leq \cdots$. Now we give the rigorous definition of $\tau(\mathcal{A})$, namely

$$\tau(\mathcal{A}) = \lim_{N \to \infty} \tau^{(N)}(\mathcal{A}).$$

Let $\tau(\mathcal{A}) = \infty$ on the exceptional set where $\tau^{(N)}(\mathcal{A})$ is not nondecreasing in N. We note that $\tau(\mathcal{A}) = \infty$ is possible even off of this exceptional set. Now we can define $\psi^{\mathcal{A}}$ as follows for $0 \leq t < \tau(\mathcal{A})$ and $x \in \mathcal{A}$

$$\psi_t^{\mathcal{A}}(x) = \lim_{N \to \infty} \psi_t^{(N)}(x).$$

We see that for $0 \leq t < \tau(\mathcal{A})$ and for $x \in \mathcal{A}$, we have

(2.4) $\qquad d\psi_t^{\mathcal{A}}(x) = f\big(\psi_t^{\mathcal{A}}(x) - B_t\big)dt$

$$\psi_0^{\mathcal{A}}(x) = x \in \mathbb{R}^n.$$

In order to define $\psi_t^{\mathcal{A}}$ for all time, we introduce a cemetery state Δ and let

$$\psi_t^{\mathcal{A}} = \Delta$$

for $t \geq \tau(\mathcal{A})$. Finally, as in the introduction, we define

$$\mathcal{A}_t = \{\psi_t(x) : \ x \in \mathcal{A}\}.$$

Our next goal is to formulate a strong Markov property for $\psi^{\mathcal{A}}$. In fact $\psi^{\mathcal{A}}$ is a nonanticipating measurable function of the Brownian path B_t, so we can define the shift operator θ_t for B_t and extend it to $\psi^{\mathcal{A}}$.

Lemma 4. *Let $a \subset \mathbb{R}^n$ and suppose that σ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t>0}$ generated by Brownian motion B_t. Assume that $\sigma < \tau(\mathcal{A})$ almost surely. Then conditioned on $\mathcal{F}_\sigma$, we have that $\psi_t(x) := (\theta_\sigma \psi^{\mathcal{A}})_t(x)$ satisfies the following modified version of (2.4). For $0 \le t < \theta_\sigma \tau(\mathcal{A}) = \tau(\mathcal{A}_\sigma)$ and $x \in \mathcal{A}_\sigma$,*

$$(2.5) \qquad d\psi_t(x) = f\big(\psi_t(x) - (\theta_\sigma B)_t\big)dt$$
$$\psi_0(x) = x.$$

This lemma follows immediately from the strong Markov property of Brownian motion.

For $x \in \mathcal{A}$ and $t < \tau(\mathcal{A})$, we can also define

$$\phi_t^{\mathcal{A}}(x) = \psi_t^{\mathcal{A}}(x) - B_t$$

and let $\phi_t^{\mathcal{A}}(x) = \Delta$ for $t \ge \tau_{\mathcal{A}}$. From (2.5) we see that $\phi_t^{\mathcal{A}}(x)$ satisfies (1.1) for $x \in \mathcal{A}$ and $t < \tau_{\mathcal{A}}$. Also, the strong Markov property for $\phi_t^{\mathcal{A}}(x)$ follows from Lemma 4, and the formulation is similar.

We will often use the following containment property, which follows by the definition of a flow. If $S_1 \subset S_2 \subset \mathcal{A}_0$ and $t \in [0, \tau_{\mathcal{A}_0})$, then we have that with probability 1,

$$(2.6) \qquad \phi_t(S_1) \subset \phi_t(S_2) \qquad \text{and} \qquad \psi_t(S_1) \subset \psi_t(S_2).$$

Finally, we record a scaling property of (1.5) which we will use repeatedly. Since this kind of scaling is standard, we leave the proof to the reader.

Lemma 5. *Let $(\psi_t, B_t, \mathcal{A}_0, \tau)$ be a solution to (1.5). Then for $\lambda > 0$,*

$$(\lambda \psi_{\lambda^{-2} t}, \lambda B_{\lambda^{-2} t}, \lambda \mathcal{A}_0, \lambda^{-2} \tau)$$

is also a solution to (1.5), but with $F(x)$ replaced by $F(x/\lambda)$.

Note that this scaling does not change any lower or upper bounds on F.

3. Proof of Theorem 1

Throughout this section we assume $n \ge 2$; as mentioned at the end of Section 1, the case $n = 1$ reduces to an easy question about the Bessel process.

Here is an outline of our strategy. We will show that if there exists a constant C^* such that if $F(x) > C^* n$ for all $x \in \mathbb{R}^n$ then the following holds. We construct a sequence of stopping times $\tau_0 \le \tau_1 \le \cdots$ such

that $\lim_{i\to\infty} \tau_i = \infty$ almost surely, and such that the distance from B_{τ_i} to $\mathcal{A}_{\tau_i}$ tends to get larger and larger. Furthermore, at times τ_i we enlarge the region $\mathcal{A}_{\tau_i}$, and call the enlarged region $\mathbb{A}_{\tau_i}$. Then we show that with probability 1, the Brownian motion B_t never hits the enlarged region $\mathbb{A}_t$. So by (2.6) and the strong Markov property, B_t cannot hit the original region. We write ρ_i for the distance from B_{τ_i} to our enlarged region $\mathbb{A}_{\tau_i}$. This would prove Theorem 1.

Now we give the details. To start with, let $\tau_0 = 0$. Considering that the initial domain $\mathcal{A}_0$ of the flow is of positive distance $\rho_0 > 0$ from the origin, we define $\mathbb{A}_0 = \mathbf{B}_{\rho_0}(0)^c$, that is, the complement of the ball of radius ρ_0 centered at the origin. Then $\mathcal{A}_0 \subset \mathbb{A}_0$. We provisionally define $\tilde{\mathbb{A}}_t = \psi_t^{\mathbb{A}_0}(\mathbb{A}_0)$ for $t < \tau(\mathbb{A}_0)$, where we recall that $\tau(\mathbb{A}_0)$ is the stopping time τ defined with respect to the initial set $\mathbb{A}_0$. Next, let $\tilde{\rho}(t)$ be the shortest distance from B_t to $\tilde{\mathbb{A}}_t$. Let τ_1 be the smallest time $t \in [0, \tau(\mathbb{A}_0))$ such that $\tilde{\rho}(t)$ equals either $\rho_0/2$ or $2\rho_0$. If there is no such time t, then B_t never hits $\tilde{\mathbb{A}}_t$, and Theorem 1 is proved. So we assume that there is such a time τ_1. Finally, let $\rho_1 = \tilde{\rho}(\tau_1)$.

For $0 = \tau_0 \le t < \tau_1$, define $\mathbb{A}_t = \tilde{\mathbb{A}}_t = \psi_t^{\mathbb{A}_0}(\mathbb{A}_{\tau_0})$. Next, define

$$\mathbb{A}_{\tau_1} = \mathbf{B}_{\rho_1}(B_{\tau_1})^c.$$

So at the end of our time stage $[\tau_0, \tau_1)$ we have changed our region so it is again the complement of a ball centered at the location of the Brownian particle B_{τ_1} and of radius ρ_1. We illustrate this setup in Figure 1.

Now we repeat the procedure, with time restarted at τ_1 and with B_{τ_1} playing the role of the origin, and proceed inductively. Here we have used the strong Markov property to restart the process. To summarize, at each time τ_i we change our region to be $\mathbb{A}_{\tau_i} = \mathbf{B}_{\rho_i}(B_{\tau_i})^c$, the complement of the ball of radius ρ_i with center B_{τ_i}. It follows from our construction that ρ_i equals either $\rho_{i-1}/2$ or $2\rho_{i-1}$. Observe that the following lemma implies Theorem 1.

Lemma 6. *There exists a constant $C^* > 0$ not depending on n such that if $F(x) \ge C^* n$ for all $x \in \mathbb{R}^n$, then*

(i) $\lim_{i\to\infty} \rho_i = \infty$ *almost surely.*
(ii) $\lim_{i\to\infty} \tau_i = \infty$ *almost surely.*

The remainder of this section is therefore dedicated to the proof of Lemma 6. Before we start with the proof, we introduce the following auxiliary lemmas, which will be proved later in this section.

The following lemma establishes a lower bound on the tail distribution of the Brownian exit time from a ball of radius $1/2$.
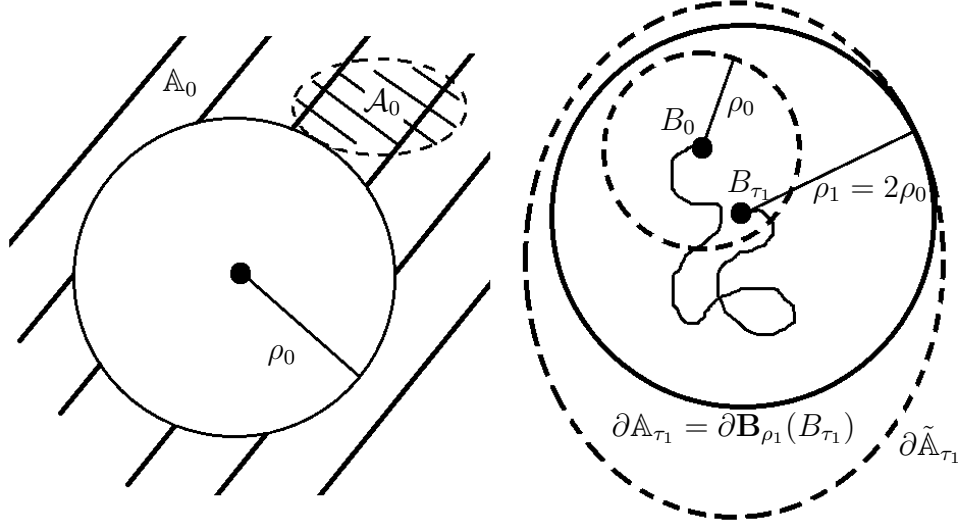
FIGURE 1. The initial set $\mathbb{A}_0$ and the case when $\rho_1 = 2\rho_0$.

Lemma 7. *Let σ be the first time that an n-dimensional standard Brownian motion B_t reaches $\|B_t\| = 1/2$, and assume $0 \leq p < 1$. There exists a constant $C_1 > 0$ not depending on the dimension $n \geq 1$ such that*

$$(3.1) \qquad P\left(\sigma > \frac{C_1}{n}\right) > p.$$

The above probability is computed explicitly in Ciesielski and Taylor [1], Theorem 2, page 444. However, their expression involves an infinite series with roots of Bessel functions, and dependence on the dimension n is not immediately clear. So we give a self-contained proof.

Here we make a comment about the difficulties in working in high-dimensional space. We want to establish a lower bound for the probability that B stays inside an n-dimensional ball of radius $1/2$. Therefore it is not enough to bound

$$P\left(\sigma > \frac{C_1}{n}\right) = P\left(\sup_{0 \leq t \leq C_1/n} \|B_t\| < \frac{1}{2}\right)$$

by the probability that B remains inside the cube $[-1/2, 1/2]^n$, due to the difference between the volume of balls and cubes in high dimensions. Also, a high-dimensional cube has a large ratio between the maximum and minimum distances between the center and the boundary. In the proof of Lemma 7, we use the radial component $D_t = \|B_t\|$

of the Brownian motion and introduce stopping times and auxiliary processes for comparison, in order to derive (3.1).

In the following lemma we establish a lower bound on the transition probability for ρ_i.

Lemma 8. *There exists a constant C^* not depending on n such that if $F(x) \geq C^* n$ for all $x \in \mathbb{R}^n$, then for all $i \in \mathbb{N}$,*

$$P\left(\rho_i = 2\rho_{i-1} \big| \mathcal{F}_{\tau_{i-1}}\right) \geq \frac{2}{3}.$$

Now we are ready to prove Lemma 6.

Proof of Lemma 6. (i) We define

$$X_i = \log_2(\rho_i / \rho_{i-1})$$

where X_i takes the values $\{-1, 1\}$. Lemma 8 states that

$$P\left(X_i = 1 \big| \mathcal{F}_{\tau_{i-1}}\right) \geq \frac{2}{3}.$$

By expanding the probability space if necessary, we can construct a $\mathcal{F}_{\tau_i}$-measurable random variable $Y_i \leq X_i$ for all i, such that Y_i also takes on the values $1, -1$, and such that

$$P\left(Y_i = 1 \big| \mathcal{F}_{\tau_{i-1}}\right) = \frac{2}{3}.$$

It follows that $Y_1, Y_2, \dots$ is a sequence of i.i.d. random variables with expectation $1/3$. The strong law of large numbers along with the fact that $X_i \geq Y_i$ implies that almost surely,

$$\liminf_{m \to \infty} \log_2\left(\frac{\rho_m}{\rho_0}\right) = \liminf_{m \to \infty} \sum_{i-1}^{m} X_i \geq \lim_{m \to \infty} \sum_{i-1}^{m} Y_i = \infty$$

and we get part (i)

(ii) Since $(\tau_i)_{i \in \mathbb{N}}$ is a nondecreasing sequence of random variables, the limit $\tau_\infty = \lim_{i \to \infty} \tau_i$ almost surely exists, and we would like to show that it is infinity a.s. By the construction of the sequence $\{\rho_i\}_{i \geq 0}$ and since $\rho_i \to \infty$ a.s. by part (i) the distance between B_t and $\mathcal{A}_t$ tends to ∞ as $t \uparrow \tau_\infty$ and is strictly positive for $t < \tau_\infty$ with probability one.

Therefore, for almost every realization ω, we can find a natural number $N(\omega)$ such that $\frac{1}{N} \leq \psi_t^{\mathbb{A}_0}(x)$ for all $0 \leq t < \tau_\infty$ and $x \in \mathbb{A}_0$. On this ω, the drift in equation (1.5) is therefore identical to f_N in (2.1). Since f_N is a Lipschitz function, the corresponding flow is finite for all time with probability one. Therefore, the distance between B_t and $\mathcal{A}_t$ tends to ∞ as $t \uparrow \tau_\infty$, only if $\tau_\infty = \infty$ a.s. It follows that $\tau_i \to \infty$ almost surely, and this proves part (ii). □

By our earlier comment, Theorem 1 follows from Lemma 6. The remainder of this section is dedicated to the proofs of Lemmas 7 and 8.

Proof of Lemma 7. We first show that there exist a natural number n^* and a constant $C_{1,n^*} > 0$ such that (3.1) holds whenever $n \geq n^*$. Then, we generalize the requirement $n \geq n^*$ to $n \geq 1$ by possibly choosing a number $C_1 > 0$ smaller than C_{1,n^*}.

Recall that $D_t = \|B_t\|$ is a Bessel process which satisfies

$$(3.2) \qquad dD_t = \frac{n-1}{2D_t} dt + d\tilde{W}_t,$$

for some one-dimensional Brownian motion $\tilde{W}_t$.

Let σ_0 be the first time t that $D_t = 3/8$, and let

$$D_t^{(1)} = D_{\sigma_0 + t}.$$

Let $\mathcal{G}_t$ be the filtration generated by $\tilde{W}_t$. By the strong Markov property of Brownian motion, conditioned on $\mathcal{G}_{\sigma_0}$, $D_t^{(1)}$ satisfies (3.2) with the initial condition $D_0^{(1)} = 3/8$. Now define $D_t^{(2)}$ to be the solution of

$$dD_t^{(2)} = f(D_t^{(2)})dt + d\tilde{W}_t,$$

$$D_0^{(2)} = \frac{3}{8},$$

where

$$f(x) = \begin{cases} (n-1)/(2x), & \text{if } 0 < x < 1/4, \\ 2(n-1), & \text{if } x \geq 1/4. \end{cases}$$

By a standard comparison result (see Theorem 1.1 in Chapter V.1 of Ikeda and Watanabe [7]), since $D_t^{(1)}$ has drift which is not greater than the drift of $D_t^{(2)}$, we conclude

$$D_t^{(1)} \leq D_t^{(2)}$$

for all $t \geq 0$ with probability 1. Actually, the result of Ikeda and Watanabe does not cover locally unbounded drift, so we must argue via a truncation argument. We leave these details to the reader.

So $D_t^{(1)}$ will reach $1/2$ later than $D_t^{(2)}$ reaches $1/2$. Now let σ_1 be the first time t that $D_t^{(1)} = 1/2$, and let σ_2 be the first time t that $D_t^{(2)} = 1/2$. It follows that $\sigma \geq \sigma_1 \geq \sigma_2$, and so

$$P(\sigma > x) \geq P(\sigma_2 > x).$$

Finally, let σ_3 be the first time t that $D_t^{(2)} = 1/4$ or $D_t^{(2)} = 1/2$. Then $\sigma_2 \geq \sigma_3$, and so

$$P(\sigma > x) \geq P(\sigma_2 > x) \geq P(\sigma_3 > x).$$

Therefore, in order to prove (3.1), we look for a lower bound on $P(\sigma_3 > x)$. To this end, let $D_t^{(3)}$ be the solution of

$$dD_t^{(3)} = 2(n-1)dt + d\tilde{W}_t,$$
$$D_0^{(3)} = \frac{3}{8},$$

and let σ_4 to be the first time t that $D_t^{(3)} = 1/4$ or $1/2$. Then $\sigma_3 = \sigma_4$. Finally, let σ_5 be the first time t that $D_t^{(3)} = 1/2$. Then we have

$$P(\sigma > x) \geq P(\sigma_3 > x) = P(\sigma_4 > x)$$
$$\geq P\left(\sigma_4 > x, D_{\sigma_4}^{(3)} = 1/2\right).$$

Note that on $D_{\sigma_4}^{(3)} = 1/2$, we have $\sigma_4 = \sigma_5$, and also $\{D_{\sigma_4}^{(3)} = 1/2\}^c = \{D_{\sigma_4}^{(3)} = 1/4\}$. Thus, we can continue the above inequality as follows.

$$(3.3) \qquad P(\sigma > x) \geq P\left(\sigma_4 > x, D_{\sigma_4}^{(3)} = 1/2\right)$$
$$= P\left(\sigma_5 > x, D_{\sigma_4}^{(3)} = 1/2\right)$$
$$= P\left(\sigma_5 > x\right) - P\left(\sigma_5 > x, D_{\sigma_4}^{(3)} = 1/4\right)$$
$$\geq P\left(\sigma_5 > x\right) - P\left(D_{\sigma_4}^{(3)} = 1/4\right).$$

We first give an upper bound on $P(D_{\sigma_4}^{(3)} = 1/4)$. Recall that

$$h(x) = \frac{e^{-2(n-1)x} - e^{-(n-1)}}{e^{-(n-1)/2} - e^{-(n-1)}}$$

is a harmonic function for $D_t^{(3)}$, since the process $D_t^{(3)}$ has generator

$$Gf(x) = 2(n-1)f'(x) + \frac{1}{2}f''(x).$$

Furthermore, $h(1/4) = 1$ and $h(1/2) = 0$. We deduce that

$$(3.4) \qquad P\left(D_{\sigma_4}^{(3)} = 1/4\right) = h(3/8) = \frac{e^{-3(n-1)/2} - e^{-2(n-1)}}{e^{-(n-1)} - e^{-2(n-1)}}$$
$$= e^{-(n-1)/2}\left(1 + o(1)\right)$$

as $n \to \infty$.

We now focus on $P(\sigma_5 > x)$. Assume that

$$x \leq 1/(8(n-1)).$$

Note that for $t \leq x$ we have $(n-1)t \leq 1/8$. Thus, for such values of x, using the reflection principle we have

$$P(\sigma_5 > x) \geq P\left(\sup_{0 \leq t \leq x} \tilde{W}(t) < 1/8\right)$$
$$= 1 - P\left(\sup_{0 \leq t \leq x} \tilde{W}(t) \geq 1/8\right)$$
$$= 1 - 2P(\tilde{W}(x) \geq 1/8).$$

Using a standard Gaussian random variable Z, we get

$$P(\tilde{W}(x) \geq 1/8) = P(\sqrt{x}Z \geq 1/8) = P\left(Z \geq \frac{1}{8\sqrt{x}}\right)$$
$$\leq \frac{4\sqrt{2}\sqrt{x}}{\sqrt{\pi}} \exp\left(-\frac{1}{128x}\right).$$

See Durrett [6], Theorem 1.2.3, page 12, for the Gaussian estimate.

With $x \leq 1/(8(n-1))$, we have

$$P(\tilde{W}(x) \geq 1/8) \leq \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{n-1}} \exp\left(-\frac{n-1}{16}\right).$$

So,

$$(3.5) \qquad P(\sigma_5 > x) \geq 1 - \frac{4}{\sqrt{\pi}} \frac{1}{\sqrt{n-1}} \exp\left(-\frac{n-1}{16}\right).$$

Combining (3.3), (3.4), and (3.5), we can conclude that for large values of n and $x \leq 1/(8(n-1))$, we get $P(\sigma > x) > p$. In particular,

$$P\left(\sigma > \frac{1}{8n}\right) > p$$

for all $n \geq n^*$ for some n^*.

It remains to lower the requirement $n \geq n^*$ to $n \geq 1$. Note that for each fixed value of n, we have $P(\sigma > C/n) \to 1$ as the constant $C \downarrow 0$. Thus for each $1 \leq n < n^*$, we can find $C_{1,n} > 0$ such that

$$P\left(\sigma > \frac{C_{1,n}}{n}\right) > p.$$

By choosing

$$C_1 = \min\left\{C_{1,1}, \ldots, C_{1,n^*-1}, \frac{1}{8}\right\},$$

we can conclude that

$$P\left(\sigma > \frac{C_1}{n}\right) > p$$

for all $n \geq 1$. $\qquad\square$

Proof of Lemma 8. Notice that by the strong Markov property applied at time τ_{i-1}, we can start afresh at that time and relabel τ_{i-1} as $\tau_0 = 0$. By translating if necessary, we may assume that

(1) $i = 1$
(2) $\tau_0 = 0$
(3) $B_0 = 0$

Furthermore, by scaling time and space via Lemma 5, we can assume that

(4) $\rho_0 = 1$.

These transformations may change our drift F, but not the lower bound on F. Thus, to prove Lemma 8, it suffices to show

$$(3.6) \qquad\qquad P(\rho_1 = 2) \geq \frac{2}{3}.$$

Our intuitive idea is as follows. We show that with high probability the time spent by the Brownian motion in $\mathbf{B}_{1/2}(0)$ is greater than C_1/n, for some constant C_1 not depending on n. If this event occurs, then the magnitude of the total drift on points $y \in \mathcal{A}_t \cap \left(\mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)\right)$ is

$$\frac{F(\|y - B_t\|)}{\|y - B_t\|} \geq \frac{C^* n}{3}$$

within the time interval $[0, C_1/n]$. The inequality was from the hypothesis $F(x) \geq C^* n$ of Lemma 8 and from the distance between a point in $\mathbf{B}_{5/2}(0)$ and the Brownian motion in $\mathbf{B}_{1/2}(0)$ is at most $5/2 + 1/2 = 3$. Furthermore, we will see that the projection of this drift onto the outward radial direction at worst reduces this drift by a multiplicative factor c. See Figure 2 for the illustration. Thus, it follows from (1.5) the total outward radial drift on a point $y \in \mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)$ for the time interval $[0, C_1/n]$ is at least

$$(3.7) \qquad \inf_{y \in \mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)} \left\{ \frac{F(\|y - B_t\|)}{\|y - B_t\|} \right\} \cdot \frac{C_1}{n} \geq \frac{C^* n}{3} \cdot c \cdot \frac{C_1}{n} > 3/2,$$

if C^* is large enough. Also observe that $3/2$ is the radial distance from $\mathbf{B}_1(0)$ to the boundary of $\mathbf{B}_{5/2}(0)$. Roughly speaking, this means that if C^* is large enough, then any point in the region $\mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)$ will now be at least at distance $5/2$ from the origin, while the Brownian motion remains in $\mathbf{B}_{1/2}(0)$. Note that the drift might be smaller than $3/2$ if the point exits the ball $\mathbf{B}_{5/2}(0)$ during the time interval $[0, C_1/n]$, but this is what we want to show anyway. Also, once it exits this ball, it never reenters. So in both cases the point in $\mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)$ exits the ball $\mathbf{B}_{5/2}(0)$, and the distance between the Brownian particle and $\mathcal{A}_t$ increases by a factor of 2.
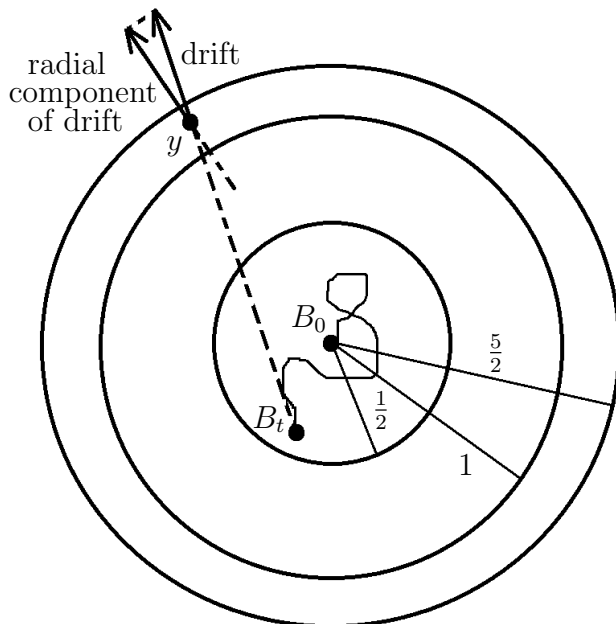
FIGURE 2

In order to make this argument precise, let **A** be the event that
$\sigma > C_1/n$. We will show that on the event **A**, we have $\rho_1 = 2\rho_0$. Since
Lemma 7 shows that $P(\mathbf{A}) > p$, choosing $p > 2/3$ will then give us
(3.6). From the reasoning which was given in the preceding paragraph
and specifically in (3.7), all that is left to verify is the statement about
the radial part of the drift. Let $a \in \mathbf{B}_{1/2}(0)$ and $b \in \mathbf{B}_{5/2}(0) \setminus \mathbf{B}_1(0)$.
We wish to show that there is a constant $c > 0$ not depending on n
such that for any such pair of points a, b the projection of $b - a$ on b has
magnitude bounded below by c. First we note that the points $0, a, b$
determine a plane passing through the origin, so we may assume that
$n = 2$, and our space is the $x - y$ plane. We may also assume that b
lies on the vertical axis, with y-coordinate between 1 and 5/2. Then
the projection is bounded by

$$(3.8) \qquad |(y - x) \cdot y| = |y \cdot y - x \cdot y|$$
$$\geq \|y\|^2 - \|x\| \cdot \|y\|$$
$$\geq (\|y\| - \|x\|)\|y\|$$
$$\geq \frac{1}{2} \cdot 1.$$

So, we may choose $c = 1/2$. This finishes the proof of Lemma 8.   □

4. Proof of Theorem 2

We again assume that $n \geq 2$. First we introduce some definitions
and notation which will help us to compare the problem of hitting
the origin for $\phi^{\mathcal{A}_0}$, to the corresponding problem for a class of biased
random walks.

Notation. Recall our previous notation $x = (x_1, ..., x_n)$. We define
$x^\perp = (x_2, ..., x_n)$. We further write $B_t = (B_t^{(1)}, ..., B_t^{(n)})$ and define
$B_t^\perp = (B_t^{(2)}, ..., B_t^{(n)})$.

As stated in Theorem 2, we assume

(4.1) $$\mathcal{A}_0 = \{x \in \mathbb{R}^n : x_1 \geq 1\}.$$

By scaling via Lemma 5, if we prove Theorem 2 for this definition of
$\mathcal{A}_0$, we have also proved it for $\mathcal{A}_0 = \{x \in \mathbb{R}^n : x_1 \geq a\}$ for any $a > 0$.

Next, we define the following objects by induction.

(i) A sequence $\tau_0 \leq \tau_1 \leq \cdots$ of stopping times with limit
$\tau_\infty = \lim_{i \to \infty} \tau_i$.

(ii) A sequence of positive random variables $\rho_0, \rho_1, \ldots$.

(iv) A collection $\{\mathbb{A}_t\}_{t < \tau_\infty}$ of random subsets of $\mathbb{R}^n$.

Here, ρ_i is the distance of B_{τ_i} to $\mathbb{A}_{\tau_i}$. Recall that τ was defined in (1.2).
Note that it is possible that $\tau < \tau_\infty$. One of the main ingredients in
the induction is to choose $\mathcal{A}_t$ such that, $\mathbb{A}_t \subset \mathcal{A}_t$ for $t < \tau \wedge \tau_\infty$.

To begin with, let $\tau_0 = 0$ and let

$$\rho_0 = \sup_{x \in \partial \mathcal{A}_0} |B_0^{(1)} - x_1| = 1,$$

where $\partial \mathcal{A}_0 = \{x \in \mathbb{R}^n : x_1 = 1\}$ is the boundary of the set $\mathcal{A}_0$. Let τ_1
be the first time $t > 0$ such that

(4.2) $$\sup_{x \in \partial \mathcal{A}_t} |B_t^{(1)} - x_1| = \frac{1}{2} \text{ or } 2$$

and note that $\tau_1 < \tau$ unless both of these times are ∞. We leave it to
the reader to check that $\tau_1 < \infty$ with probability one. For $0 \leq t < \tau_1$,
let

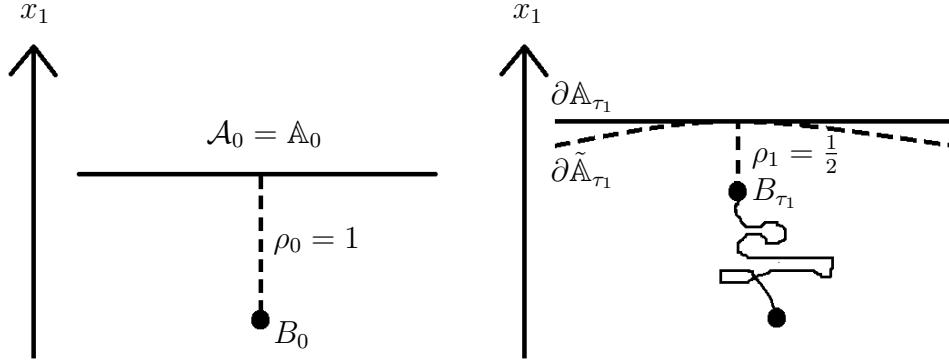$$\mathbb{A}_t = \mathcal{A}_t.$$

We provisionally define

$$\tilde{\mathbb{A}}_{\tau_1} = \mathcal{A}_{\tau_1}.$$

Finally, we let

$$\rho_1 = \sup_{x \in \partial \tilde{\mathbb{A}}_{\tau_1}} |B_{\tau_1}^{(1)} - x_1|.$$

See Figure 3 for the setup.

FIGURE 3. The initial set $\mathbb{A}_0$ and the case when $\rho_1 = 1/2$.

Now assume that we have defined τ_m, ρ_m, and $\{\mathbb{A}_t\}_{t < \tau_m}$ such that $\mathbb{A}_t \subset \mathcal{A}_t$ for $t < \tau_m$. First we define

$$\mathbb{A}_{\tau_m} = \{x \in \mathbb{R}^n : x_1 \geq B_{\tau_m}^{(1)} + \rho_m\}$$

and note that $\mathbb{A}_{\tau_m} \subset \mathcal{A}_{\tau_m}$.

Recall that the shift operator θ_t was defined before Lemma 4. Next we define τ_{m+1}, so that $\tau_{m+1} - \tau_m$ is the first time $t > 0$ such that

$$\sup_{x \in \partial(\theta_{\tau_m} \psi^{\mathbb{A}_{\tau_m}})_t(\mathbb{A}_{\tau_m})} |B_t^{(1)} - x_1| = \frac{1}{2}\rho_m \text{ or } 2\rho_m.$$

Then, for $0 \leq t < \tau_{m+1} - \tau_m$, let

$$\mathbb{A}_{t+\tau_m} = (\theta_{\tau_m} \psi^{\mathbb{A}_{\tau_m}})_t(\mathbb{A}_{\tau_m}).$$

Again, we provisionally define

$$\tilde{\mathbb{A}}_{\tau_{m+1}} = (\theta_{\tau_m} \psi^{\mathbb{A}_{\tau_m}})_{\tau_{m+1}}(\mathbb{A}_{\tau_m}).$$

and define

$$\rho_{m+1} = \sup_{x \in \partial \tilde{\mathbb{A}}_{\tau_{m+1}}} |B_{\tau_{m+1}}^{(1)} - x_1|.$$

This finishes our inductive definition. Our next goal is to show the following proposition, which essentially proves Theorem 2. The rest of this section is devoted to the proof of this proposition.

Proposition 9. *We have*

(i) $\lim_{i \to \infty} \rho_i = 0$ *almost surely.*
(ii) $P(\tau_\infty < \infty) = 1$.

Proof. The proof of Proposition 9 is rather long, hence we divide it into a few steps.

But before presenting the details, we give a brief overview of the proof. A point x moves in the vertical direction (the x_1 direction) because Brownian motion B_t spends time with $B^{(1)}$ close to x_1, and with $B^\perp$ close to $x^\perp$. To take account of $|x^\perp - B^\perp|$, we cover the Brownian path $B_t^\perp$ with balls of various sizes, and consider the amount of time the Brownian motion spends in (i) balls of the smallest size and (ii) annuli consisting of set differences between two balls of the same center. Secondly, as time progresses, the point x may move not only in the vertical direction, but also in the lateral direction (the $x^\perp$ direction). As x moves in the lateral direction, it may get closer to certain parts of the Brownian path which did not contribute much drift previously. Thus, we need to control the lateral distance over which x might move. Using the same balls as before, we will control the amount of lateral drift experienced by x, and thus the distance which $x^\perp$ can travel. The upward drift experienced by x_1 will then be the sum of contributions from all the balls, and over all possible positions of $x^\perp$.

Step 1: a random walk comparison. We set up a random walk comparison as we did at the beginning of Section 3. Recall that the vertical direction refers to the first coordinate of $\mathbb{R}^n$, and the horizontal direction refers to the remaining coordinates. We will show that the vertical distance between the Brownian motion and $\mathcal{A}_t$ is bounded by the distance from the origin of a one dimensional biased random walk initiated at $+1$.

In order to do that we consider $\overline{\mathcal{A}}_t(x)$ which satisfies (1.5), with $\overline{F}(x) \equiv \|F\|_\infty$ for all $x \in \mathbb{R}$ instead of F. For every $i = 0, 1, ...,$ let $\overline{\tau}_i$ and $\overline{\rho}_i$ be the equivalents of τ_i and ρ_i with $\overline{\mathcal{A}}$ instead of $\mathcal{A}$. Moreover, let $\overline{\mathbb{A}}_t$ be the equivalent of $\mathbb{A}_t$ with $\overline{\mathcal{A}}$ instead of $\mathcal{A}$. Then from the construction of $\{(\tau_i, \rho_i)\}_{i \geq 0}$ and $\mathbb{A}_t$, it follows that for every $i = 0, 1, ...,$

$$P\big(\overline{\rho}_{i+1} = \overline{\rho}_i/2\big) \leq P\big(\rho_{i+1} = \rho_i/2\big),$$

and

$$P(\overline{\tau}_\infty < \infty) \leq P(\tau_\infty < \infty).$$

From the preceding paragraph we conclude that it is enough to prove Proposition 9 for $\overline{\mathcal{A}}$. In order to simplify notation, we will write $(\mathcal{A}_t(x), \tau_i, \rho_i, \mathbb{A}_t)$ instead of $(\overline{\mathcal{A}}_t(x), \overline{\tau}_i, \overline{\rho}_i, \overline{\mathbb{A}}_t)$, when there is no ambiguity.

Using scaling via Lemma 5 with $\|F\|_\infty$ in place of F, we find that there is a constant $p \in (0, 1)$ such that, conditioned on $\mathcal{F}_{\tau_{i-1}}$,

$$\log_2 \rho_i = \begin{cases} \log_2 \rho_{i-1} + 1 & \text{with probability } p \\ \log_2 \rho_{i-1} - 1 & \text{with probability } 1 - p, \end{cases}$$

for $i = 0, 1, \ldots,$ and the increments $\rho_{i+1} - \rho_i$ are independent. It follows that $\{\log_2 \rho_i\}_{i \in \mathbb{N}}$ is a nearest-neighbor random walk on the integers, and is biased if $p \neq 1/2$. So, $\rho_i \to 0$ as $i \to \infty$ with probability 1 if and only if $p < 1/2$. Since the probability p is the same for each time stage $[\tau_m, \tau_{m+1}]$, we prove that $p < 1/2$ only on the first time stage $t \in [\tau_0, \tau_1]$, and this will finish the proof of Proposition 9(i).

In addition to ρ_1, we define the maximum displacement of $\tilde{\mathbb{A}}_{\tau_1}$ in the x_1-direction D_{τ_1} as

$$(4.3) \qquad\qquad D_{\tau_1} = \sup_{x \in \partial \tilde{\mathbb{A}}_{\tau_1}} (x_1 - 1) = \rho_1 + B_{\tau_1}^{(1)} - 1.$$

We are interested in finding a bound of the probability $p = P\big(\rho_1 = 2\big)$. First note that if we set $D_{\tau_1} \equiv 0$, from (4.3) and by the gambler's ruin, we would have $p = 1/3$, since it is the probability that $1 - B_t^{(1)}$, a one-dimensional Brownian motion starting at 1, hits 2 before hitting $1/2$. That is,

$$p = \frac{1/2}{1/2 + 1} = 1/3.$$

Since $D_{\tau_1} > 0$, P-a.s., in order to get a bound on p, we assume the following hypothesis, which will be verified later. For every $\delta > 0$ sufficiently small,

$$(4.4) \qquad\qquad P\left(D_{\tau_1} < \frac{1}{4} - 2\delta\right) > 1 - \delta.$$

From (4.3) it follows that under $\{D_{\tau_1} < \frac{1}{4} - 2\delta\}$, p is bounded from above by the probability that a Brownian motion, starting from $1 + \frac{1}{4} - 2\delta$, exits the interval $[1/2, 2]$ from the right boundary. It follows that

$$q \equiv 1 - p \geq \frac{2 - (1 + \frac{1}{4} - 2\delta)}{2 - (1 + \frac{1}{4} - 2\delta) + |1/2 - (1 + \frac{1}{4} - 2\delta)|} > 1/2 + \delta.$$

Therefore, in order to prove that B_t hits $\mathbb{A}_t$, we just need to verify (4.4). The remainder of the proof is dedicated to deriving an inequality similar to (4.4).

Step 2: construction of a cover on the Brownian path. Recall that $B_0 = 0$. In order to bound the vertical and lateral drifts we first need to fix a cover of the Brownian path $\{B_t^\perp : t \in [0, T]\}$ in $\mathbb{R}^{n-1}$, for some arbitrary $T > 0$, and specify some of its properties.

In what follows $\lceil \cdot \rceil$ is the ceiling function which gives the smallest integer greater than or equal to the number inside.

Proposition 10. *For any $\varepsilon > 0$ there exists a constant C_2 not depending on n such that the following holds. If $K \in (0, 1/8)$ and $r = e^k$ with integer $1 \le k \le k_\infty$, where $k_\infty = \lceil (\log n)/2 \rceil$, then with probability at least $1 - \varepsilon$, we can cover $\{B_t^\perp : t \in [0, T]\} \subset \mathbb{R}^{n-1}$ with a number $\lceil C_2 \frac{nT}{Ke^{2k}} \rceil$ of $(n-1)$-dimensional balls of radius r.*

We prove Proposition 10 in Section 5.

Recall that τ_1 was defined in (4.2). We define the following event:

$$(4.5) \qquad \mathcal{E}_1 = \{\tau_1 \le T\}.$$

The following Lemma, which is proved in Section 6, helps us to bound the probability of $\mathcal{E}_1^c$.

Lemma 11. *There exist constants $C_3, C_4 > 0$ such that*

$$P(\tau_1 > T) \le C_3 \exp(-C_4 T).$$

Let $\varepsilon > 0$ be arbitrarily small. By Lemma 11, we can fix T large enough so that

$$(4.6) \qquad P(\mathcal{E}_1^c) < \varepsilon.$$

We also fix $K \in (0, 1/8)$ and let

$$(4.7) \qquad m_k^*(n) = \left\lceil C_2 \frac{nT}{Ke^{2k}} \right\rceil,$$

where C_2 is defined as in Proposition 10.

We define $\mathcal{E}_2$ to be the event that there is a cover of $\{B_t^\perp : t \in [0, T]\}$ with $m_k^*(n)$ balls of of radius e^k for any $k = 1, ..., \lceil (\log n)/2 \rceil$. From Proposition 10 we have

$$(4.8) \qquad P(\mathcal{E}_2^c) < \varepsilon.$$

Step 3: uniform bound on the Brownian occupation time. In order to bound the vertical and lateral drift, we will bound the amount of time spent by the Brownian motion in each ball of the cover which was constructed in the previous step. Let $L_t^{(n-1)}(\cdot)$ be the occupation measure of the $n - 1$ dimensional Brownian motion $B_t^\perp$. Define $L^{(n-1)}(\cdot) = \lim_{t \to \infty} L_t^{(n-1)}(\cdot)$, where we often omit the dependence in n and write $L_t(\cdot)$ and $L(\cdot)$. We will need the following proposition.

Proposition 12. *Assume that $n \ge 4$. Let $\mathbf{B}_r(0)$ be a ball of radius $r > 0$ centered at the origin. Then, there exist constants $C_5, C_6 > 0$ such that*

$$P\big(L^{(n-1)}(\mathbf{B}_r(0)) > sr^2\big) \le C_5 \exp\big(-C_6 n^2 s\big)$$

for all $s > \frac{8}{n}$.

The proof of Proposition 12 is given in Section 6.

From now on assume that $n \geq 4$. We will deal with the case where $n = 2, 3$ later. Assume further that $\mathcal{E}_2$ is satisfied. Denote by $\mathbf{B}_i^k$, $i = 1, ..., m_k^*(n)$, the balls of radius e^k in the random cover of $\{B_t^\perp : t \in [0, T]\}$. We also write $C \cdot \mathbf{B}_i^k$ as the ball with the same center as $\mathbf{B}_i^k$ with radius C times the radius of $\mathbf{B}_i^k$.

In what follows we fix $1 \leq k \leq k_\infty$, where $k_\infty = \lceil (\log n)/2 \rceil$. Suppose that $p \in \mathbf{R}^{n-1}$, and define

(4.9)
$$\bar{\tau}(p) = \inf\{t \geq 0 : |B_t - p| < 2e^k\}.$$

Assume that $\bar{\tau}(p) \leq T$. Let $\tau_0, \tau_1, \tau_2, \ldots$ be the times at which we choose new balls in the cover of $\{B_t^\perp : t \in [0, T]\}$. That is, $\tau_0 = 0$, and $\mathbf{B}_0^k$ is centered at the origin. Then, τ_1 is the exit time from $\mathbf{B}_0^k$ and $\mathbf{B}_1^k$ is centered at B_{τ_1}. We continue in the same manner, so $\mathbf{B}_i^k$ has its center at B_{τ_i} with radius e^k. By our assumption $\bar{\tau}(p) \leq T$, and since $\mathbf{B}_i^k$ cover $B_{[0,T]}$, it follows that there must be a first index $i(p)$ such that

(4.10)
$$|B_{\tau_{i(p)}} - B_{\bar{\tau}(p)}| < e^k$$

and hence

(4.11)
$$|B_{\tau_{i(p)}} - p| < 3e^k.$$

Note that $\tau_{i(p)}$ is a stopping time with respect to the Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$. Also, we have $\tau_{i(p)} < \bar{\tau}(p)$ by the construction of our cover. By this definition, $B_t \notin 2\mathbf{B}_{e^k}(p)$ for $t < \bar{\tau}(p)$, and so $B_t \notin 2\mathbf{B}_{e^k}(p)$ for $t < \tau_{i(p)}$ also. From (4.10) and (4.11), we have

(4.12)
$$2\mathbf{B}_{e^k}(p) \subset 5\mathbf{B}_{i(p)}^k.$$

Next, we consider the occupation measure of the Brownian path after time $\tau_{i(p)}$. Using the time shift operator $\{\theta_t\}_{t \geq 0}$, we define

$$L_{\tau_{i(p)}}^{(n-1)}(2\mathbf{B}_{e^k}(p)) = \theta_{\tau_{i(p)}} \cdot L^{(n-1)}(2\mathbf{B}_{e^k}(p)).$$

Since $B_t \notin 2\mathbf{B}_{e^k}(p)$ for $t < \tau_i(p)$, it follows from (4.12) that,

(4.13)
$$L^{(n-1)}(2\mathbf{B}_{e^k}(p)) = L_{\tau_{i(p)}}^{(n-1)}(2\mathbf{B}_{e^k}(p))$$
$$\leq L_{\tau_{i(p)}}^{(n-1)}(5\mathbf{B}_{i(p)}^k).$$

By the strong Markov property of Brownian motion with respect to the stopping time $\tau_i(p)$ and using Proposition 12, it follows that if $s > 8/n$, then there are $C_5, C_6 > 0$ such that

(4.14)
$$P\big(L_{\tau_{i(p)}}^{(n-1)}(5\mathbf{B}_{i(p)}^k) > 25se^{2k}\big) \leq C_5 \exp\big(-C_6 n^2 s\big).$$

Now let $\mathcal{A}_{k,i}$ be the event that

$$(4.15) \qquad\qquad L^{(n-1)}_{\tau_i(p)}(5\mathbf{B}^k_i) \le \rho\frac{e^{2k}}{n}$$

for some positive constant ρ independent of i and k, which will be fixed later. From (4.14) we have for $\rho/(25n) > 8/n$ (that is $\rho > 200$),

$$(4.16) \qquad\qquad P(A^c_{k,i}) \le C_5 \exp\left(-C_6 n\rho/25\right).$$

Let us define the good event where all bounds such as (4.15) are satisfied as

$$\mathcal{E}_3 = \bigcap_{k=1}^{k_\infty} \bigcap_{i=1}^{m^*_k(n)} A_{k,i}.$$

Note that from (4.15) and (4.13), it follows that if $\rho > 200$, then we have on $\mathcal{E}_3$

$$(4.17) \qquad\qquad \sup_{p\in\mathbf{R}^{n-1}} L^{(n-1)}(2\mathbf{B}_{e^k}(p)) \le \rho\frac{e^{2k}}{n}.$$

From (4.17) we get that under $\mathcal{E}_3$ we have a uniform bound in p on the occupation measure $L^{(n-1)}$, which is a stronger statement than the local bound in Proposition 12.

We now show that we can choose ρ large enough, so the probability of $\mathcal{E}_3$ will be arbitrary close to 1. Recall that $k_\infty = \lceil (\log n)/2 \rceil$. From (4.16) and (4.7) it follows that

$$
\begin{aligned}
P(\mathcal{E}^c_3) &\le \sum_{k=1}^{k_\infty} \sum_{i=1}^{m^*_k(n)} P(A^c_{k,i}) \\
&\le \sum_{k=1}^{k_\infty} m^*_k(n) C_5 e^{-C_6 n\rho/25} \\
&= \sum_{k=1}^{k_\infty} \left\lceil C_2\frac{nT}{Ke^{2k}}\right\rceil C_5 e^{-C_6 n\rho/25}.
\end{aligned}
$$

Using (4.7), we continue with

$$
P(\mathcal{E}_3^c) \leq C \frac{nT}{K} e^{-C_6 n\rho/25} \sum_{k=1}^{k_\infty} \frac{1}{e^{2k}}
$$

$$
\leq C \frac{nT}{K} e^{-C_6 n\rho/25} \left(\frac{1}{e^2} + \int_1^{k_\infty} \frac{1}{e^{2x}} dx \right)
$$

$$
\leq C \frac{nT}{K} e^{-C_6 n\rho/25} \left(\frac{1}{e^2} + \int_1^{(\log n)/2+1} \frac{1}{e^{2x}} dx \right)
$$

$$
= C \frac{nT}{K} e^{-C_6 n\rho/25} \frac{1}{2e^2} \left(3 - \frac{1}{n} \right)
$$

$$
= C' \frac{T}{K} e^{-C_6 n\rho/25} (3n - 1).
$$

Note that the right hand side is positive since $n \geq 1$. In order to have $P(\mathcal{E}_3^c) < \varepsilon$, we need ρ to satisfy

(4.18) $$ C' \frac{T}{K} e^{-C_6 n\rho/25} (3n - 1) < \varepsilon. $$

From the conditions on (4.16) we can choose $\rho > 200$ such that (4.18) is satisfied. For such ρ, we get

(4.19) $$ P(\mathcal{E}_3^c) < \varepsilon. $$

In the next steps we derive an upper bound of the lateral drift (in the $x^\perp$–direction) and the vertical drift (x_1–direction), during the time interval $[0, \tau_1]$, on point p which initially is in $\mathcal{A}_0$. Our bound will be independent of the initial position p.

We write

$$
p = (p_1, \ldots, p_n) = (p_1, p^\perp),
$$

where $p^\perp = (p_2, \ldots, p_n) \in \mathbf{R}^{n-1}$.

Step 4: bound on the lateral drift. We use the cover of the Brownian path on $[0, T]$ and the bounds on the Brownian occupation time which were established in the previous steps, in order to bound the lateral drift (i.e. the displacement of $p^\perp$). We will use the notation $\{p^\perp(t)\}_{t \geq 0}$ to denote the position of $p^\perp$ at time t with $p^\perp(0) \in \mathcal{A}_0$ being its original position.

We first define concentric balls centered at $p(0)^\perp$ in $\mathbb{R}^{n-1}$. Let $V_{p(0)}^k$ be a ball of radius $2e^k$ centered at $p(0)^\perp$ for $k = 1, ..., k_\infty$. Then, we define the set of the concentric annuli $A_{p(0)}^k = V_{p(0)}^k \setminus V_{p(0)}^{k-1}$, for $k = 2, 3, ...$, with the first set equal to the first ball: $A_{p(0)}^1 := V_{p(0)}^1$. So, these annuli (except for the first) have inner radii $2e^k$ and outer radii $2e^{k+1}$.

Let $D_k^\perp$ be the maximal lateral drift experienced by a point $q = (q_1, q^\perp)$ such that

(4.20)
$$q_1 \le 0 \quad \text{and} \quad q^\perp \in \mathbf{B}_{e^k}(p^\perp(0))$$

assuming that the Brownian particle is located at $r = (r_1, r^\perp)$. Recall that by the hypothesis of Theorem 2, $\|F\|_\infty \le c^* n^{3/4}$. On the first interval $[0, \tau_1]$, the vertical distance (along the x_1-axis) from $\mathbb{A}_t$ to B_t is on the interval $[1/2, 2]$ and so it is bounded below by $1/2$. The lateral distance (along $x^\perp$) from B_t to a point q in $\mathbb{A}_t$ is $\|q^\perp - B_t^\perp\|$. From (1.5) it follows that,

(4.21)
$$D_k^\perp \le \|F\|_\infty \frac{1}{\sqrt{1/4 + \|q^\perp - r^\perp\|^2}}$$
$$\le c^* n^{3/4} \frac{1}{\sqrt{1/4 + \|q^\perp - r^\perp\|^2}}.$$

We distinguish between the following two cases:

Case 1: $r^\perp \in A_{p(0)}^1$. Then from (4.21) it follows that there exists a constant $C > 0$ independent of n and p, such that

(4.22)
$$D_1^\perp \le C c^* n^{3/4}.$$

Case 2: $r^\perp \in A_{p(0)}^{k+1}$, for $1 \le k \le k_\infty$. Then from (4.21) it follows that there exists a constant $C > 0$ independent of n and p, such that

(4.23)
$$D_k^\perp \le \|F\|_\infty \frac{1}{\sqrt{1/4 + e^{2k}}}$$
$$\le C c^* n^{3/4} e^{-k}.$$

We define k_0 by the equation
$$e^{k_0} = \left(e^{k_\infty}\right)^{1/2} = n^{1/4}.$$

First we assume that the process $\{p^\perp(t)\}_{t \ge 0}$ stops when it exits from $\mathbf{B}_{e^{k_0}}(p^\perp(0))$. If, under this assumption, we find that $p^\perp(t)$ in fact does not exit from $\mathbf{B}_{e^{k_0}}(p^\perp(0))$ over time $[0, T]$, then the unmodified process must not exit either.

We will use our estimates on the occupation measure for Brownian motion from step 3, in the various balls $2\mathbf{B}_{e^{2k}}(p^\perp(0))$, in order to bound the horizontal drift experienced by points in $\mathcal{A}_t$ where $t \in [0, \tau_1]$. We distinct between the following 3 cases: $k = 1$, $1 < k < k_0$, and $k_0 \le k \le k_\infty$. In the first two cases $k < k_0$, we have that $p^\perp(t)$ traverses at most $e^{k_0 - k}$ balls. For $k > k_0$, we have that $p^\perp(t)$ traverses at most one ball. Then, recalling that $e^{k_0} = n^{1/4}$, we have the following. In each term, we first write our upper bound for the drift from (4.22) and (4.23), then the number of balls traversed, and then the occupation measure L^{n-1}

of each ball from (4.17). At first, we separate these terms by dots. Let $D_{\tau_1}^{\perp}$ be the total horizontal drift experienced by $p(t)$ between $[0, \tau_1]$. It follows that

$$D_{\tau_1}^{\perp} \leq Cc^* n^{3/4} \cdot e^{k_0} \cdot L^{(n-1)} \big[A_1(p^{\perp}(0)) \big]$$

$$+ \sum_{k=2}^{k_0-1} Cc^* n^{3/4} e^{-k} \cdot e^{k_0-k} \cdot L^{(n-1)} \big[A_k(p^{\perp}(0)) \big]$$

$$+ \sum_{k=k_0}^{k_\infty} Cc^* n^{3/4} e^{-k} \cdot 1 \cdot L^{(n-1)} \big[A_k(p^{\perp}(0)) \big]$$

$$\leq Cc^* n^{3/4} e^{k_0} \rho \frac{1}{n} + Cc^* \sum_{k=0}^{k_0} n^{3/4} e^{k_0-2k} \rho \frac{e^{2k}}{n} + Cc^* \sum_{k=k_0}^{k_\infty} n^{3/4} e^{-k} \rho \frac{e^{2k}}{n}$$

$$\leq Cc^* \rho n^{-1/4} e^{k_0} + Cc^* \sum_{k=0}^{k_0} n^{3/4} e^{k_0} \rho \frac{1}{n} + Cc^* \sum_{k=k_0}^{k_\infty} n^{3/4} e^{-k} \rho \frac{e^{2k}}{n}$$

$$\leq Cc^* \rho + Cc^* \rho k_0 + Cc^* \rho n^{-1/4} e^{k_\infty}$$

$$\leq n^{1/4},$$

for c^* small enough. Note that we used $e^{k_0} = n^{1/4}$, $e^{k_\infty} = n^{1/2}$, and $k_0 = C \log n$ in the last inequality.

Because $\mathbf{B}_{e^{k_0}}(p^{\perp}(0))$ has radius $n^{1/4}$, we find that for c^* small enough, $p^{\perp}(t)$ does not exit from this ball, as required.

Step 5: bound on the downward drift. We use the same strategy as in Step 4 to bound the total downward drift. Let D_k^1 be the maximal downward drift experienced by a point $q = (q_1, q^{\perp})$ as in (4.20), assuming that the Brownian particle is located at $r = (r_1, r^{\perp})$. As in step 4, on the first interval $[0, \tau_1]$, the vertical distance (along the x_1-axis) from $\mathbb{A}_t$ to B_t is on the interval $[1/2, 2]$ and so it is bounded below by $1/2$. The lateral distance (along $x^{\perp}$) from B_t to q is $\|q^{\perp} - r^{\perp}\|$. We want to compute an upper bound of the vertical drift on the point q, so we multiply the factor $\frac{1}{\sqrt{1/4 + \|q^{\perp} - r^{\perp}\|^2}}$, which bounds the projection of the drift vector from B_t to q onto the x_1–direction, to the diagonal drift. We get that

$$(4.24) \quad \begin{aligned} D_k^1 &\leq \|F\|_\infty \frac{1}{\sqrt{1/4 + \|q^{\perp} - r^{\perp}\|^2}} \cdot \frac{1}{\sqrt{1/4 + \|q^{\perp} - r^{\perp}\|^2}} \\ &\leq c^* n^{3/4} \frac{1}{1/4 + \|q^{\perp} - r^{\perp}\|^2}. \end{aligned}$$

We again distinguish between the following cases:

Case 1: $r^\perp \in A^1_{p(0)}$. Then from (4.24) it follows that there exists a constant $C > 0$ independent of n and p, such that

$$(4.25) \qquad D^1_1 \le \|F\|_\infty \le Cc^* n^{3/4}.$$

Case 2: $r^\perp \in A^k_{p(0)}$, for $2 \le k \le k_\infty$. Then from (4.24) it follows that there exists a constant $C > 0$ independent of n and p, such that

$$(4.26) \qquad \begin{aligned} D^1_k &\le c^* n^{3/4} \frac{1}{1/4 + e^{2k}} \\ &\le Cc^* n^{3/4} e^{-2k}. \end{aligned}$$

Let $D^1_{\tau_1}$ be the total vertical drift experienced by $p(t)$ between $[0, \tau_1]$. As in the previous step, in each term, we first write our upper bound for the drift from (4.25) and (4.26), then the number of balls traversed, and then the occupation measure of each ball from (4.17). It follows that

$$\begin{aligned}
D^1_{\tau_1} &\le Cc^* n^{3/4} \cdot e^{k_0} \cdot L^{(n-1)}\big[A_1(p^\perp(0))\big] \\
&\quad + \sum_{k=2}^{k_0-1} Cc^* n^{3/4} e^{-2k} \cdot e^{k_0-k} \cdot L^{(n-1)}\big[A_k(p^\perp(0))\big] \\
&\quad + \sum_{k=k_0}^{k_\infty} Cc^* n^{3/4} e^{-2k} \cdot 1 \cdot L^{(n-1)}\big[A_k(p^\perp(0))\big] \\
&\le Cc^* n^{3/4} \cdot n^{1/4} \cdot \rho \frac{1}{n} + Cc^* \sum_{k=0}^{k_0} n^{3/4} e^{k_0-3k} \rho \frac{e^{2k}}{n} \\
&\quad + Cc^* \sum_{k=k_0}^{k_\infty} n^{3/4} e^{-2k} \rho \frac{e^{2k}}{n} \\
&\le Cc^* \rho + Cc^* \rho n^{-1/4} e^{k_0} \sum_{k=0}^{k_0} e^{-k} + Cc^* \rho n^{-1/4} k_\infty \\
&\le Cc^* \rho + Cc^* \rho n^{-1/4} e^{k_0} + Cc^* \rho n^{-1/4} k_\infty.
\end{aligned}$$

Since $e^{k_0} = n^{1/4}$ and $k_\infty = (\log n)/2$, we have

$$D \le Cc^*.$$

By choosing a small c^* we get

$$(4.27) \qquad D^1_{\tau_1} \le 1/8,$$

on the event $\cap_{i=1}^3 \mathcal{E}_i$. Recall D_{τ_1} was defined in (4.3). From (4.27), (4.6), (4.8) and (4.19), we get (4.4) and we finish the proof for $n > 3$.

Step 6: bound D_{τ_1} for $n = 2, 3$. In this case we can bound the total downward drift at x due to $B^\perp_{[0,T]}$ by $c^* C_7$, where C_7 is some positive constant, since the Brownian motion is at a positive distance from x on $[0, \tau_1]$. Again, by choosing c^* small enough, we get

$$D_{\tau_1} < c^* C_7 T \leq 1/8$$

on the event $\mathcal{E}_1$. From the above equation and (4.6) we get (4.4) and we finish the proof for $n = 2, 3$. □

5. Proof of Proposition 10

Recall that our goal is to get an upper bound, with a high probability, on the number of balls of radius r needed to cover the Brownian path $B^\perp_{[0,T]}$.

Let $\tau_{r,1}$ be the first time t that the $(n-1)$-dimensional Brownian motion $B^\perp_t$ initiated at the origin exits the ball $B(0, r)$ centered at 0 of radius r. At $\tau_{r,1}$, we start the Brownian motion from the origin over again and wait till the next time $\tau_{r,2}$ at which the restarted Brownian motion reaches distance r from its starting point. We also write $\tau_{r,0} = 0$. Let

$$\sigma_i = \sigma_{r,i} = \tau_{r,i} - \tau_{r,i-1}.$$

Then $\sigma_i : i = 1, 2, \ldots$ are i.i.d. random variables.

Let $N = N_{r,T}$ be the number of balls of radius r needed to cover the Brownian path $B^\perp_{[0,T]}$. By Brownian scaling, N equals in law to the number of balls of radius 1 needed to cover the Brownian path $B^\perp_{[0,T/r^2]}$. So we can redefine σ_i and N to reflect this Brownian scaling, that is, σ_i are i.i.d. exit times from a unit ball.

We expect σ_i to be about C/n, since the Bessel process $\|B^\perp_t\|$ has drift about n for $1/2 \leq \|B^\perp_t\| \leq 1$ and larger when $\|B^\perp_t\| \leq 1/2$. So, roughly speaking we expect,

$$(5.1) \qquad\qquad N \approx \frac{nT}{r^2}.$$

The following lemma is an important ingredient in the proof of Proposition 10.

Lemma 13. *For any $m \geq 1$, there exist constants $C_8, C_9 > 0$ such that*

$$P\left(\inf_{i=1,\ldots,m} \sigma_i < K/n\right) \leq m C_8 \exp\left(-C_9 n/K\right)$$

for all $0 < K < 1/8$.

Proof. Note that for any $K > 0$ and $m \geq 1$ we have,

$$P\left(\inf_{i=1,...,m} \sigma_i < K/n\right) \leq mP\left(\sigma_1 < K/n\right).$$

Hence we need to show that $mP(\sigma_1 < K/n)$ is small enough.

For simplicity, we let $\sigma = \sigma_1$. Recall that the Bessel process $Y_t = \|B_t^\perp\|$ satisfies the following SDE

$$dY_t = \frac{n-2}{2Y_t} dt + dW_t,$$

where W_t is a standard one dimensional Brownian motion.

Note that σ is greater than $\tilde{\sigma}$, which is the time needed for Y_t to exit the interval $[1/2, 1]$ starting from $Y_0 = 3/4$. Therefore, it suffices to bound

$$mP\left(\tilde{\sigma} < K/n\right).$$

Suppose now that

$$K < 1/8$$

so that within time $t < K/n$, the drift term in Y_t is at most $1/8$. Then, in order for $\tilde{\sigma} < K/n$, we must have

$$\sup_{0 \leq t \leq K/n} |W_t| \geq 1/8.$$

However, a standard Brownian estimate gives

$$P\left(\sup_{0 \leq t \leq K/n} |W_t| \geq 1/8\right) \leq C_8 \exp\left(-C_9 n/K\right),$$

where C_8, C_9 are positive constants.

It follows that

$$P\left(\inf_{i=1,...,m} \sigma_i < K/n\right) \leq mC_8 \exp\left(-C_9 n/K\right).$$

$\square$

Proof of Proposition 10. Recall that $r = e^k$, $k = 1, ..., k_\infty$, so we are covering the Brownian path with balls of radius e^k.

By a scaling argument this is equivalent to a cover the path of $B^\perp$, between 0 to $T/r^2 = Te^{-2k}$ with balls of radius 1. Note that if $\sigma_i \geq K/n$ for all $i = 1, 2, ..$, then we will need

$$m^* = m^*(n) = \lceil \frac{nT}{Ke^{2k}} \rceil,$$

such balls. Let $0 < K < 1/8$. From Lemma 13 it follows that the probability to cover the path with m^* balls of radius 1 is

$$P_k := P\left(\inf_{i=1,\dots,m*} \sigma_i < K/n\right) \leq C_8 \lceil \frac{nT}{Ke^{2k}}\rceil \exp\left(-C_9 n/K\right).$$

The sum of the probabilities P_k over k is bounded by

$$\sum_{k=1}^{\lceil\frac{1}{2}\log n\rceil} P_k \leq C_8 \exp\left(-C_9 n/K\right) \sum_{k=1}^{\lceil\frac{1}{2}\log n\rceil} \lceil \frac{nT}{Ke^{2k}}\rceil$$

$$\leq \widetilde{C}_8 \frac{nT}{K} \exp\left(-C_9 n/K\right).$$

It follows that for any $\varepsilon \in (0,1)$, there exists $N(\varepsilon, K)$ such that if $n > N(\varepsilon, K)$ then for any choice of $r = e^k$, $k = 1, \dots \lceil(\log n)/2\rceil$, we have $\sigma_i \geq K/n$ for all $i = 0, \dots, m^*$ with probability $1 - \varepsilon$. From the preceding paragraph we get the result for $n > N(\varepsilon, K)$.

Note that similar covers in dimensions $n \leq N(\varepsilon, K)$ will hold by projection of the covers when $n > N(\varepsilon, K)$, possibly with an additional multiplicative constant in $m_k^*(n)$. We therefore proved the result for all $n \geq 1$. □

6. Proof of Proposition 12 and Lemma 11

Before we prove Proposition 12, we recall Theorem 2 (in page 444) from Ciesielski and Taylor [1].

Theorem 14 (Paraphrased). *Suppose $n \geq 1$. Let τ_n be the first time that an n dimensional Brownian motion leaves a ball of radius 1 in $\mathbb{R}^n$. Let L_n be the amount of time that the same Brownian motion spends in this ball. Then*

$$\tau_n \stackrel{\mathcal{D}}{=} L_{n+2}.$$

Proof of Proposition 12. Recall that $B^\perp$ is an $n-1$ dimensional Brownian motion. From Theorem 14 we have for dimensions $n \geq 4$,

$$(6.1) \qquad\qquad P(L_{n-1} > s) = P(\tau_{n-3} > s).$$

We will derive an upper bound on the right side of (6.1). For notational convenience, we will first establish an upper bound on $P(\tau_n > s)$ as a function of n and replace n by $n - 3$ later.

First of all, note that this bound only depends on $Y(t) = \|B(t)\|$ for $Y \in (0, 1]$. In this case, the drift of Y satisfies

$$(6.2) \qquad\qquad \frac{n-1}{2Y(t)} \geq \frac{n-1}{2}.$$

Let
$$X(t) = \frac{(n-1)t}{2} + W(t),$$
where $W(t)$ is a one dimensional standard Brownian motion.

From (6.2) and a standard comparison result (see Theorem 1.1 in Chapter V.1 of Ikeda and Watanabe [7]), we compare the drifts of Y and X, we see that $Y(t) > X(t)$ for $0 \le t \le \tau_n$ with probability 1. Note that the result of Ikeda and Watanabe does not cover locally unbounded drift, but this could be argued via a standard truncation argument.

Therefore, from (6.1) we get,
$$P(L_{n+2} > s) \le P(\tau_n^X > s),$$
where τ_n^X denotes the first time t that $X_t = 1$.

Simple calculations give us
$$\begin{aligned} P(\tau_n^X > s) &\le P(X(s) \le 1) \\ &= P((n-1)s/2 + W(s) \le 1) \\ &= P(W(s) \le 1 - (n-1)s/2). \end{aligned}$$

Note that for $s > 4/(n-1)$,
$$1 - \frac{(n-1)s}{2} \le -\frac{(n-1)s}{4}.$$

So, we conclude that

(6.3)
$$\begin{aligned} P(L_{n+2} > s) &\le P(\tau_n^X > s) \\ &\le P\left(W(s) \le -\frac{(n-1)s}{4}\right) \\ &\le C_5' \exp\left(-C_6' n^2 s\right) \end{aligned}$$

for some constants C_5', C_6'. Replacing n by $n-3$ only changes the constants C_5', C_6' in (6.3) but nothing else. $\qquad\square$

6.1. **Proof of Lemma 11.** Recall that we must show that there exist constants $C_3, C_4 > 0$ such that
$$P(\tau_1 > T) \le C_3 \exp(-C_4 T).$$

We distinguish between the following two cases.

Case 1. $F \equiv 0$. Then $P(\tau_1 > T) \le P\left(\sup_{t \in [0,T]} |B_t^{(1)}| \le 2\right)$, and the result follows by a standard Gaussian tail estimate.

Case 2. $\|F\|_\infty > 0$. Note that ρ_t, which is the maximal vertical distance between $\mathcal{A}_t$ and W_t, is given by
$$D_t = \rho_t + B_t^{(1)} - 1,$$

where D_t is the maximal vertical drift accumulated up to time t. From (1.5) with $\|F\|_\infty$ instead of F and (4.2) we get

$$D_t \geq \frac{t}{2}\|F\|_\infty \quad \text{for all } 0 \leq t \leq \tau_1.$$

It follows that

$$\rho_t \geq 1 + \frac{t}{2}\|F\|_\infty + W_t \quad \text{for all } 0 \leq t \leq \tau_1,$$

where W_t is a one dimensional Brownian motion. Let Z be a standard Gaussian random variable. It follows that there exist $C_3, C_4 > 0$ independent from n such that

$$P(\tau_1 > T) \leq P\left(\sup_{0 \leq t \leq T} \rho_t \leq 2\right)$$

$$\leq P\left(1 + \frac{1}{2}\|F\|_\infty T + W_T \leq 2\right)$$

$$= P\left(Z \leq \frac{1}{\sqrt{T}} - \frac{1}{2}\|F\|_\infty \sqrt{T}\right)$$

$$\leq C_3 \exp\left(-C_4 T\right).$$

$\square$

7. Proof of Theorem 3

We use our usual notation for a point $a \in \mathbb{R}^n$, that is $a = (a_1, \ldots, a_n)$ and $a^\perp = (a_2, \ldots, a_n)$. Let $\mathbf{B}_r^{(n-1)}(a^\perp)$ be a ball in $\mathbb{R}^{n-1}$ of radius r and center $a^\perp$, for $a \in \mathbb{R}^n$.

Let $\mathbb{F}$ denote the set of Lipschitz functions $F : \mathbb{R}^n \to \mathbb{R}$ satisfying the hypotheses of Theorems 2 and 3, namely $0 \leq F(x) \leq c^* n^{3/4}$ for all $x \in \mathbb{R}^n$. The following Lemma will help us to prove Theorem 3.

Lemma 15. *Let*

$$\mathcal{A}_0^{F,m} = \{1\} \times \mathbf{B}_m^{(n-1)}(0)$$

and let $\mathcal{A}_t^{F,m}$ be the image of $\mathcal{A}_0^{F,m}$ under our flow $\psi_t^{\mathcal{A}_0^{F,m}}$. Then

$$\lim_{m \to \infty} \inf_{F \in \mathbb{F}} P(B_t \text{ hits } \mathcal{A}_t^{F,m}) = 1.$$

Note that Theorem 2 implies that the above limit is 1 for $F \in \mathbb{F}$ fixed. Lemma 15 states that this convergence is uniform in $F \in \mathbb{F}$.

Proof of Lemma 15. Let (ψ_t^F, B_t) be a solution to (1.5) with drift $F \in \mathbb{F}$, which is initiated by $\mathcal{A}_0 = \{1\} \times \mathbb{R}^{n-1}$. We define σ^F to be the first time that B_t hits $\mathcal{A}_t = \psi_t^F(\mathcal{A}_0)$. Define $\overline{F} \equiv c^* n^{3/4}$, and note that $F \leq \overline{F}$ for all $F \in \mathbb{F}$.

Let $\varepsilon > 0$ be arbitrary small. From the statement of Theorem 2 it follows that there exists T_0 large enough, such that for all $T > T_0$, we have
$$P(\sigma^{\overline{F}} > T) < \varepsilon.$$
We observe that in the proof of Theorem 2 the only bound on F which was used is $\|F\|_\infty \leq c^* n^{3/4}$, see the second paragraph of Step 1 in Proposition 9 and the bounds on $\mathcal{D}_{\tau_1}^\perp$ and $\mathcal{D}_{\tau_1}^1$ in Steps 4 and 5 of Proposition 9 (respectively). We therefore conclude that for all $T > T_0$

(7.1)
$$\sup_{F \in \mathbb{F}} P(\sigma^F > T) < \varepsilon.$$

Now choose $m > 0$ depending on ε and the dimension n such that

(7.2)
$$P\left(\sup_{0 \leq t \leq T} \|B_t^{(n-1)}\| > m \right) < \varepsilon.$$

Let A_m^F be the event that B_t hits $\mathcal{A}_t^{F,m} = \psi_t^F(\mathcal{A}_0^{F,m})$. Also, let $\partial \mathcal{A}_t^{F,m}$ denote the image of the boundary of the original ball $\mathbf{B}_m^{(n-1)}(0)$ under the same drift. We also note that the drift experienced by $\mathcal{A}_t^{F,m}$ is continuous as long as B_t has not yet hit $\mathcal{A}_t^{F,m}$.

Note that as long as $\|B_t^{(n-1)}\| < m$, the drift induced on $\partial \mathcal{A}_t^{F,m}$ by B_t, when projected on $\mathbb{R}^{n-1}$, and further projected onto the radial direction in $\mathbb{R}^{n-1}$, points away from the origin. It follows that the projection of $\partial \mathcal{A}_t^{F,m}$ onto $\mathbb{R}^{n-1}$ lies in the exterior of the ball $\mathbf{B}_m(0)$, and in fact encloses this ball. By continuity, it follows that as long as $\|B_t^{(n-1)}\| < m$, we have $\mathcal{A}_t^{F,m} \supset \mathbf{B}_m^{(n-1)}(0)$.

By the above and (7.1), we have
$$\sup_{F \in \mathbb{F}} P(A_m^F) > 1 - 2\varepsilon$$
and this completes the proof of Lemma 15. $\qquad\square$

Proof of Theorem 3. By the hypothesis of Theorem 3 we can assume that $\mathcal{A}_0$ contains a cylinder
$$\mathbf{C}_\delta(a) = (a_1, a_1 + \delta) \times \mathbf{B}_\delta^{(n-1)}(a^\perp)$$
for some $a \in \mathbb{R}^n$ and some $\delta > 0$. Furthermore, we can assume that $a_1 > 0$.

Let $0 < \varepsilon < a_1/2$, $\varepsilon_1 > 0$, and let σ be the first time t that $B_t^{(1)} = a_1 - \varepsilon/2$. It is easy to see that there is a positive probability that $\|B_\sigma^\perp\| < \varepsilon$ and $\sigma < \varepsilon_1$. Thus, if ε_1 is small enough and $\sigma < \varepsilon_1$, the integral of the drift from time 0 to time σ, acting on $\mathbf{C}_\delta(a)$ up to time σ will be arbitrarily small. It follows that there is a positive probability that $\|B_\sigma^\perp\| < \varepsilon$ and $\psi_\sigma^{\mathcal{A}_0}(\mathbf{C}_\delta(a))$ contains the disc $(a_1+\varepsilon/2) \times \mathbf{B}_{\delta/2}^{(n-1)}(a^\perp)$.

So, $B_\sigma^{(1)} = a_1 - \varepsilon/2$ has distance ε from the line $x_1 = a_1 + \varepsilon/2$, and if $\varepsilon < \delta/4$ then

$$\mathbf{B}_{\delta/2}^{(n-1)}(a^\perp) \supset \mathbf{B}_{\delta/4}^{(n-1)}(B_\sigma^\perp).$$

Using Lemma 15, choose a natural number m such that

$$(7.3) \qquad \inf_{F \in \mathbb{F}} P(B_t \text{ hits } \mathcal{A}_t^{F,m}) > 0.$$

Using the strong Markov property, let us start over at time σ. Next, by translation and by our scaling (Lemma 5), we find that our new Brownian motion starts at 0, and that $(a_1 + \varepsilon/2) \times \mathbf{B}_{\delta/4}^{(n-1)}(B_\sigma^\perp)$ has been transformed to a set containing $\{1\} \times \mathbf{B}_K^{(n-1)}(0)$ with $K > \delta/(4\varepsilon)$. If $\varepsilon > 0$ is small enough, we conclude that $K > m$ and so

$$\{1\} \times \mathbf{B}_K^{(n-1)}(0) \supset \{1\} \times \mathbf{B}_m^{(n-1)}(0).$$

Using (7.3), we conclude that with positive probability, B_t hits $\mathcal{A}_t^{F,K}$. Since our original region $\mathcal{A}_t$, after scaling, contains $\mathcal{A}_t^{F,K}$, it follows that B_t hits $\mathcal{A}_t$ with positive probability.

This completes the proof of Theorem 3. $\qquad \square$

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References

[1] Z. Ciesielski and S. J. Taylor. First passage times and sojourn times for brownian motion in space and the exact hausdorff measure of the sample path. *Transactions of the American Mathematical Society*, 103(3):434–450, 1962.

[2] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[3] R. C. Dalang, D. Khoshnevisan, and E. Nualart. Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, 3:231–271, 2007.

[4] R. C. Dalang and M. Sanz-Solé. Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli*, 16(4):1343–1368, 2010.

[5] R. C. Dalang and M. Sanz-Solé. Hitting probabilities for nonlinear systems of stochastic waves. *Mem. Amer. Math. Soc.*, 237(1120):v+75, 2015.

[6] R. Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[7] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North Holland/Kodansha, Amsterdam, Oxford, New York, 1989.

[8] Makoto Katori, *Bessel processes, Schramm-Loewner evolution, and the Dyson model*, SpringerBriefs in Mathematical Physics, vol. 11, Springer, [Singapore], 2015.

[9] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[10] G. F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[11] P. Mörters and Y. Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[12] C. Mueller. Long-time existence for signed solutions to the heat equation with a noise term. *Ann. Probab.*, 24(1):377–398, 1997.

[13] C. Mueller and E. Pardoux. The critical exponent for a stochastic PDE to hit zero. In *Stochastic analysis, control, optimization and applications*, pages 325–338. Birkhäuser Boston, Boston, MA, 1999.

[14] C. Mueller and R. Tribe. Hitting properties of a random string. *Electronic J. Prob.*, 7:1–29, 2002. Paper no. 10.

[15] E. Nualart and F. Viens. The fractional stochastic heat equation on the circle: Time regularity and potential theory. *Stochastic Processes and their Applications*, 119(5):1505 – 1540, 2009.

[16] Boris Tsirelson and Wendelin Werner, *Lectures on probability theory and statistics*, Lecture Notes in Mathematics, vol. 1840, Springer-Verlag, Berlin, 2004, Lectures from the 32nd Probability Summer School held in Saint-Flour, July 7–24, 2002, Edited by Jean Picard.

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