Multiple points of the Brownian sheet in critical dimensions

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Abstract

It is well-known that an N-parameter d-dimensional Brownian sheet has no k-multiple points when $(k-1)d > 2kN$, and does have such points when $(k-1)d < 2kN$. We complete the study of the existence of k-multiple points by showing that in the critical cases where $(k-1)d = 2kN$, there are a.s. no k-multiple points.

Abbreviated title: Multiple points of the Brownian sheet

1 Introduction and main theorems

Let d and N be positive integers, and let $B = (B^1, \dots, B^d)$ denote an N-parameter Brownian sheet with values in $\mathbb{R}^d$, that is, B is a centered $\mathbb{R}^d$-valued Gaussian random field with continuous sample paths, defined on a probability space $(\Omega, \mathcal{F}, P)$, with parameter set $\mathbb{R}_+^N$ and covariances

$$\mathrm{Cov}(B^i(\mathbf{s}),\, B^j(\mathbf{t})) = \delta_{i,j} \prod_{\ell=1}^{N} (s_\ell \wedge t_\ell),$$

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where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$, $\mathbf{s} = (s_1, \ldots, s_N)$ and $\mathbf{t} = (t_1, \ldots, t_N)$.

The Brownian sheet is perhaps the most studied extension to multiparameter Gaussian processes of classical Brownian motion, to which it reduces when $N = 1$. Khoshnevisan devotes a chapter to this process in his book [6]. The CIME Summer School lectures [1] contain a presentation of the history of the study of this random field, and its connections to statistics, Markov properties, level sets, stochastic partial differential equations, potential theory and Malliavin calculus.

Here, we are interested in a fundamental sample path property of this random field, namely multiple points, or self-intersections. For $\omega \in \Omega$ and integers $k \geq 2$, a point $x \in \mathbb{R}^d$ is a *k-multiple point of* $\mathbf{t} \mapsto B(\mathbf{t}, \omega)$ if there exist distinct parameters $\mathbf{t}^1, \ldots, \mathbf{t}^k \in]0, \infty[^N$ such that $B(\mathbf{t}^1, \omega) = \cdots = B(\mathbf{t}^k, \omega) = x$. We denote the (random, possibly empty) set of all k-multiple points of $\mathbf{t} \mapsto B(\mathbf{t}, \omega)$ by $M_k(\omega)$. Note that $M_{k+1}(\omega) \subset M_k(\omega)$.

Typically, for d small and N large, the set of k-multiple points is a.s. nonempty, while for d large and N small, M_k is empty a.s. See [2] for the history of this problem in the case of Brownian motion ($N = 1$).

When $N > 1$ and $k \geq 2$, it was shown in [5] that k-multiple points exist if $(k-1)d < 2kN$ and do not exist if $(k-1)d > 2kN$. The critical case $k = 2$ and $d = 4N$ was handled in [2], where it was shown, via quantitative estimates on the conditional distribution of a pinned Brownian sheet and a decoupling method, that there are no double points in the critical case.

In this paper, we solve the remaining critical cases, where $N > 1$, $k \geq 2$ and $(k-1)d = 2kN$. The main result of this paper is the following statement concering the absence of k-multiple points in these critical cases.

Theorem 1.1. *Fix $N > 1$ and $k \geq 2$. If N, d and k are such that $(k-1)d = 2kN$, then an N-parameter d-dimensional Brownian sheet has no k-multiple points, that is, $P\{M_k \neq \emptyset\} = 0$.*

The proof of this theorem relies on known results for hitting probabilities of the Brownian sheet, due to Khoshnevisan and Shi [7], on results for intersections of k independent Brownian sheets, due to Peres [10], and a decoupling idea. While [2] used quantitative estimates to obtain their decoupling, we will achieve our decoupling here by using Girsanov's theorem. Our decoupling result is the following.

Let $\mathcal{T}_N^k$ denote the set of parameters $(\mathbf{t}^1,\ldots,\mathbf{t}^k)$ with $\mathbf{t}^i \in]0,\infty[^N$ such that no two $\mathbf{t}^i$ and $\mathbf{t}^j$ $(i \neq j)$ share a common coordinate:

$$\mathcal{T}_N^k = \{(\mathbf{t}^1,\ldots,\mathbf{t}^k) \in (]0,\infty[^N)^k : t_\ell^i \neq t_\ell^j, \text{ for all } \ell = 1,\ldots,N \\ \text{and } 1 \leq i < j \leq k\}$$

(here, $\mathbf{t}^i = (t_1^i,\ldots,t_N^i)$, so in our notation, the coordinates t_ℓ^i of $\mathbf{t}^i$ inherit the superscript).

Theorem 1.2. *Let $A \subset \mathbb{R}^d$ be a Borel set. For all $k \in \{2,3,\ldots\}$, we have*

$$P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k) \in \mathcal{T}_N^k : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \in A\} > 0$$

if and only if

$$P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k) \in \mathcal{T}_N^k : W_1(\mathbf{t}^1) = \cdots = W_k(\mathbf{t}^k) \in A\} > 0,$$

where $W_1,\ldots,W_k$ are independent N-parameter Brownian sheets with values in $\mathbb{R}^d$.

The proof of this theorem uses an explicit formula for the conditional expectation $\tilde{B}(t)$ of $B(t)$ given the values of the sheet in a product of $N-1$ complements of intervals and a single interval (see Lemma 3.2), together with the fact that Girsanov's theorem can be used to show that the law of the process $B(t) - \tilde{B}(t)$ is mutually absolutely continuous with respect to the law of B (see Lemma 3.5).

In order to deal with the possibility of a k-multiple point arising from parameters $\mathbf{t}^1,\ldots,\mathbf{t}^k$ that share a common coordinate, define

$$\mathcal{H}_N^k(i,j;\ell) = \left\{(\mathbf{t}^1,\ldots,\mathbf{t}^k) \in (]0,\infty[^N)^k : t_\ell^i = t_\ell^j\right\}.$$

That is, $\mathcal{H}_N^k(i,j;\ell)$ is the set of $(\mathbf{t}^1,\ldots,\mathbf{t}^k)$ for which $\mathbf{t}^i$ and $\mathbf{t}^j$ share their ℓ-th coordinate.

Our next theorem states that in the critical case $(k-1)d = 2kN$, there are (with probability one) no k-multiple points arising from parameters in $\mathcal{H}_N^k(i,j;\ell)$.

Theorem 1.3. *Suppose $(k-1)d = 2kN$, $1 \leq i < j \leq k$ and $1 \leq \ell \leq N$. Then*

$$P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k) \in \mathcal{H}_N^k(i,j;\ell) : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k)\} = 0.$$

This theorem is proved by using a covering argument. It requires checking that certain finite-dimensional distributions of increments of the Brownian sheet have a uniformly bounded density, provided the increments are taken at points that are at least δ units apart ($\delta > 0$): see Lemma 2.4. This uses an explicit formula for the conditional expectation $\bar{B}(t)$ of $B(t)$ given the values of the sheet in a product of N complements of intervals (see Lemma 2.1).

The paper is structured as follows. First, in Section 2, assuming Theorems 1.2 and 1.3, we easily deduce Theorem 1.1 from the results of Khoshnevisan and Shi [7] and Peres [10]. Then we prove Theorem 1.3 via an argument based on Hausdorff dimension, as just mentioned. Finally, in Section 3, we show how to use Girsanov's theorem in order to prove Theorem 1.2.

2 Proof of Theorems 1.1 and 1.3

We first prove Theorem 1.1, assuming Theorems 1.2 and 1.3.

Proof of Theorem 1.1. Clearly,

$$\begin{aligned} &P\{M_k \neq \emptyset\} \\ &\qquad \leq P\{\exists (\mathbf{t}^1, \ldots, \mathbf{t}^k) \in \mathcal{T}_N^k : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k)\} \\ &\qquad\quad + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{\ell=1}^{N} P\{\exists (\mathbf{t}^1, \ldots, \mathbf{t}^k) \in \mathcal{H}_N^k(i,j;\ell) : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k)\}. \end{aligned}$$

By Theorem 1.3, the second term vanishes, and by Theorem 1.2, the first term vanishes if and only if

$$P\{\exists (\mathbf{t}^1, \ldots, \mathbf{t}^k) \in \mathcal{T}_N^k : W_1(\mathbf{t}^1) = \cdots = W_k(\mathbf{t}^k)\} = 0, \tag{2.1}$$

where $W_1, \ldots, W_k$ are independent N-parameter Brownian sheets with values in $\mathbf{R}^d$. According to [7], for all sets of the form $R = \prod_{\ell=1}^N [s_\ell^0, s_\ell^1] \subset]0, \infty[^N$, there is a finite constant $C \geq 1$ such that for all nonrandom Borel sets $A \subset \mathbb{R}^d$ contained in a fixed compact subset of $\mathbb{R}^d$,

$$C^{-1} \operatorname{Cap}_{d-2N}(A) \leq P\{\exists \mathbf{t} \in R : W^i(\mathbf{t}) \in A\} \leq C \operatorname{Cap}_{d-2N}(A),$$

where $\operatorname{Cap}(\cdot)$ denotes Bessel-Riesz capacity. We recall that $\operatorname{Cap}(A)$ is defined as follows. Let $\mathcal{P}(K)$ denote the collection of all probability measures that are supported by the Borel set $K \subseteq \mathbb{R}^d$, and define the β-dimensional *capacity* of A by

$$\mathrm{Cap}_\beta(A) := \left[\inf_{\substack{\mu \in \mathcal{P}(K): \\ K \subset A \text{ is compact}}} \mathrm{I}_\beta(\mu) \right]^{-1},$$

where $\inf \varnothing := \infty$, and $\mathrm{I}_\beta(\mu)$ is the β-dimensional *energy* of μ, defined as follows for all $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\beta \in \mathbb{R}$:

$$\mathrm{I}_\beta(\mu) := \iint \kappa_\beta(x-y)\, \mu(dx)\, \mu(dy).$$

In this formula, the function $\kappa_\beta : \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$\kappa_\beta(x) := \begin{cases} \|x\|^{-\beta} & \text{if } \beta > 0, \\ \log_+(\|x\|^{-1}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0, \end{cases}$$

where, as usual, $1/0 := \infty$ and $\log_+(z) := 1 \vee \log(z)$ for all $z \geq 0$.

Since $d - 2N > 0$ because $(k-1)d = 2kN$, it follows from [10, Corollary 15.4] that (2.1) is equivalent to $\mathrm{Cap}_{k(d-2N)}(\mathbb{R}^d) = 0$. According to [6, Appendix C, Corollary 2.3.1], this is indeed the case since $k(d-2N) = d$, because we are in the critical dimension where $(k-1)d = 2kN$. □

Before proving Theorem 1.3, we need some preliminary lemmas. For $U \subset \mathbb{R}_+^N$, we set $\mathcal{F}(U) = \sigma(B(\mathbf{t}),\, \mathbf{t} \in U)$.

Lemma 2.1. *For $\ell = 1, \ldots, N$, fix $0 < s_\ell^0 < s_\ell^1$, and set*

$$R = \prod_{\ell=1}^N [s_\ell^0, s_\ell^1] \qquad \text{and} \qquad S = \prod_{\ell=1}^N]s_\ell^0, s_\ell^1[^c.$$

Let $\mathcal{J}$ denote the set of functions from $\{1, \ldots, N\}$ into $\{0, 1\}$. Then for $\mathbf{t} \in R$, set

$$\bar{B}(\mathbf{t}) = \sum_{\gamma \in \mathcal{J}} \left(\prod_{\ell \in \gamma^{-1}(\{1\})} \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} \right) \left(\prod_{\ell \in \gamma^{-1}(\{0\})} \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right) B(s_1^{\gamma(1)}, \ldots, s_N^{\gamma(N)}) \tag{2.2}$$

(we use the convention that a product over an empty set of indices is equal to 1). Then $\bar{B}(\mathbf{t}) = E(B(\mathbf{t}) \mid \mathcal{F}(S))$.

Remark 2.2. *The set of corners (extreme points) of R is*

$$\mathcal{C} = \{(s_1^{\gamma(1)}, \ldots, s_N^{\gamma(N)}) : \gamma \in \mathcal{J}\},$$

so the sum over γ in (2.2) *involves B evaluated at each corner of R.*

Proof of Lemma 2.1. Since the components of B are independent, we may and will assume in this proof that $d = 1$. In this case, since we are working with Gaussian random variables, it suffices to prove that for each $\mathbf{s} \in S$,

$$E(\bar{B}(\mathbf{t})B(\mathbf{s})) = E(B(\mathbf{t})B(\mathbf{s})). \tag{2.3}$$

The right-hand side of (2.3) is equal to $\prod_{\ell=1}^N (t_\ell \wedge s_\ell)$, so we compute the left-hand side of (2.3). Clearly,

$$\begin{aligned} E(\bar{B}(\mathbf{t})B(\mathbf{s})) &= \sum_{\gamma \in \mathcal{J}} \left(\prod_{\ell \in \gamma^{-1}(\{1\})} \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} \right) \left(\prod_{\ell \in \gamma^{-1}(\{0\})} \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right) \prod_{\ell=1}^N (s_\ell^{\gamma(\ell)} \wedge s_\ell) \\ &= \prod_{\ell=1}^N \left[(s_\ell^1 \wedge s_\ell) \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + (s_\ell^0 \wedge s_\ell) \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right]. \end{aligned}$$

Therefore, (2.3) will be proved if we show that for each $\ell \in \{1, \ldots, N\}$,

$$t_\ell \wedge s_\ell = (s_\ell^1 \wedge s_\ell) \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + (s_\ell^0 \wedge s_\ell) \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0}. \tag{2.4}$$

There are two cases to distinguish:

Case 1. $s_\ell \leq s_\ell^0$. In this case, $s_\ell^k \wedge s_\ell = s_\ell$ for $k \in \{0, 1\}$ and $t_\ell \wedge s_\ell = s_\ell$, since $s_\ell^0 \leq t_\ell \leq s_\ell^1$, so the right-hand side of (2.4) is equal to

$$s_\ell \, \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + s_\ell \, \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} = s_\ell,$$

which is also the left-hand side of (2.4).

Case 2. $s_\ell \geq s_\ell^1$. In this case, $s_\ell^k \wedge s_\ell = s_\ell^k$ for $k \in \{0, 1\}$ and $t_\ell \wedge s_\ell = t_\ell$, so the right-hand side of (2.4) is equal to

$$s_\ell^1 \, \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + s_\ell^0 \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} = t_\ell,$$

and which is also the left-hand side of (2.4).

This completes the proof of Lemma 2.1. $\square$

Remark 2.3. *We note that the right-hand side of* (2.2) *is in fact a convex combination of the values of B at the corners of R, since each coefficient is non-negative and*

$$\sum_{\gamma\in\mathcal{J}}\left(\prod_{\ell\in\gamma^{-1}(\{1\})}\frac{t_\ell-s_\ell^0}{s_\ell^1-s_\ell^0}\right)\left(\prod_{\ell\in\gamma^{-1}(\{0\})}\frac{s_\ell^1-t_\ell}{s_\ell^1-s_\ell^0}\right)$$

$$=\prod_{\ell=1}^{N}\left[\frac{t_\ell-s_\ell^0}{s_\ell^1-s_\ell^0}+\frac{s_\ell^1-t_\ell}{s_\ell^1-s_\ell^0}\right]=1.$$

Lemma 2.4. *Fix $\delta>0$ (small), $K\in\mathbb{N}$ (positive and large), and $k\in\mathbb{N}$, $k\geq 2$.*

(a) There is $C>0$ such that for all $\mathbf{t}^1,\dots,\mathbf{t}^k$ such that $\|\mathbf{t}^i-\mathbf{t}^j\|\geq\delta$, for all $i\neq j$ with $i,j\in\{1,\dots,k\}$, and $K\geq t_\ell^i\geq\delta$, for all $\ell=1,\dots,N$ and $i\in\{1,\dots,k\}$, the random vector $(B(\mathbf{t}^1),\dots B(\mathbf{t}^k))$ has a joint probability density function that is bounded by C.

(b) For the same choices of $\mathbf{t}^1,\dots,\mathbf{t}^k$, the $(\mathbb{R}^d)^{k-1}$-valued random vector

$$(B(\mathbf{t}^1)-B(\mathbf{t}^2),B(\mathbf{t}^2)-B(\mathbf{t}^3),\dots B(\mathbf{t}^{k-1})-B(\mathbf{t}^k))$$

has a bounded probability density function (with bound depending only on δ, K and k, as well as d and N).

Proof. Since the $B^1,\dots,B^d$ are independent Brownian sheets, we may and will assume in this proof that $d=1$.

We first deduce (b) from (a). Let

$$Y=(B(\mathbf{t}^1)-B(\mathbf{t}^2),\dots,B(\mathbf{t}^{k-1})-B(\mathbf{t}^k),B(\mathbf{t}^k)).$$

Then Y is obtained from $(B(\mathbf{t}^1),\dots,B(\mathbf{t}^k))$ by applying an invertible linear transformation from $(\mathbb{R}^d)^k$ into $(\mathbb{R}^d)^k$. Therefore, by (a), Y has a bounded joint probability density function. It follows that the probability density function of $(B(\mathbf{t}^1)-B(\mathbf{t}^2),\dots,B(\mathbf{t}^{k-1})-B(\mathbf{t}^k))$, which is a marginal density of Y, is bounded by the same constant. This proves (b).

We now prove (a). Set

$$n = \inf\left\{n \in \mathbb{N} : 2^{-n} < \frac{\delta}{3\sqrt{N}}\right\},$$

and consider a dyadic grid in $\mathbb{R}_+^N$ with edges of length 2^{-n}. We let $G_{\delta,K}$ denote the set of such grid points with all coordinates $\leq K$.

By construction, each closed box in this grid contains at most one of the $\mathbf{t}^i$, and we denote by R^i the box containing $\mathbf{t}^i$. Suppose that

$$R^i = \prod_{\ell=1}^{N}[s_\ell^{i,0}, s_\ell^{i,1}], \qquad \text{and set} \qquad S^i = \prod_{\ell=1}^{N}]s_\ell^{i,0}, s_\ell^{i,1}[^c.$$

Because of our choice of n, the set $\mathcal{C}^i$ of corners of R^i is distinct from $\mathcal{C}^j$ when $i \neq j$.

Define

$$Y^i = E(B(\mathbf{t}^i)|\mathcal{F}(S^i)), \qquad i = 1, \ldots, k.$$

Then $B(\mathbf{t}^i) - Y^i$ is orthogonal to Y_i, and for $j \neq i$, since Y^j is a linear combination of values of B at elements of S^i (because $\mathcal{C}^j \cap \mathcal{C}^i = \emptyset$), $B(\mathbf{t}^i) - Y^i$ is orthogonal to Y^j. Letting $Y = (Y^1, \ldots, Y^k)$ and $Z = (B(\mathbf{t}^1) - Y^1, \ldots, B(\mathbf{t}^k) - Y^k)$, we see that the Gaussian vectors Y and Z are independent, and

$$(B(\mathbf{t}^1), \ldots, B(\mathbf{t}^k)) = Y + Z.$$

Using properties of convolution, we see that it suffices to show that the joint probability density function of Y is bounded (uniformly over the $(\mathbf{t}^1, \ldots, \mathbf{t}^k)$).

Since Y is a Gaussian random vector, let M be its variance-covariance matrix. It suffices to show that

$$\det M > c > 0, \tag{2.5}$$

where c depends only on δ, K and k, as well as d and N.

Consider the random vector $(B(\mathbf{r}),\, \mathbf{r} \in G_{\delta,K})$. Observe that this random vector can be obtained by applying an invertible linear transformation, from $\mathbb{R}^{((2^n K)^N)}$ into itself (recall that $d = 1$), to the random vector $(W(R)$, R a box in the grid), which has i.i.d. components, each with variance $(2^{-n})^N > 0$. Therefore, $(B(\mathbf{r}), \mathbf{r} \in G_{\delta,K})$ has a bounded density, where the bound depends only on δ and K (and d and N). This implies that $(B(\mathbf{t}),\, t \in \mathcal{C}^i,\, i = 1, \ldots, k)$

has a joint probability density function that is bounded, since it is a marginal density of $(B(\mathbf{r}),\, \mathbf{r} \in G_{\delta,K})$.

Let $\tilde{M}$ be the variance-covariance matrix of the Gaussian random vector $(B(\mathbf{t}),\, \mathbf{t} \in \mathcal{C}^i,\, i = 1,\dots,k)$. Then by the above, there is $c > 0$ such that $\det \tilde{M} > C$. In particular, there is $c_0 > 0$ such that

$$\lambda^T \tilde{M} \lambda \geq c_0 \|\lambda\|^2, \qquad \text{for all } \lambda \in \mathbb{R}^{k2^N}.$$

Note that c_0 depends only on (δ, K, k, d, N).

Let $\mu \in \mathbb{R}^k$. Then

$$\begin{aligned} \mu^T M \mu &= \operatorname{Var}\left(\sum_{i=1}^k \mu_i Y^i\right) \\ &= \operatorname{Var}\left(\sum_{i=1}^k \mu_i \sum_{\mathbf{s}^{i,j} \in \mathcal{C}^i} a_{i,j} B(\mathbf{s}^{i,j})\right) \\ &\geq c_0 \sum_{i=1}^k \sum_{\mathbf{s}^{i,j} \in \mathcal{C}^i} \mu_i^2 a_{i,j}^2, \end{aligned}$$

where the $a_{i,j}$ are the coefficients obtained in formula (2.2) of Lemma 2.1. According to Remark 2.3, $\sum_{s^{i,j} \in \mathcal{C}^i} a_{i,j} = 1$ and $a_{i,j} \geq 0$, therefore, there is $\alpha > 0$ such that $\sum_{s^{i,j} \in \mathcal{C}^i} a_{i,j}^2 > \alpha$. We conclude that

$$\mu^T M \mu \geq c_0\, \alpha \sum_{i=1}^k \mu_i^2,$$

and this implies that $\det M > c_1 > 0$, where c_1 depends only on (δ, K, k, d, N). In turn, this proves (2.5) and completes the proof of (a) in Lemma 2.4. □

Proof of Theorem 1.3. It suffices to prove the theorem in the case where $i = 1, j = 2$ and $\ell = 1$. Therefore, we write $\mathcal{H}_N^k$ instead of $\mathcal{H}_N^k(1,2;1)$.

For $\delta > 0$, set

$$\begin{aligned} \mathcal{H}_N^k(\delta) = \{(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{H}_N^k : t_\ell^i \geq \delta,\ \|\mathbf{t}^i - \mathbf{t}^j\| \geq \delta, \\ \text{for all } i \neq j,\ \ell = 1, \dots, N,\ i, j \in \{1, \dots, k\}\}. \end{aligned}$$

Since $\mathcal{H}_N^k = \cup_{n=1}^{\infty} \mathcal{H}_N^k\left(\frac{1}{n}\right)$, it suffices to prove that for fixed $\delta > 0$,

$$P\{\exists(\mathbf{t}^1, \ldots, \mathbf{t}^k) \in \mathcal{H}_N^k(\delta) : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k)\} = 0.$$

Consider the random field indexed by $(]0, \infty[^N)^k$ with values in $(\mathbb{R}^d)^{k-1}$ defined by

$$X(\mathbf{t}^1, \ldots, \mathbf{t}^k) = (B(\mathbf{t}^1) - B(\mathbf{t}^2), B(\mathbf{t}^2) - B(\mathbf{t}^3), \ldots, B(\mathbf{t}^{k-1}) - B(\mathbf{t}^k)).$$

Then

$$B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \iff X(\mathbf{t}^1, \ldots, \mathbf{t}^k) = 0,$$

so parameters which give rise to a k-multiple point of B are k-tuples at which X hits 0 ($\in (\mathbb{R}^d)^{k-1}$). Therefore, it will suffice to show that

$$P\{\exists(\mathbf{t}^1, \ldots, \mathbf{t}^k) \in \mathcal{H}_N^k(\delta) : X(\mathbf{t}^1, \ldots, \mathbf{t}^k) = 0\} = 0. \tag{2.6}$$

Let $D(K) = \mathcal{H}_N^k(\delta) \cap ([0, K]^N)^k$. Since $\mathcal{H}_N^k$ is a vector space of dimension $kN - 1$, there is $C > 0$ such that for all large $n \geq 1$, we can cover $D(K)$ by $C(2^{2n})^{kN-1}$ dyadic boxes in $(\mathbb{R}^N)^k$ with edges of length 2^{-2n}. Let $\mathcal{D}_n$ be the set of boxes in such a covering, and for $D \in \mathcal{D}_n$, let $t_n(D)$ be the corner of D for which all coordinates are smallest possible.

For $(\mathbf{t}^1, \ldots, \mathbf{t}^k) \in D$, let $p_{(\mathbf{t}^1,\ldots,\mathbf{t}^k)}(z_1, \ldots, z_{k-1})$ be the value of the joint probability density function of $X(\mathbf{t}^1, \ldots, \mathbf{t}^k)$ at $(z_1, \ldots, z_{k-1}) \in (\mathbb{R}^d)^{k-1}$. By Lemma 2.4, there is $C < +\infty$ such that

$$p_{(\mathbf{t}^1,\ldots,\mathbf{t}^k)}(z_1, \ldots, z_{k-1}) \leq C. \tag{2.7}$$

Let $B(0, n2^{-n})$ denote the ball in $(\mathbb{R}^d)^{k-1}$ centered at 0 with radius $n2^{-n}$. By (2.7),

$$P\{X(\mathbf{t}^1, \ldots, \mathbf{t}^k) \in B(0, n2^{-n})\} \leq C(n2^{-n})^{d(k-1)}. \tag{2.8}$$

In order to prove (2.6), it suffices to prove (2.6) with $\mathcal{H}_N^k(\delta)$ replaced by

$D(K)$. So we compute

$$\begin{aligned}
&P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k)\in D(K): X(\mathbf{t}^1,\ldots,\mathbf{t}^k)=0\}\\
&\quad\le P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k)\in D(K): X(\mathbf{t}^1,\ldots,\mathbf{t}^k)\in B(0,2^{-n})\}\\
&\quad\le \sum_{D\in\mathcal{D}_n} P\{\exists(\mathbf{t}^1,\ldots,\mathbf{t}^k)\in D: X(\mathbf{t}^1,\ldots,\mathbf{t}^k)\in B(0,2^{-n})\}\\
&\quad\le \sum_{D\in\mathcal{D}_n} P\Big(\{X(t_n(D))\in B(0,n2^{-n})\}\\
&\qquad\qquad\qquad \cup\left\{\sup_{t\in D}\|X(t)-X(t_n(D))\|\ge(n-1)2^n\right\}\Big).
\end{aligned}$$

We now use (2.8) to bound this by

$$2^{2n(kN-1)}\left(C(n2^{-n})^{d(k-1)}+P\left\{\sup_{t\in D}\|X(t)-X(t_n(D))\|\ge(n-1)2^{-n}\right\}\right).$$

We will show below that

$$\lim_{n\to+\infty} 2^{2n(kN-1)}P\left\{\sup_{t\in D}\|X(t)-X(t_n(D))\|\ge(n-1)2^{-n}\right\}=0, \qquad (2.9)$$

so it remains to examine the term $n^{d(k-1)}(2^{-n})^{d(k-1)-2kN+2}$. Since we are in the critical case, $2kN=(k-1)d$, so the exponent of 2^{-n} is equal to 2, and therefore

$$n^{d(k-1)}(2^{-n})^{d(k-1)-2kN+2}=n^{d(k-1)}2^{-2n}\to 0$$

as $n\to+\infty$. This will prove (2.8) and complete the proof of Theorem 1.3 once we establish (2.9)), to which to now turn.

We can write $D=D_1\times\cdots\times D_k$, where each D_i is a box in $\mathbb{R}^N$ with edges of length 2^{-2n}, and we can write $t_n(D)=(t_n^1(D_1),\ldots,t_n^k(D_k))$. Clearly,

$$\|X(t)-X(t_n(D))\|\le 2\sum_{i=1}^k\|B(\mathbf{t}^i)-B(t_n^i(D_i))\|,$$

so it suffices to prove that for each $i\in\{1,\ldots,k\}$ and n sufficiently large, there are constants $C<\infty$ and $c>0$ such that

$$P\left\{\sup_{\mathbf{t}^i\in D_i}\|B(\mathbf{t}^i)-B(t_n^i(D_i))\|\ge\frac{(n-1)2^{-n}}{2k}\right\}\le Ce^{-c^2(n-1)^2}. \qquad (2.10)$$

In order to simplify the notation, we assume that $D_i = [1, 1 + 2^{-2n}]^N$, so $t_n^i(D_i) = (1, \dots, 1)$, and we write $\mathbf{t}^i = (t_1^i, \dots, t_N^i)$. We use the decomposition of the Brownian sheet presented in [4, Proof of Theorem (1,1)], to write

$$B(\mathbf{t}^i) - B(t_n^i(D_i)) = \sum_{m=1}^{N} \sum_{1 \leq \ell_1 < \cdots < \ell_m \leq N} W_{\ell_1, \dots, \ell_m}(t_{\ell_1}^i - 1, \dots, t_{\ell_m}^i - 1),$$

where the $W_{\ell_1, \dots, \ell_m}$ are mutually independent Brownian sheets. There are $2^N - 1$ terms in this decomposition, so, using the scaling property of the Brownian sheet [11, Chapter 1], we see that

$$P\left\{ \sup_{\mathbf{t}^i \in D_i} \|B(\mathbf{t}^i) - B(t_n^i(D_i))\| \geq \frac{(n-1)2^{-n}}{2k} \right\}$$
$$\leq \sum_{m=1}^{N} \sum_{1 \leq \ell_1 < \cdots < \ell_m \leq N} P\left\{ \sup_{\mathbf{t} \in [0,1]^m} W_{\ell_1, \dots, \ell_m}(\mathbf{t}) \geq \frac{(n-1)2^{(m-1)n}}{2k2^N} \right\}.$$

Using [9, Lemma 1.2], we see that the largest probability in this sum is obtained when $m = 1$, and in this case it is bounded by $4^N P\{Z \geq c(n-1)\}$, where Z is a standard normal random variable and $c = 2^{-N-1}/k$. Therefore,

$$P\left\{ \sup_{\mathbf{t}^i \in D_i} \|B(\mathbf{t}^i) - B(t_n^i(D_i))\| \geq \frac{(n-1)2^{-n}}{2k} \right\} \leq 8^N e^{-c^2(n-1)^2},$$

which proves (2.10) and completes the proof of Theorem 1.3. □

3 Proof of Theorem 1.2

The main ingredient in the proof of Theorem 1.2 is the following result.

Theorem 3.1. *Let $W_1, \dots, W_k$ be independent Brownian sheets. Fix $M > 0$ and let $\mathcal{R}_M$ denote the set of k-tuples of boxes $(R_1, \dots, R_k)$, where each box R_i is contained in $[M^{-1}, M]^N$ and for each coordinate axis, the projections of the R_i onto this coordinate axis are pairwise disjoint. Then, for all $(R_1, \dots, R_k) \in \mathcal{R}_M$, the random vectors*

$$(B|_{R_1}, \dots, B|_{R_k}) \qquad \text{and} \qquad (W_1|_{R_1}, \dots, W_k|_{R_k})$$

(with values in $(C(R_1, \mathbb{R}^d) \times \cdots \times C(R_k, \mathbb{R}^d))$) have mutually absolutely continuous probability distributions.

Before proving Theorem 3.1, we show that it readily implies Theorem 1.2.

Proof of Theorem 1.2. Let $A \subset \mathbb{R}^d$ be a Borel set. Fix $M > 0$ and set $\mathcal{T}_N^k(M) = \mathcal{T}_N^k \cap [M^{-1}, M]^N$. Then $\mathcal{T}_N^k = \cup_{M=1}^{\infty} \mathcal{T}_N^k(M)$. Therefore,

$$P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{T}_N^k : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \in A\} = 0 \tag{3.1}$$

is equivalent to

$$\forall M \in \mathbb{N}^*,\ P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{T}_N^k(M) : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \in A\} = 0,$$

and this in turn is equivalent to

$$\begin{aligned} &\forall M \in \mathbb{N}^*,\ \forall(R_1, \dots, R_k) \in \mathcal{R}_M, \\ &\quad P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in R_1 \times \cdots \times R_k : B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \in A\} = 0. \end{aligned} \tag{3.2}$$

Similarly, the property

$$P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{T}_N^k : W_1(\mathbf{t}^1) = \cdots = W_k(\mathbf{t}^k) \in A\} = 0 \tag{3.3}$$

is equivalent to

$$\begin{aligned} &\forall M \in \mathbb{N}^*, \forall(R_1, \dots, R_k) \in \mathcal{R}_M : \\ &\quad P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in R_1 \times \cdots \times R_k : W_1(\mathbf{t}^1) = \cdots = W_k(\mathbf{t}^k) \in A\} = 0. \end{aligned} \tag{3.4}$$

According to Theorem 3.1, properties (3.2) and (3.4) are equivalent, and therefore (3.1) and (3.3) are also equivalent. This proves Theorem 1.2. □

For Theorem 3.1, we will need a variant of Lemma 2.1.

Lemma 3.2. *For $\ell = 1, \dots, N$, fix $0 < s_\ell^0 < s_\ell^1$ and set*

$$R = \prod_{\ell=1}^{N} [s_\ell^0, s_\ell^1] \qquad \textit{and} \qquad S = \left(\prod_{\ell=1}^{N-1}]s_\ell^0, s_\ell^1[^c \right) \times [0, s_N^0].$$

Let $\mathcal{J}_N$ denote the set of functions from $\{1, \dots, N-1\}$ into $\{0, 1\}$ and set

$$\mathcal{C}_N = \left\{ \left(s_1^{\gamma(1)}, s_2^{\gamma(2)}, \dots, s_N^{\gamma(N-1)}, s_N^0 \right) : \gamma \in \mathcal{J}_N \right\}.$$

For $\mathbf{t} \in R$, *set*

$$\tilde{B}(\mathbf{t}) = \sum_{\gamma \in \mathcal{J}_N} \left(\prod_{\ell \in \gamma^{-1}(\{1\})} \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} \right) \left(\prod_{\ell \in \gamma^{-1}(\{0\})} \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right) \times B\left(s_1^{\gamma(1)}, \dots, s_{N-1}^{\gamma(N-1)}, s_N^0 \right).$$

Then $\tilde{B}(\mathbf{t}) = E(B(\mathbf{t}) \mid \mathcal{F}(S))$.

Remark 3.3. $\mathcal{C}_N$ *is the set of corners of* R *with the smallest of the two possible* N*-th coordinates, and* S_N *is in the "past" of* R *if we define the "past" using the (partial) order* $\mathbf{s} \leq_N \mathbf{t}$ *if and only if* $s_N \leq t_N$.

Proof of Lemma 3.2. Since the components of B are independent, we may and will assume in this proof that $d = 1$. In this case, as in the proof of Lemma 2.1, it suffices to prove that for each $\mathbf{s} \in S$,

$$E(\tilde{B}_N(\mathbf{t})B(\mathbf{s})) = E(B(\mathbf{t})B(\mathbf{s})). \tag{3.5}$$

The right-hand side of (3.5) is equal to $s_N \prod_{\ell=1}^{N-1}(t_\ell \wedge s_\ell)$, so we compute the left-hand side of (3.5). Clearly,

$$\begin{aligned} E(\tilde{B}(\mathbf{t})B(\mathbf{s})) &= s_N \sum_{\gamma \in \mathcal{J}_N} \left[\prod_{\ell \in \gamma^{-1}(\{1\})} \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} \right] \left[\prod_{\ell \in \gamma^{-1}(\{0\})} \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right] (s_\ell^{\gamma(\ell)} \wedge s_\ell) \\ &= s_N \prod_{\ell=1}^{N-1} \left[(s_\ell^1 \wedge s_\ell) \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + (s_\ell^0 \wedge s_\ell) \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0} \right], \end{aligned}$$

so (3.5) will be proved if we check that for each $\ell \in \{1, \dots, N-1\}$,

$$t_\ell \wedge s_\ell = (s_\ell^1 \wedge s_\ell) \frac{t_\ell - s_\ell^0}{s_\ell^1 - s_\ell^0} + (s_\ell^0 - s_\ell) \frac{s_\ell^1 - t_\ell}{s_\ell^1 - s_\ell^0}.$$

But this is simply equality (2.4), and the proof of Lemma 3.2 is complete. □

We will need the following form of Girsanov's theorem for the Brownian sheet, which is essentially the version given in [8, Proposition 1.6]. Fix $M > 0$. Define the one-parameter filtration $\mathcal{G} = (\mathcal{G}_u,\ u \in [0, M])$ by

$$\mathcal{G}_u = \sigma\left\{ B(t_1, \dots, t_{N-1}, v) : (t_1, \dots, t_{N-1}) \in \mathbb{R}_+^{N-1},\ v \in [0, u] \right\} \tag{3.6}$$

(the filtration is completed and made right-continuous). Let $(Z(\mathbf{s}), \mathbf{s} \in \mathbb{R}_+^{N-1} \times [0,M])$ be a (jointly measurable) $\mathbb{R}^d$-valued random field that is adapted to $\mathcal{G}$, that is, for all $\mathbf{s} \in \mathbb{R}_+^{N-1} \times [0,M]$, $Z(\mathbf{s})$ is $\mathcal{G}_{s_N}$-measurable. Suppose that

$$E\left(\int_{\mathbb{R}_+^{N-1}\times[0,M]} \|Z(\mathbf{s})\|^2 \, d\mathbf{s}\right) < +\infty. \tag{3.7}$$

For $u \in [0,M]$, define

$$L_u = \exp\left(\int_{\mathbb{R}_+^{N-1}\times[0,u]} Z(\mathbf{s}) \cdot dB(\mathbf{s}) - \frac{1}{2}\int_{\mathbb{R}_+^{N-1}\times[0,u]} \|Z(\mathbf{s})\|^2 \, d\mathbf{s}\right),$$

where "$\cdot$" denotes the Euclidean inner product and, for each component, the stochastic integral $\int Z^i(\mathbf{s})\, dB^i(\mathbf{s})$ is defined in the sense of [11], with the N-th coordinate playing the role of the time variable and the other coordinates playing the role of the spatial variables.

Theorem 3.4. *(Girsanov) If $(Z(\mathbf{s}), \mathbf{s} \in \mathbb{R}_+^{N-1} \times [0,M])$ is such that $(L_u, u \in [0,M])$ is a martingale with respect to the filtration $\mathcal{G}$, then the process $(\tilde{B}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^{N-1} \times [0,M])$ defined by*

$$\tilde{B}(t_1,\dots,t_N) = B(t_1,\dots,t_N) - \int_{[0,t_1]\times\cdots\times[0,t_N]} Z(s_1,\dots,s_N)\, ds_1 \cdots ds_N$$

is an $\mathbb{R}^d$-valued Brownian sheet under the probability measure Q, where Q is defined by

$$\frac{dQ}{dP} = L_M.$$

We now fix $k \geq 2$ and consider k boxes $R_1, \dots, R_k$ as in the statement of Theorem 3.1:

$$R_j = \prod_{\ell=1}^{N} [s_{j,\ell}^0, s_{j,\ell}^1], \qquad j = 1,\dots,k,$$

where, for $\ell = 1,\dots,N$, the intervals

$$[s_{1,\ell}^0, s_{1,\ell}^1],\ [s_{2,\ell}^0, s_{2,\ell}^1], \dots, [s_{k,\ell}^0, s_{k,\ell}^1]$$

are pairwise disjoint (that is, the projection of the R_j onto each coordinate axis are pairwise disjoint). Without loss of generality, we assume that

$$s_{j-1,N}^1 < s_{j,N}^0, \qquad j = 2,\dots,N$$

(that is, the projections of the R_j onto the N^{th}-coordinate axis are in increasing order).

Let

$$R = \left(\prod_{\ell=1}^{N-1} [s^0_{k,\ell}, s^1_{k,\ell}]\right) \times [s^1_{k-1,N}, s^1_{k,N}],$$

$$S = \left(\prod_{\ell=1}^{N-1}]s^0_{k,\ell}, s^1_{k,\ell}[^c\right) \times [0, s^1_{k-1,N}].$$

Notice that $R_k \subset R$ and for $j = 1, \ldots, k$, $R_j \subset S$.

Lemma 3.5. *Let M be as in Theorem 3.1. There is a process $(\hat{B}_{\mathbf{t}}, \mathbf{t} \in [0, M]^N)$ with law mutually equivalent to the law of $(B_{\mathbf{t}}, \mathbf{t} \in [0, M]^N)$ such that*

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}), \qquad \textit{for } \mathbf{t} \in [0, M]^{N-1} \times [0, s^1_{k-1,N}]$$

and

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}) - E(B(\mathbf{t}) \mid \mathcal{F}(S)), \qquad \textit{for } \mathbf{t} \in R_k.$$

In particular, $\hat{B}|_{R_k}$ and $\big(B|_{R_1}, \ldots, B|_{R_{k-1}}\big)$ are independent.

Proof. We apply Lemma 3.2 to the sets R and S, yielding the process $(\tilde{B}(\mathbf{t}), \mathbf{t} \in R)$, such that $\tilde{B}(\mathbf{t}) = E(B(\mathbf{t}) \mid \mathcal{F}(S))$, $\mathbf{t} \in R_k$. In particular, if we set

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}), \qquad \text{for } \mathbf{t} \in [0, M]^{N-1} \times [0, s^1_{k-1,N}], \tag{3.8}$$

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}) - \tilde{B}(\mathbf{t}), \quad \text{for } \mathbf{t} \in R_k, \tag{3.9}$$

then $\hat{B}|_{R_k}$ and $(B|_{R_1}, \ldots, B|_{R_{k-1}})$ are independent, since B is a Gaussian process. The main point of this lemma is to establish, after extending the definition of $\hat{B}(\mathbf{t})$ to $\mathbf{t} \in [0, M]^N$, that the law of $(\hat{B}(\mathbf{t}), \mathbf{t} \in [0, M]^N)$ is mutually equivalent to the law of $(B(\mathbf{t}), \mathbf{t} \in [0, M]^N)$.

For this, we will use Girsanov's theorem (Theorem 3.4), by constructing a process $(Z(\mathbf{s}))$ satisfying the assumption of Theorem 3.4 and such that

$$B(\mathbf{t}) - \int_{[0,t_1]\times\cdots\times[0,t_N]} Z(s_1, \ldots, s_N)\, ds_1 \cdots ds_N, \qquad \mathbf{t} \in \mathbb{R}^{N-1} \times [0, M], \tag{3.10}$$

agrees with $\hat{B}(\mathbf{t})$ on $[0, M]^{N-1} \times [0, s^1_{k-1,N}]$ and on R_k. Using the formula in (3.10) to define $\hat{B}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^{N-1} \times [0, M]$, this immediately implies that the laws of $(\hat{B}(\mathbf{t}), \mathbf{t} \in [0,M]^N)$ and $(B(\mathbf{t}), \mathbf{t} \in [0,M]^N)$ are mutually equivalent.

We note that for $\mathbf{t} = (t_1, \dots, t_N) \in R$,

$$\tilde{B}(\mathbf{t}) = \tilde{B}(t_1, \dots, t_{N-1}, t_N) = \tilde{B}(t_1, \dots, t_{N-1}, s^1_{k-1,N}),$$

so $\tilde{B}(\mathbf{t})$ does not depend explicitly on the N^{th}-coordinate of t.

We now construct $Z(\mathbf{s})$. Let

$$U = \left(\prod_{\ell=1}^{N-1} [0, s^1_{k,\ell}] \right) \times [s^1_{k-1,N}, s^0_{k,N}].$$

We set

$$Z(\mathbf{s}) \equiv 0 \qquad \text{for } \mathbf{s} \notin U, \tag{3.11}$$

and we define $Z(\mathbf{s})$ for $\mathbf{s} \in U$ as follows. For $\mathbf{t} \in U \cup R$, define

$$p_\ell(\mathbf{t}) = s^0_{k,\ell} \vee t_\ell, \qquad \ell = 1, \dots, N-1,$$

$p_N(\mathbf{t}) = s^1_{k-1,N}$, and $p(\mathbf{t}) = (p_1(\mathbf{t}), \dots, p_N(\mathbf{t}))$. Now let

$$F(\mathbf{t}) = \begin{cases} \dfrac{t_N - s^1_{k-1,N}}{s^0_{k,N} - s^1_{k-1,N}} \left(\displaystyle\prod_{\ell=1}^{N-1} \frac{t_\ell \wedge s^0_{k,\ell}}{s^0_{k,\ell}} \right) \tilde{B}(p(\mathbf{t})) & \text{if } \mathbf{t} \in U, \\ 0 & \text{otherwise,} \end{cases} \tag{3.12}$$

so that $F(\mathbf{t})$ is an $\mathbb{R}^d$-valued multilinear interpolation of $\tilde{B}(p(\mathbf{t}))$ with the process which vanishes on the coordinate hyperplanes 1 to $N-1$, and on the hyperplane $\mathbb{R}^{N-1} \times \{s^1_{k-1,N}\}$. In particular, for $\mathbf{t} \in U$,

$$F(\mathbf{t}) = 0 \qquad \text{if } t_1 = 0 \text{ or } \cdots \text{ or } t_{N-1} = 0 \text{ or } t_N = s^1_{k-1,N}, \tag{3.13}$$

and

$$F(\mathbf{t}) = \frac{t_N - s^1_{k-1,N}}{s^0_{k,N} - s^1_{k-1,N}} \tilde{B}(t_1, \dots, t_{N-1}, s^1_{k-1,N}) \qquad \text{if } \mathbf{t} \in R. \tag{3.14}$$

We note that $\mathbf{t} \mapsto F(\mathbf{t})$ is piecewise C^∞, and we set

$$Z(s_1, \dots, s_N) = \frac{\partial^N}{\partial s_1 \cdots \partial s_N} F(s_1, \dots, s_N).$$

It is clear that $Z(\mathbf{s})$ is a linear combination of the random variables $B(s_{k,1}^{j(1)}, \ldots, s_{k,N-1}^{j(N-1)}, s_{k-1,N}^1)$ that come from Lemma 3.2. Explicit formulas can be given, for instance, letting $\dot{B}$ denote the white noise associated to B,

$$Z(\mathbf{s}) = \dot{B}([s_{k-1}^0, s_{k-1}^1] \times \cdots \times [s_{k,N-1}^0, s_{k,N-1}^1] \times [0, s_{k-1,N}^1], \qquad \text{if } \mathbf{s} \in R,$$

but we will not need them. We note however, that $(Z(\mathbf{s}))$ is adapted to the filtration $(\mathcal{G}_u)$ defined in (3.6).

For $\mathbf{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N$, let

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}) - \int_{[0,t_1]\times\cdots\times[0,t_N]} Z(s_1, \ldots, s_N)\, ds_1 \cdots ds_N.$$

Then (3.8) is clearly satisfied by (3.11), and (3.9) is satisfied since for $\mathbf{t} \in R_k$, by (3.13) and (3.14),

$$\begin{aligned}
&\int_{[0,t_1]\times\cdots\times[0,t_N]} Z(s_1, \cdots, s_N)\, ds_1 \cdots ds_N \\
&\quad = \int_0^{t_1} ds_1 \cdots \int_0^{t_{N-1}} ds_{N-1} \int_{s_{k-1,N}^1}^{s_{k,N}^0} ds_N \; \frac{\partial^N}{\partial s_1 \cdots \partial s_N} F(s_1, \ldots, s_N) \\
&\quad = \frac{s_{k,N}^0 - s_{k-1,N}^1}{s_{k,N}^0 - s_{k-1,N}^1} \tilde{B}(t_1, \ldots, t_{N-1}, s_{k-1,N}^1) \\
&\quad = \tilde{B}(p(\mathbf{t})) \\
&\quad = \tilde{B}(\mathbf{t}).
\end{aligned}$$

In order to complete the proof, it remains to check that the assumption of Theorem 3.4 is satisfied, and, in particular, that the process

$$L_u = \exp\left[\int_{\mathbb{R}_+^{N-1}\times[0,u]} Z(\mathbf{s}) \cdot dB(\mathbf{s}) - \frac{1}{2}\int_{\mathbb{R}_+^{N-1}\times[0,u]} \|Z(\mathbf{s})\|^2\, d\mathbf{s}\right], \quad u \in [0, M],$$

is a martingale. Since Z vanishes on $\mathbb{R}^N \setminus U$, it suffices, according to the extension of Novikov's criterion presented in [3, Chapter 3.5, Corollary 5.14], to check that for n sufficiently large and $t_i = s_{k-1,N}^1 + \frac{i}{n}(s_{k,N}^0 - s_{k-1,N}^1)$, $i = 0, \ldots, n$,

$$E\left[\exp\left(\frac{1}{2}\int_0^{s_{k,1}^1} ds_1 \;\cdots \int_0^{s_{k,N-1}^1} ds_{N-1} \int_{t_{i-1}}^{t_i} ds_N\, \|Z(\mathbf{s})\|^2\right)\right] < +\infty.$$

But this follows from the fact that the integral is bounded by

$$\frac{C}{n} \sup_{j \in \mathcal{J}_N} \|(B(s_{k,1}^{j(1)}, \cdots, s_{k,N-1}^{j(N-1)}, s_{k-1,N}^0))\|^2,$$

for some constant C that depends only on R_{k-1} and R_k, and this random variable has a finite exponential moment if n is sufficiently large. The proof of Lemma 3.5 is complete. □

Proof of Theorem 3.1. We proceed by induction on k. For $k = 1$, there is nothing to prove. So assume that $k \geq 2$ and that we have proved the statement for $k - 1$.

We consider the two independent Brownian sheets B and W_k. We apply Lemma 3.5 to both of these processes, producing processes $\hat{B}$ and $\hat{W}_k$ such that, in particular,

(1) $\hat{B}|_{R_1} = B|_{R_1}, \ldots, \hat{B}|_{R_{k-1}} = B|_{R_{k-1}}$;

(2) $\hat{B}|_{R_k}$ and $(B|_{R_1}, \ldots, B|_{R_{k-1}})$ are independent;

(3) $B|_{[0,M]^N}$ and $\hat{B}|_{[0,M]^N}$ have mutually equivalent probability laws;

(4) $\hat{W}_k|_{R_k}$ and $W_k|_{R_k}$ have mutually equivalent probability laws;

(5) $\hat{B}|_{R_k}$ and $\hat{W}_k|_{R_k}$ have the *same* probability law.

We write $\mathcal{L}(B|_{R_1}, \ldots, B|_{R_k})$ for the probability law of the random vector $(B|_{R_1}, \ldots, B|_{R_k})$, and use "$\sim$" to indicate mutually equivalent probability laws. Then, by (3) and (1),

$$\begin{aligned} \mathcal{L}(B|_{R_1}, \ldots, B|_{R_k}) &\sim \mathcal{L}(\hat{B}|_{R_1}, \ldots, \hat{B}|_{R_{k-1}}, \hat{B}|_{R_k}) \\ &= \mathcal{L}(B|_{R_1}, \ldots, B|_{R_{k-1}}, \hat{B}|_{R_k}). \end{aligned}$$

By (2) and (5), and since B and W_k are independent,

$$\mathcal{L}(B|_{R_1}, \ldots, B|_{R_{k-1}}, \hat{B}|_{R_k}) = \mathcal{L}(B|_{R_1}, \ldots, B|_{R_{k-1}}, \hat{W}_k|_{R_k}).$$

Let $W_1, \ldots, W_{k-1}$ be independent Brownian sheets independent of W_k and B. Since B and W_k are independent, we can use the induction hypothesis to see that

$$\mathcal{L}(B|_{R_1}, \ldots, B|_{R_{k-1}}, \hat{W}|_{R_k}) \sim \mathcal{L}(W_1|_{R_1}, \ldots, W_{k-1}|_{R_{k-1}}, \hat{W}_k|_{R_k})$$

By (4) and the independence of $(W_1, \ldots, W_{k-1})$ and W_k, we conclude that

$$\mathcal{L}(W_1|_{R_1}, \ldots, W_{k-1}|_{R_{k-1}}, \hat{W}_k|_{R_k}) \sim \mathcal{L}(W_1|_{R_1}, \ldots, W_{k-1}|_{R_{k-1}}, W_k|_{R_k}),$$

and this proves Theorem 3.1. □

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