L-functions of symmetric powers of the generalized Airy family of exponential sums: ℓ-adic and p-adic methods

C. Douglas Haessig   Antonio Rojas León*

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Abstract

For ψ a nontrivial additive character on the finite field $\mathbb{F}_q$, observe that the map $t \mapsto \sum_{x \in \mathbb{F}_q} \psi(f(x) + tx)$ is the Fourier transform of the map $t \mapsto \psi(f(t))$. As is well-known, this has a cohomological interpretation, producing a continuous ℓ-adic Galois representation. This paper studies the L-function attached to the k-th symmetric power of this representation using both ℓ-adic and p-adic methods. Using ℓ-adic techniques, we give an explicit formula for the degree of this L-function and determine the complex absolute values of its roots. Using p-adic techniques, we study the p-adic absolute values of the roots.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and characteristic p. Let Y be a smooth, geometrically connected open variety defined over $\mathbb{F}_q$; for instance, take Y to be affine s-space $\mathbb{A}^s_{\mathbb{F}_q}$ or the torus $\mathbb{G}^s_m$. Denote its function field by K, and its corresponding absolute Galois group by $G_K := \mathrm{Gal}(K^{sep}/K)$. Let V be a finite dimensional vector space over a finite extension field of $\mathbb{Q}_\ell$, where $\ell \neq p$. Let $\rho : G_K \to GL(V)$ be a continuous ℓ-adic representation unramified on Y, and let $\mathcal{F}$ be the corresponding lisse sheaf on Y. Define the L-function of ρ on Y by

$$L(Y, \rho, T) := \prod_{x \in |Y|} \frac{1}{\det(1 - \rho(\mathrm{Frob}_x) T^{\deg(x)})}. \tag{1}$$

By the Lefschetz trace formula, this is a rational function whose zeros and poles may be described using étale cohomology with compact support:

$$L(Y, \rho, T) = \prod_{i=0}^{2\dim(Y)} \det(1 - \mathrm{Frob}_q T | H^i_c(Y \otimes \overline{\mathbb{F}}_q, \mathcal{F}))^{(-1)^{i+1}}$$

Given such a representation, we may construct new L-functions via operations such as tensor, symmetric, or exterior products. Natural questions about these new L-functions concern the determination of their degrees (Euler characteristic) and describing various properties about their zeros and poles. In this paper, we will focus on the symmetric powers of a particular family of exponential sums called the *generalized Airy family*. Other families whose symmetric powers have been investigated are the Legendre family of elliptic curves [3] [8] and the hyperKloosterman family [10] [11] [17].

The study of these L-functions has attracted some attention recently due to their relation with Dwork's unit root zeta function. More precisely, the unit-root zeta function of an ordinary overconvergent F-crystal on $\mathbb{A}^n/\mathbb{F}_q$ is a certain p-adic limit of the L-functions of some Adams powers of the F-crystal, which in turn can be expressed as an alternating product of the L-functions of its symmetric and alternating powers. See [18], [19] for details.

Given a polynomial f of degree d, the generalized Airy family of exponential sums is defined by the polynomial $f(x) + tx$ as follows. Let d be a positive integer such that $p \nmid d$. Let ψ be a nontrivial additive character on $\mathbb{F}_q$. For each $\bar{t} \in \overline{\mathbb{F}}_q$ define its degree by $\deg(\bar{t}) := [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$. It is well-known that the associated L-function of the sequence of exponential sums

$$S_m(\bar{t}) := \sum_{x \in \mathbb{F}_{q^{m \deg(\bar{t})}}} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{m \deg(\bar{t})}}/\mathbb{F}_q}(f(x) + \bar{t}x) \quad \text{for } m = 1, 2, 3, \ldots$$

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is a polynomial of degree $d-1$:

$$L(f, \mathbb{A}^1, \bar{t}; T) := \exp\left(\sum_{m=1}^{\infty} S_m(\bar{t}) \frac{T^m}{m}\right) = (1-\pi_1(\bar{t})T)\cdots(1-\pi_{d-1}(\bar{t})T).$$

As we will describe later, the relative cohomology of this family may be represented either ℓ-adically as a lisse sheaf of rank $d-1$ over $\mathbb{A}^1$ via Fourier transform, or p-adically as a free module of rank $d-1$ over an overconvergent power series ring in t equipped with a connection and Frobenius. Let us denote both by Ai_f. The L-function of the k-th symmetric power of Ai_f takes the form:

$$M_k(f, T) := L(\mathbb{A}^1, \mathrm{Sym}^k(\mathrm{Ai}_f), T) := \prod_{t \in |\mathbb{A}^1|} \prod_{a_1+\cdots+a_{d-1}=k} (1-\pi_1(t)^{a_1}\cdots\pi_{d-1}(t)^{a_{d-1}} T^{deg(t)})^{-1},$$

where $|\mathbb{A}^1|$ denotes the set of closed points on $\mathbb{A}^1$. By the Lefschetz trace formula, $M_k(f, T)$ is a rational function. The ℓ-adic sheaf Ai_f was extensively studied by N. Katz in [13], where its monodromy group is determined and, as a consequence, an equidistribution result is obtained for the exponential sums in the family ([13, Corollary 20]). From these results it follows that, for $p > 2d-1$, $M_k(f, T)$ is in fact a polynomial. For $d = 3$, a study of the monodromy group may be avoided using Adolphson's [4].

Our first main result is the computation of the degree of $M_k(f, T)$ for $p > d$. The degree of the rational function $M_k(f, T)$ equals the k-th coefficient of a generating series which is explicitly given in Corollary 2.5. Simplified formulas are given in section 2.3 for some particularly nice values of f and p.

As an example of this theorem, consider the family generated by $f(x) = x^d$. Then the degree of $M_k(x^d, T)$ may be described as follows. Let ζ be a primitive $(d-1)$-th root of unity. Denote by $N_{d-1,k}$ the number of $(d-1)$-tuples $(a_0, a_1, \ldots, a_{d-2})$ of nonnegative integers such that $a_0 + a_1 + \cdots + a_{d-2} = k$ and $a_0 + a_1\zeta + \cdots + a_{d-2}\zeta^{d-2} = 0$ in $\overline{\mathbb{F}}_q$.

Theorem 1. *With the notation defined above, we have*

$$\deg M_k(x^d, T) = \frac{1}{d-1}\left[\binom{k+d-2}{d-2} - dN_{d-1,k}\right].$$

It was conjectured in [12] that $M_k(x^3, T)$ is a polynomial for all $p > 3$ since it was shown, in that paper, that $M_k(x^3, T)$ is a polynomial for every odd integer k, and also for every k even with $k < 2p$. Surprisingly, $M_k(x^3, T)$ is *not* a polynomial when $p = 5$ for all k. This was communicated to the first author by N. Katz. As he pointed out, when $p = 5$ the geometric monodromy group of Ai_{x^3} is finite, and so for infinitely many symmetric powers k there will indeed be a denominator.

Theorem 2. *Suppose $p > 2d-1$. Then $M_k(f, T)$ is a polynomial which may be factored into a product $Q_k(f, T)P_k(f, T)$, where $P_k(f, T)$ satisfies the functional equation*

$$P_k(f, T) = cT^{\deg(P_k)}\overline{P_k(f, 1/q^{k+1}T)} \qquad \text{with } |c| = q^{\deg(P_k)(k+1)/2}$$

and $Q_k(f, T)$ has reciprocal roots of weight $\leq k$. Furthermore, writing $f(x) = \sum_{i=0}^{d} c_i x^i$, if we assume $\mathbb{F}_q$ contains the $2(d-1)$-th roots of $-dc_d$ then an explicit description of $Q_k(f, T)$ may be given; see Corollary 2.9.

Lastly, we wish to describe the p-adic behavior of the reciprocal roots of $M_k(f, T)$. We will first focus on the case $f(x) = x^3$. In this case, we will prove:

Theorem 3. *Assume $p \geq 7$, k is odd, and $k < p$. Write*

$$M_k(x^3, T) = 1 + c_1 T + \cdots + c_r T^r.$$

Then

$$ord_q(c_m) \geq \frac{1}{3}(m^2 + m + km) \qquad \text{for } m = 0, 1, 2, \ldots, r. \tag{2}$$

Furthermore, as a consequence of the functional equation, the endpoints of the q-adic Newton polygon of $M_k(x^3, T)$ coincide with the lower bound (2). If $p = 5$, then the numerator of $M_k(x^3, T)$ satisfies (2).

We note that the lower bound (2) is expected to hold for all symmetric powers k (see [12] for details). Now, there is reason to believe that the lower bound (2) may be optimal in the sense that the q-adic Newton polygon will in fact equal this lower bound under certain conditions on p. As supporting evidence we note that the lower bound has the following symmetric property. Consider the points $P_m \in \mathbb{R}^2$ defined by the lower bound:

$$P_m := (m, \frac{1}{3}(m^2 + m + km)).$$

The slope of the line segment joining P_m and P_{m+1} is given by $s_m := \frac{1}{3}(2m+2+k)$. If we set $m' := \frac{k-1}{2} - m$, then we have the symmetry

$$(k+1) - s_m = s_{m'}.$$

In other words, for every slope s_m there is a corresponding slope $s_{m'}$. This is precisely a consequence of the functional equation for $M_k(x^3, T)$. That is, if α is a reciprocal root of $M_k(x^3, T)$ of slope s, then p^{k+1}/α is another reciprocal root whose slope is $(k+1) - s$.

In section 3, we will investigate the Newton polygon of $M_k(f, T)$ for a slightly more general polynomial given by

$$f(t, x) := x^d + P_{d-1}(t)x^{d-1} + \cdots + P_i(t)x^i,$$

where $(d, p) = 1$, $i \geq 1$, and each $P_j \in t\mathbb{F}_q[t]$ satisfies the following condition. For $i < j < d$, each $\delta_j := deg(P_j)$ satisfies

$$\frac{\delta_j}{d-j} < \frac{\delta_i}{d-i}.$$

Notice that the polynomial $f(x) + tx$ satisfies these conditions.

To describe our results for the p-adic estimates, define $W : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{d-1} \to \frac{1}{d\delta_i}\mathbb{Z}_{\geq 0}$ by

$$W(n, (i_1, \ldots, i_{d-1})) := \left(\frac{d-i}{d\delta_i}\right) n + \frac{1}{d}(i_1 + 2i_2 + \cdots + (d-1)i_{d-1}).$$

Theorem 4. *Under certain conditions, described in detail in Theorem 5, if we write the numerator of $M_k(f, T)$ as $\sum_{m=0}^{r} c_m T^m$, then there is a finite set A consisting of elements $(n, (i_1, \ldots, i_{d-1})) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{d-1}$ such that $i_1 + \cdots + i_{d-1} = k$ which satisfies:*

$$ord_q(c_m) \geq \min\{\sum_{j=1}^{m} W(n^{(j)}, \mathbf{i}^{(j)})\} \qquad \textit{for every } m = 0, 1, \ldots, r,$$

where the minimum runs over all sets consisting of m distinct elements of A.

We expect that these "certain conditions" hold under rather generic conditions on p, k, and f. With an eye toward upcoming work, an initial, detailed investigation into these conditions is given in Section 3.4.

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2 The ℓ-adic point-of-view

In this section we will study the generalized Airy family of exponential sums from the point of view of ℓ-adic cohomology. We will do so by studying the sheaf Ai_f that represents this family on the affine line $\mathbb{A}^1$ over the given finite field $\mathbb{F}_q$. We begin by observing that the map $\mathbb{F}_q \to \mathbb{C}$ given by $t \mapsto \sum_{x \in \mathbb{F}_q} \psi(f(x) + tx)$ is the Fourier transform with respect to ψ, in the classical sense, of the map $t \mapsto \psi(f(t))$. This will translate, in the cohomological sense, to the fact that Ai_f is the Fourier transform, in the sheaf-theoretical sense, of the $\overline{\mathbb{Q}}_\ell$-sheaf that represents the latter map, which is just the pull-back of the Artin-Schreier sheaf associated to ψ via the map given by f. Let us be more precise.

The polynomial f naturally defines a morphism, also denoted by $f : \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$. Let $\mathcal{L}_\psi$ be the Artin-Schreier sheaf on $\mathbb{A}^1_{\mathbb{F}_q}$ associated to ψ (cf. [5, 1.7]). For every finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, every $t \in \mathbb{A}^1(\mathbb{F}_{q^m}) = \mathbb{F}_{q^m}$ and every geometric point $\bar{t}$ over t, we have $\mathrm{Trace}(\mathrm{Frob}_t | \mathcal{L}_{\psi, \bar{t}}) = \psi(\mathrm{Trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(t))$, where Frob_t denotes a geometric Frobenius element at t. Consider the pullback $\mathcal{L}_{\psi(f)} := f^\star \mathcal{L}_\psi$.

By [13, Theorem 17], for $d \geq 2$ the Fourier transform with respect to ψ of $\mathcal{L}_{\psi(f)}$ (which, in principle, is an element of the derived category $\mathcal{D}^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$) is in fact a (shifted) lisse sheaf on $\mathbb{A}^1$, of rank $d-1$ and with $d/(d-1)$ as its single slope at infinity. Its Swan conductor is therefore d. Let us denote this sheaf by $\mathrm{Ai}_f = \mathrm{R}^1\pi_{t!}\mathcal{L}_{\psi(f(x)+tx)}$, where $\pi_t : \mathbb{A}^2 \to \mathbb{A}^1$ is the projection $(x, t) \mapsto t$. For every finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, every $t \in \mathbb{F}_{q^m}$ and every geometric point $\bar{t}$ over t we have, denoting $\psi_m = \psi \circ \mathrm{Trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$:

$$\mathrm{Trace}(\mathrm{Frob}_t | (\mathrm{Ai}_f)_{\bar{t}}) = -\sum_{x \in \mathbb{F}_{q^m}} \psi_m(f(x) + tx).$$

The characteristic polynomial of the action of a geometric Frobenius element Frob_t at t on the stalk of Ai_f at a geometric point over t has the form

$$L(\mathrm{Ai}_f, t, T) = (1 - \pi_1(t)T) \cdots (1 - \pi_{d-1}(t)T)$$

where $\pi_i(t)$ is a Weil algebraic number of weight 1 (i.e. all its complex conjugates have absolute value $q^{1/2}$) and $\sum_{x\in\mathbb{F}_{q^m}} \psi_m(f(x)+tx) = -\sum_i \pi_i(t)^m$ for all $m \geq 1$. Its k-th "symmetric power" is given by

$$L(k; \mathrm{Ai}_f, t, T) := \prod_{a_1+\cdots+a_{d-1}=k} (1 - \pi_1(t)^{a_1} \cdots \pi_{d-1}(t)^{a_{d-1}} T).$$

These are the local factors of the L-function of the k-th symmetric power of Ai_f, which is given by the infinite product

$$M_k(f,T) := \prod_{t\in|\mathbb{A}^1|} L(k; \mathrm{Ai}_f, t, T^{\deg(t)})^{-1}$$

The Lefschetz trace formula demonstrates that the zeros and poles of $M_k(f,T)$ may be described in terms of cohomology:

$$M_k(f,T) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob}\, T|\mathrm{H}^i_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))^{(-1)^{i+1}}.$$

Since $\mathrm{Sym}^k\mathrm{Ai}_f$ is a lisse sheaf on the affine line, we have $\mathrm{H}^0_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f) = 0$, and the previous formula simplifies to

$$M_k(f,T) = \frac{\det(1 - \mathrm{Frob}\, T|\mathrm{H}^1_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))}{\det(1 - \mathrm{Frob}\, T|\mathrm{H}^2_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))}.$$

On the other hand, $\mathrm{H}^2_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f)$ is just the space of coinvariants of the sheaf $\mathrm{Sym}^k\mathrm{Ai}_f$, regarded as a representation of the fundamental group $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}}_q})$, which is the k-th symmetric power of Ai_f regarded as a representation of the same group. This is the same as the space of coinvariants for its monodromy group, which is defined to be the Zariski closure of its image in the group of automorphisms of the generic stalk of Ai_f, isomorphic to $\mathrm{GL}(d-1) := \mathrm{GL}(d-1, \overline{\mathbb{Q}}_\ell)$. By [13, Theorem 19], for $p > 2d-1$ the geometric monodromy group of Ai_f is either $\mathrm{SL}(d-1)$ for d even, or $\mathrm{Sp}(d-1)$ for d odd if $c_{d-1} = 0$ and $\mu_p \cdot \mathrm{SL}(d-1)$ for d even or $\mu_p \cdot \mathrm{Sp}(d-1)$ for d odd if $c_{d-1} \neq 0$ (where $f(x) = \sum_{i=0}^d c_i x^i$). In either case, its k-th symmetric power is still an irreducible representation of rank $\binom{d+k-2}{d-2}$ of the monodromy group (because it is an irreducble representation of its subgroup $\mathrm{SL}(d-1)$ or $\mathrm{Sp}(d-1)$), and in particular the space of coinvariants vanishes. Consequently, we have the following:

Theorem 2.1. *For $p > 2d-1$, the L-function of the k-th symmetric power of Ai_f is a polynomial:*

$$M_k(f,T) = \det(1 - \mathrm{Frob}\, T|\mathrm{H}^1_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))$$

While it is tempting to believe that $M_k(f,T)$ is always a polynomial, this is not true, as mentioned in the introduction. In fact, the monodromy group can be finite in certain cases; for instance when $p = 5$ and $f(x) = x^3$, as proven in [15]. In such cases, $\mathrm{H}^2_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f)$ will be non-trivial for infinitely many values of k, and consequently $M_k(f,T)$ will have a denominator.

2.1 Computation of the degree of the L-function

We will now study the degree of $M_k(f,T)$ when $p > d$. From the formula above we have

$$\deg(M_k(f,T)) = \dim(\mathrm{H}^1_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f)) - \dim(\mathrm{H}^2_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f)) = -\chi_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k\mathrm{Ai}_f),$$

where χ_c denotes the Euler characteristic with compact supports. Using the Grothendieck-Néron-Ogg-Shafarevic formula, we have then

$$\begin{aligned} \deg(M_k(f,T)) &= \mathrm{Swan}_\infty(\mathrm{Sym}^k\mathrm{Ai}_f) - \mathrm{rank}(\mathrm{Sym}^k\mathrm{Ai}_f) \\ &= \mathrm{Swan}_\infty(\mathrm{Sym}^k\mathrm{Ai}_f) - \binom{k+d-2}{d-2}. \end{aligned} \tag{3}$$

In order to compute the Swan conductor of $\mathrm{Sym}^k\mathrm{Ai}_f$ we have to study the sheaf Ai_f as a representation of the inertia group I_∞ of $\mathbb{A}^1_{\overline{\mathbb{F}}_q}$ at infinity. Since $\mathcal{L}_{\psi(f)}$ is lisse on $\mathbb{A}^1$, as a representation of the decomposition group at infinity we have $\mathrm{Ai}_f \cong \mathcal{F}_{\infty,\infty}(\mathcal{L}_{\psi(f)})$, where $\mathcal{F}_{\infty,\infty}$ is the local Fourier transform as defined in [16].

Recently, Fu [9] and, independently, Abbes and Saito [1] have given an explicit description of the different local Fourier transforms for a wide class of ℓ-adic sheaves. We will mainly be using the description given in [1], which works over an arbitrary (not necessarily algebraically closed) perfect base field, and therefore gives an explicit formula for Ai_f as a representation of the decomposition group D_∞.

If $S_{(\infty)}$ is the henselization of the local ring of $\mathbb{P}^1_{\mathbb{F}_q}$ at infinity with uniformizer $1/t$, the triple $(\mathcal{L}_{\psi(f(t))}, t, -f'(t))$ is a Legendre triple in the sense of [1, Definition 2.16]. Therefore by [1, Theorem 3.9] we conclude that, as a representation of D_∞, Ai_f is isomorphic to

$$(-f')_\star(\mathcal{L}_{\psi(f(t))} \otimes \mathcal{L}_{\psi(-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) = (-f')_\star(\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q})$$

where ρ is the unique character $I_\infty \to \overline{\mathbb{Q}}_\ell^\star$ of order 2, $\mathcal{L}_\rho$ the corresponding Kummer sheaf and $\mathcal{Q}$ is the pull-back of the character $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ mapping the geometric Frobenius to the quadratic Gauss sum $g(\psi, \rho) := -\sum_{t\in\mathbb{F}_q^\star} \psi(t)\rho(t)$.

Write $f(t) = \sum_{i=0}^d c_i t^i$. For simplicity, from now on we will assume that $\mathbb{F}_q$ contains the $2(d-1)$-th roots of $-dc_d$ (which can always be achieved by a finite extension of the base field). Following [9, Proposition 3.1] we can find an invertible power series $\sum_{i\geq 0} r_i t^{-i} \in \mathbb{F}_q[[t^{-1}]]$ with $r_0^{d-1} = -dc_d$ such that $u(t) := t\sum_{i\geq 0} r_i t^{-i}$ is a solution to $f'(t) + u(t)^{d-1} = 0$ (the other solutions being $\zeta u(t)$ for every $(d-1)$-th root of unity ζ). The map $\phi : 1/t \mapsto 1/u(t)$ defines an automorphism $S_{(\infty)} \to S_{(\infty)}$, and by construction $-f' = [d-1] \circ \phi$, where $[d-1]$ is the $(d-1)$-th power map. So Ai_f is isomorphic to

$$\begin{aligned}
[d-1]_\star\phi_\star(\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) &= [d-1]_\star(\phi^{-1})^\star(\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) \\
&= [d-1]_\star(\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))} \otimes \mathcal{Q}) \\
&= [d-1]_\star(\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))}) \otimes \mathcal{Q}
\end{aligned}$$

since $[d-1]^\star\mathcal{Q} = \mathcal{Q}$, where $v(t) := \phi^{-1}(t) = t\sum_{i\geq 0} s_i t^{-i}$.

Let $g(t)$ be the polynomial of degree d obtained from $f(v(t)) + v(t)t^{d-1}$ by removing the terms with negative powers of t. It is important to notice that the coefficients of g are polynomials in the coefficients of f. More precisely, if we write $g(t) = \sum b_i t^i$, the coefficient b_i is a polynomial in the coeficients $a_i, a_{i+1}, \ldots, a_d$ of f. Since $\mathcal{L}_{\psi(h(t))}$ is trivial as a representation of D_∞ for any $h(t) \in t^{-1}\mathbb{F}_q[[t^{-1}]]$, we have an isomorphism $\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \cong \mathcal{L}_{\psi(g(t))}$ as representations of D_∞.

On the other hand, from $f'(v(t)) + t^{d-1} = 0$ we get $f''(v(t))v'(t) + (d-1)t^{d-2} = 0$, so $\mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))} = \mathcal{L}_{\rho(-\frac{d-1}{2}v'(t)t^{d-2})}$. Since $v'(t) = \sum_{i\geq 0}(1-i)s_i t^{-i} = s_0(1+\sum_{i\geq 2}(1-i)\frac{s_i}{s_0}t^{-i})$ and $1+\sum_{i\geq 2}(1-i)\frac{s_i}{s_0}t^{-i}$ is a square in $\mathbb{F}_q[[t^{-1}]]$, we have $\mathcal{L}_{\rho(-\frac{d-1}{2}v'(t)t^{d-2})} = \mathcal{L}_{\rho(-\frac{d-1}{2}s_0 t^{d-2})} = \mathcal{L}_{\rho(\frac{d(d-1)}{2}c_d(s_0 t)^{d-2})}$ (since $s_0^{d-1} = -1/dc_d$). So we finally get

$$\mathrm{Ai}_f \cong [d-1]_\star(\mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho^d(s_0 t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}. \tag{4}$$

We can now easily compute the Swan conductor at infinity of its symmetric powers. By [14, 1.13.1],

$$\mathrm{Swan}_\infty \mathrm{Sym}^k \mathrm{Ai}_f = \frac{1}{d-1}\mathrm{Swan}_\infty[d-1]^\star\mathrm{Sym}^k\mathrm{Ai}_f = \frac{1}{d-1}\mathrm{Swan}_\infty\mathrm{Sym}^k[d-1]^\star\mathrm{Ai}_f$$

Lemma 2.2. *Let ζ be a primitive $(d-1)$-th root of unity if $\mathbb{F}_q$, I_∞^{d-1} the unique closed subgroup of I_∞ of index $d-1$. As a representation of I_∞^{d-1}, the restriction $[d-1]^\star\mathrm{Ai}_f$ of Ai_f is isomorphic to the direct sum*

$$\bigoplus_{i=0}^{d-2} \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)} \cong \bigoplus_{i=0}^{d-2} \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(t)}$$

.

Proof. Since $(\zeta^i)^\star\mathcal{L}_{\psi(g)} = \mathcal{L}_{\psi(g(\zeta^i t))}$, $(\zeta^i)^\star\mathcal{L}_{\rho^d(s_0 t)} = \mathcal{L}_{\rho^d(s_0\zeta^i t)}$ and $[d-1] \circ \zeta^i = [d-1]$ for every i, we have $[d-1]_\star(\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)}) = [d-1]_\star(\mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho^d(s_0 t)})$, and therefore by Frobenius reciprocity $\mathrm{Hom}_{I_\infty^{d-1}}([d-1]^\star\mathrm{Ai}_f, \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)}) = \mathrm{Hom}_{I_\infty}(\mathrm{Ai}_f, [d-1]_\star(\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)})) = \mathrm{Hom}_{I_\infty}(\mathrm{Ai}_f, \mathrm{Ai}_f) \cong \overline{\mathbb{Q}}_\ell$ since the latter is an irreducible representation of I_∞. So for every i, $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)}$ is a subrepresentation of $[d-1]^\star\mathrm{Ai}_f$.

Now $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)}$ and $\mathcal{L}_{\psi(g(\zeta^j t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^j t)}$ are isomorphic if and only if $\mathcal{L}_{\psi(g(\zeta^i t))}$ and $\mathcal{L}_{\psi(g(\zeta^j t))}$ are, if and only if $g(\zeta^i t) - g(\zeta^j t) = h^p - h$ for some $h \in \overline{\mathbb{F}}_q[t]$. Since $p > d$, this can only happen if $g(\zeta^i t) = g(\zeta^j t)$. Comparing the highest degree coefficients we conclude that ζ^i and ζ^j must be equal. Therefore the direct sum of the $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0\zeta^i t)}$ for $i = 0, \ldots, d-2$ injects into $[d-1]^\star\mathrm{Ai}_f$ and we conclude that it must be isomorphic to it, since they have the same rank. ∎

Consequently, we have an isomorphism of $\overline{\mathbb{Q}}_\ell[I_\infty]$-modules

$$\mathrm{Sym}^k[d-1]^\star \mathrm{Ai}_f \cong \bigoplus_{a_0+a_1+\cdots+a_{d-2}=k} \mathcal{L}_{\psi(\sum_{i=0}^{d-2} a_i g(\zeta^i t))} \otimes \mathcal{L}_{\rho^{dk}(t)}.$$

For every finite subset $I \subset \mathbb{Z}$ and every integer $k \geq 0$ define

$$S_{d-1}(k,I) := \{(a_0,\ldots,a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1} | a_0 + a_1 + \cdots + a_{d-2} = k, a_0 + a_1\zeta^i + \cdots + a_{d-2}\zeta^{i(d-2)} = 0 \text{ for every } i \in I\}$$

It is clear from the definition that $S_{d-1}(k,I) = S_{d-1}(k,I')$ if $\phi(I) = \phi(I')$, where $\phi : \mathbb{Z} \to \mathbb{Z}/(d-1)\mathbb{Z}$ is reduction modulo $d-1$. Also, $S_{d-1}(k,I) = \emptyset$ if p does not divide k and $I \cap (d-1)\mathbb{Z} \neq \emptyset$. The number of elements in $S_{d-1}(k,I)$ can be conveniently expressed in terms of a generating function:

Lemma 2.3. *Let $F_{d-1}(I;T) := \sum_{k=0}^\infty \#S_{d-1}(k,I)T^k$. Then*

$$F_{d-1}(I;T) = \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \prod_{j=0}^{d-2} (1 - \psi(\sum_{i \in I} \gamma_i \zeta^{ji})T)^{-1}$$

where ψ is any non-trivial additive character of $\mathbb{F}_q$.

Proof. From the definition,

$$F_{d-1}(I;T) = \sum_{(a_0,\ldots,a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \prod_{i \in I} \delta(a_0 + a_1\zeta^i + \cdots + a_{d-2}\zeta^{i(d-2)}) T^{a_0+a_1+\cdots+a_{d-2}}$$

where $\delta(a) = 1$ if $a = 0$, 0 otherwise. Equivalently, $\delta(a) = \frac{1}{q}\sum_{\gamma \in \mathbb{F}_q} \psi(\gamma a)$. So we get

$$\begin{aligned}
F_{d-1}(I;T) &= \sum_{(a_0,\ldots,a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \prod_{i \in I} \frac{1}{q} \sum_{\gamma_i \in \mathbb{F}_q} \psi(\gamma_i(a_0 + a_1\zeta^i + \cdots + a_{d-2}\zeta^{i(d-2)})) T^{a_0+a_1+\cdots+a_{d-2}} \\
&= \sum_{(a_0,\ldots,a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \sum_{\gamma \in (\mathbb{F}_q)^I} \frac{1}{q^{\#I}} \left(\prod_{i \in I} \psi(\gamma_i a_0)\right) T^{a_0} \left(\prod_{i \in I} \psi(\gamma_i a_1 \zeta^i)\right) T^{a_1} \cdots \left(\prod_{i \in I} \psi(\gamma_i a_{d-2}\zeta^{(d-2)i})\right) T^{a_{d-2}} \\
&= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \sum_{(a_0,\ldots,a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \psi(\sum_{i \in I} \gamma_i)^{a_0} T^{a_0} \psi(\sum_{i \in I} \gamma_i \zeta^i)^{a_1} T^{a_1} \cdots \psi(\sum_{i \in I} \gamma_i \zeta^{(d-2)i})^{a_{d-2}} T^{a_{d-2}} \\
&= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \left(\sum_{a_0 \in \mathbb{Z}_{\geq 0}} \psi(\sum_{i \in I} \gamma_i)^{a_0} T^{a_0}\right) \left(\sum_{a_1 \in \mathbb{Z}_{\geq 0}} \psi(\sum_{i \in I} \gamma_i \zeta^i)^{a_1} T^{a_1}\right) \cdots \left(\sum_{a_{d-2} \in \mathbb{Z}_{\geq 0}} \psi(\sum_{i \in I} \gamma_i \zeta^{(d-2)i})^{a_{d-2}} T^{a_{d-2}}\right) \\
&= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \prod_{j=0}^{d-2} (1 - \psi(\sum_{i \in I} \gamma_i \zeta^{ji})T)^{-1}.
\end{aligned}$$

□

Write $g(t) = \sum_{j=0}^d b_j t^j$, and let $J = \{1 \leq j \leq d | b_j \neq 0\}$ and $J_{\geq j} := J \cap \{j, j+1, \ldots, d\}$ for every $j \in \{1,\ldots,d,d+1\}$. We have

$$\begin{aligned}
\mathrm{Swan}_\infty \mathrm{Sym}^k[d-1]^\star \mathrm{Ai}_f &= \sum_{a_0+a_1+\cdots+a_{d-2}=k} \mathrm{Swan}_\infty \mathcal{L}_{\psi(\sum_{i=0}^{d-2} a_i g(\zeta^i t))} \otimes \mathcal{L}_{\rho^{dk}(t)} \\
&= \sum_{a_0+a_1+\cdots+a_{d-2}=k} \deg(\sum_{i=0}^{d-2} a_i g(\zeta^i t))
\end{aligned}$$

and

$$\sum_{i=0}^{d-2} a_i g(\zeta^i t) = \sum_{i=0}^{d-2} a_i \sum_{j=0}^{d} b_j \zeta^{ij} t^j = \sum_{j=0}^{d} (b_j \sum_{i=0}^{d-2} \zeta^{ij}) t^j$$

so its degree is the greatest j such that $b_j \sum_{i=0}^{d-2} \zeta^{ij} \neq 0$. Therefore we get

$$\begin{aligned}
(d-1)\mathrm{Swan}_\infty \mathrm{Sym}^k \mathrm{Ai}_f &= \mathrm{Swan}_\infty \mathrm{Sym}^k[d-1]^\star \mathrm{Ai}_f \\
&= \sum_{j \in J} j \cdot (\#S_{d-1}(k, J_{\geq j+1}) - \#S_{d-1}(k, J_{\geq j})) \\
&= d\binom{k+d-2}{d-2} - \sum_{j \in J} h(j) \cdot \#S_{d-1}(k, J_{\geq j})
\end{aligned}$$

where $h(j) := j - \sup(J - J_{\geq j})$ is the "gap" between the t^j term and the next lower degree term in $g(t)$. Taking the corresponding generating function we get the formula

Corollary 2.4. *Let $G(f;T) := \sum_{k=0}^{\infty}(\mathrm{Swan}_\infty \mathrm{Sym}^k \mathrm{Ai}_f)T^k$, then*

$$G(f;T) = \frac{d}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1}\sum_{j\in J} h(j)\cdot F_{d-1}(J_{\geq j};T)$$

Using the previous formula for the degree, we deduce

Corollary 2.5. *The degree of $M_k(f;T)$ is the k-th coefficient of the power series expansion of*

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1}\sum_{j\in J} h(j)\cdot F_{d-1}(J_{\geq j};T).$$

Corollary 2.6. *For every $J \subset \{1,\dots,d-1\}$, let $\mathcal{P}_d(J)$ be the subspace of the affine space $\mathcal{P}_d$ of polynomials of degree d over k such that $b_j = 0$ if and only if $j \in J$. The sets $\{\mathcal{P}_d(J)|J \subseteq \{1,\dots,d-1\}\}$ define a stratification of $\mathcal{P}_d$ such that the degree of $M_k(f;T)$ is constant in each stratum.*

2.2 The trivial factor

We will now study the weights of the (reciprocal) roots of the polynomial $M_k(f,T)$ for $p > 2d-1$. Let us first consider the easier case where d is even, and therefore Ai_f is isomorphic to $[d-1]_\star\mathcal{L}_{\psi(g(t))}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}\otimes\mathcal{Q}$ as a representation of D_∞. Let $D_\infty^{d-1} = \mathrm{Gal}(\overline{\mathbb{F}_q((1/t))}/\mathbb{F}_q((1/t^{1/(d-1)})))$, denote by $\alpha : D_\infty^{d-1} \to \overline{\mathbb{Q}}_\ell^\star$ the character corresponding to the sheaf $\mathcal{L}_{\psi(g)}$, and let $b \in I_\infty$ be a generator of the cyclic group $D_\infty/D_\infty^{d-1} \cong I_\infty/I_\infty^{d-1}$. By the explicit description of induced representations, there is a basis $\{v_0,\dots,v_{d-2}\}$ of the underlying vector space V such that $a\cdot v_0 = \alpha(a)v_0$ for every $a \in I_\infty^{d-1}$ and $b\cdot v_i = v_{i+1}$ for $i = 0,\dots,d-3$. Then $b\cdot v_{d-2} = b^{d-1}\cdot v_0 = \alpha(b^{d-1})v_0$. Replacing b by $a^{-1}b$, where $a \in I_\infty^{d-1}$ is an element such that $\alpha(a)^{d-1} = \alpha(b^{d-1})$ (which is always possible since the values of α are the p-th roots of unity and $d-1$ is prime to p since $p > 2d-1$) we may assume without loss of generality that $\alpha(b^{d-1}) = 1$.

Furthermore, for any $a \in I_\infty^{d-1}$ we have $a\cdot v_i = (ab^i)\cdot v_0 = (b^ib^{-i}ab^i)\cdot v_0 = b^i\cdot\alpha(b^{-i}ab^i)v_0 = \alpha(b^{-i}ab^i)v_i$. So the restriction of Ai_f to D_∞^{d-1} is the direct sum of the characters $a \mapsto \alpha_i(a) := \alpha(b^{-i}ab^i)$. But we already know that it is the direct sum of the characters associated to the sheaves $\mathcal{L}_{\psi(g(\zeta^i t))}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}\otimes\mathcal{Q}$, so these two sets of characters are identical. Replacing b by a suitable power of itself we may assume that α_i is the character associated to $\mathcal{L}_{\psi(g(\zeta^i t))}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}\otimes\mathcal{Q}$. In particular, $\prod_{i=0}^{d-2}\alpha_i^{a_i}$ is geometricaly trivial (that is, trivial on I_∞^{d-1}) if and only if $\sum a_i g(\zeta^i t)$ is a constant in $\mathbb{F}_q[t]$, that is, if and only if $\sum a_i\zeta^{ij} = 0$ for every $j \in J$.

We turn now to the case d odd. Let χ be a multiplicative character of $\mathbb{F}_q$ of order $2(d-1)$ (which exists, since we are assuming that $\mathbb{F}_q$ contains the $2(d-1)$-th roots of unity). Then by the projection formula Ai_f is isomorphic to $[d-1]_\star(\mathcal{L}_{\psi(g(t))}\otimes\mathcal{L}_{\rho(s_0t)})\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}\otimes\mathcal{Q} \cong ([d-1]_\star\mathcal{L}_{\psi(g(t))})\otimes\mathcal{L}_{\chi(s_0t)}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}\otimes\mathcal{Q}$. Let $\alpha_i : D_\infty^{d-1} \to \overline{\mathbb{Q}}_\ell^\star$ (respectively $\beta : D_\infty \to \overline{\mathbb{Q}}_\ell^\star$) be the character corresponding to the sheaf $\mathcal{L}_{\psi(g(\zeta^i t))}$ (resp. $\mathcal{L}_{\chi(s_0t)}$). Proceeding as in the d even case, we find a generator $b \in I_\infty$ of D_∞/D_∞^{d-1} and a basis $\{v_0,\dots,v_{d-2}\}$ of V such that $a\cdot v_i = \alpha_i(a)\beta(a)v_i$ for $a \in D_\infty^{d-1}$ and $b\cdot v_i = \beta(b)v_{i+1}$ for $i = 0,\dots,d-3$, $b\cdot v_{d-2} = \beta(b)v_0$. In this case, $\prod_{i=0}^{d-2}\alpha_i^{a_i}\beta^{a_i}$ is trivial on I_∞^{d-1} if and only if $\sum a_i g(\zeta^i t)$ is a constant in $\mathbb{F}_q[t]$ and $\sum a_i$ is even (since α_i has order p and β restricted to I_∞^{d-1} has order 2).

We can now compute the dimension of the invariant subspace of the action of I_∞ on $\mathrm{Sym}^k\mathrm{Ai}_f$, in very much the same way it is done for the Kloosterman sheaf in [10, Lemma 2.1]. Its underlying vector space is Sym^kV. An element w is given by a linear combination

$$w = \sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_0^{a_0}\cdots v_{d-2}^{a_{d-2}}.$$

In the d even case we have

$$a\cdot\sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_0^{a_0}\cdots v_{d-2}^{a_{d-2}} = \sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}(\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}})(a)v_0^{a_0}\cdots v_{d-2}^{a_{d-2}}$$

for $a \in I_\infty^{d-1}$ and

$$b\cdot\sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_0^{a_0}\cdots v_{d-2}^{a_{d-2}} = \sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_1^{a_0}v_2^{a_1}\cdots v_0^{a_{d-2}}.$$

So w is fixed by I_∞ if and only if the character $\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}}$ is trivial whenever $c_{a_0\cdots a_{d-2}}\neq 0$ and $c_{a_0\cdots a_{d-2}} = c_{a_{d-2}a_0\cdots a_{d-3}}$ for all $a_0,\ldots,a_{d-2}$. A basis for the invariant subspace is thus given by all distinct sums of the form (setting $v_{d-1+l} := v_l$ for all $l\geq 0$):

$$\sum_{j=0}^{d-2} v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$$

for all $a_0,\ldots,a_{d-2}$ such that $\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}}$ is trivial, that is, such that $\sum a_i\zeta^{ij}=0$ in $\mathbb{F}_q$ for every $j\in J$.

In the d odd case we get

$$g\cdot\sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_0^{a_0}\cdots v_{d-2}^{a_{d-2}} = \sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}(\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}})(g)\beta^k(g)v_0^{a_0}\cdots v_{d-2}^{a_{d-2}}$$

for $g\in I_\infty^{d-1}$ and

$$h\cdot\sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}v_0^{a_0}\cdots v_{d-2}^{a_{d-2}} = \sum_{a_0+\cdots+a_{d-2}=k} c_{a_0\cdots a_{d-2}}\beta(h)^k v_1^{a_0}v_2^{a_1}\cdots v_0^{a_{d-2}}.$$

So w is fixed by I_∞ if and only if the character $\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}}\beta^k$ of I_∞^{d-1} is trivial whenever $c_{a_0\cdots a_{d-2}}\neq 0$ and $c_{a_0\cdots a_{d-2}} = c_{a_{d-2}a_0\cdots a_{d-3}}\beta(h)^k$ for all $a_0,\ldots,a_{d-2}$. Since all α_i's have order p and the restriction of β to I_∞^{d-1} has order 2, $\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}}\beta^k$ is trivial if and only if both $\alpha_0^{a_0}\cdots\alpha_{d-2}^{a_{d-2}}$ and β^k are trivial as characters of I_∞^{d-1}, that is, if and only if $\sum a_i\zeta^{ij}=0$ in $\mathbb{F}_q$ for every $j\in J$ and k is even. In particular, there are no non-zero invariants for I_∞ if k is odd. If k is even, a generating set for the invariant subspace is given by all distinct sums of the form

$$\sum_{j=0}^{d-2}\beta(h)^{jk}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$$

for all $a_0,\ldots,a_{d-2}$ such that $\sum a_i\zeta^{ij}=0$ in $\mathbb{F}_q$ for every $j\in J$. Let r be the size of the orbit of $(a_0,\ldots,a_{d-2})$ under the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ by cyclic permutations. If $r\neq d-1$, we can write

$$\sum_{j=0}^{d-2}\beta(h)^{jk}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}} = \sum_{j=0}^{r-1}\beta(h)^{jk}(1+\beta(h)^{rk}+\cdots+\beta(h)^{(\frac{d-1}{r}-1)rk})v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}.$$

Notice that k must be a multiple of $\frac{d-1}{r}$, since $k=\sum_{i=0}^{d-2}a_i = \frac{d-1}{r}\sum_{i=0}^{r-1}a_i$. If $\frac{rk}{d-1}$ is odd we have

$$1+\beta(h)^{rk}+\cdots+\beta(h)^{(\frac{d-1}{r}-1)rk} = \frac{1-\beta(h)^{(d-1)k}}{1-\beta(h)^{rk}} = 0,$$

so the above sum vanishes. On the other hand, if $\frac{rk}{d-1}$ is even it is clear that the element

$$\sum_{j=0}^{d-2}\beta(h)^{jk}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}} = \frac{d-1}{r}\sum_{j=0}^{r-1}\beta(h)^{jk}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$$

is non-zero, and to different orbits correspond different elements. To summarize, we have

Proposition 2.7. *Let $T_{d-1}(k,J)$ be the set of orbits of the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ on the set $S_{d-1}(k,J)$ by cyclic permutations, and let $U_{d-1}(k,J)$ be the subset of orbits such that $\frac{rk}{d-1}$ is even, where r is their cardinality. If d is even, the invariant subspace of the representation $\mathrm{Sym}^k\mathrm{Ai}_f$ of I_∞ has dimension $\#T_{d-1}(k,J)$. If d is odd and k is even, it has dimension $\#U_{d-1}(k,J)$. If d and k are odd, the representation has no non-zero invariants.*

The sequences $\#T_{d-1}(k,J)$ and $\#U_{d-1}(k,J)$ can also be described by means of generating functions. By Burnside's lemma, the dimension of the invariant subspace for d even is given by

$$\#T_{d-1}(k,J) = \frac{1}{d-1}\sum_{r=1}^{d-1}\#\{(a_0,a_1,\ldots,a_{d-2})|a_i = a_{i+r \bmod d-1}\} = \frac{1}{d-1}\sum_{r|d-1}\phi(\frac{d-1}{r})\#S_r(\frac{kr}{d-1},J)$$

where $S_r(k,J)=\emptyset$ if k is not an integer and ϕ is Euler's totient function. So the generating function for the sequence $\{\#T_{d-1}(k,J)|k\geq 0\}$ is

$$\begin{aligned}
G_{d-1}(J;T) :&= \sum_{k=0}^{\infty} \#T_{d-1}(k,J)T^k \\
&= \sum_{k=0}^{\infty} \frac{1}{d-1}T^k \sum_{r|d-1} \phi(\frac{d-1}{r})\#S_r(\frac{kr}{d-1},J) \\
&= \frac{1}{d-1}\sum_{r|d-1}\phi(\frac{d-1}{r})\sum_{\frac{d-1}{r}|k}\#S_r(\frac{kr}{d-1},J)T^k \\
&= \frac{1}{d-1}\sum_{r|d-1}\phi(\frac{d-1}{r})\sum_{s=0}^{\infty}\#S_r(s,J)T^{\frac{d-1}{r}s} \\
&= \frac{1}{d-1}\sum_{r|d-1}\phi(\frac{d-1}{r})F_r(J;T^{\frac{d-1}{r}})
\end{aligned}$$

Next, suppose that d is odd, and let $(a_0,\ldots,a_{d-2})\in S_{d-1}(k,J)$. Let r be the number of elements in its orbit. Then $\sum_{i=0}^{r-1}a_i=\frac{kr}{d-1}$. We want to count the number of orbits such that this value is even. Since $k=\frac{kr}{d-1}\cdot\frac{d-1}{r}$, if the largest power of 2 that divides $d-1$ is smaller than the largest power of 2 dividing k, $\frac{kr}{d-1}$ must always be even. Suppose that the largest power of 2 that divides k, $2^{\alpha(k)}$, divides $d-1$. Then $\frac{kr}{d-1}$ is odd if and only if $2^{\alpha(k)}$ divides $\frac{d-1}{r}$, if and only if r divides $\frac{d-1}{2^{\alpha(k)}}$. Therefore $\#U_{d-1}(k,J)=\#T_{d-1}(k,J)$ if $2^{\alpha(k)}$ does not divide $d-1$ and $\#T_{d-1}(k,J)-\#T_{\frac{d-1}{2^{\alpha(k)}}}(\frac{k}{2^{\alpha(k)}},J)$ if it does. The generating function is then

$$\begin{aligned}
\sum_{k=0}^{\infty}\#U_{d-1}(k,J)T^k &= \sum_{k=0}^{\infty}\#T_{d-1}(k,J)T^k-\sum_{j\geq 1;2^j|d-1}\sum_{l\text{ odd}}\#T_{\frac{d-1}{2^j}}(l,J)T^{2^jl} \\
&= G_{d-1}(J;T)-\sum_{j\geq 1;2^j|d-1}H_{\frac{d-1}{2^j}}(J;T^{2^j})
\end{aligned}$$

where
$$H_r(J;T):=\frac{1}{2}(G_r(J;T)-G_r(J;-T)).$$

Let $F\in D_\infty^{d-1}\subset D_\infty$ be a geometric Frobenius element, and $w=\sum_{j=0}^{d-2}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$ (resp. $w=\sum_{j=0}^{d-2}\beta(h)^{jk}v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$) a generator of the I_∞-invariant subspace of Sym^kV. F acts on $v_j^{a_0}v_{j+1}^{a_1}\cdots v_{j+d-2}^{a_{d-2}}$ via the character corresponding to $\mathcal{L}_{\psi(\sum a_ig(\zeta^{j+i}t))}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}^{\otimes k}\otimes\mathcal{Q}^{\otimes k}$ (resp. $\mathcal{L}_{\psi(\sum a_ig(\zeta^{j+i}t))}\otimes\mathcal{L}_{\rho(\prod(s_0\zeta^{j+i}t)^{a_i})}\otimes\mathcal{L}_{\rho(d(d-1)c_d/2)}^{\otimes k}\otimes\mathcal{Q}^{\otimes k}$). Since $\sum a_ig(\zeta^{j+i}t)$ must be a constant polynomial, we have $\mathcal{L}_{\psi(\sum a_ig(\zeta^{j+i}t))}\cong\mathcal{L}_{\psi(kb_0)}$. Additionally, if d is odd and k even, $\mathcal{L}_{\rho(\prod(s_0t)^{a_i})}=\mathcal{L}_{\rho(s_0t)^k}$ is trivial. We conclude:

Proposition 2.8. *A Frobenius geometric element at infinity acts on the I_∞-invariant subspace of $\mathrm{Sym}^k\mathrm{Ai}_f$ by multiplication by $\psi(kb_0)\rho(d(d-1)c_d/2)^kg(\psi,\rho)^k$.*

As an immediate consequence we get

Corollary 2.9. *The local L-function of $\mathrm{Sym}^k\mathrm{Ai}_f$ at infinity $\det(1-\mathrm{Frob}\,T|(\mathrm{Sym}^k\mathrm{Ai}_f)^{I_\infty})$ is given by $(1-\psi(kb_0)\rho(d(d-1)c_d/2)^kg(\psi,\rho)^kT)^{\#T_{d-1}(k,J)}$ if d is even, $(1-\psi(kb_0)\rho(d(d-1)c_d/2)^kg(\psi,\rho)^kT)^{\#U_{d-1}(k,J)}$ if d is odd and k is even, and 1 if d and k are odd.*

Theorem 2.10. *The polynomial $M_k(f,T)$ decomposes as a product $P_k(f,T)Q_k(f,T)$, where $Q_k(f,T)$ is given by the formula in Corollary 2.9 and $P_k(d,T)$ satisfies a functional equation*

$$P(T)=cT^r\overline{P(1/q^{k+1}T)}$$

where $|c|=q^{r(k+1)/2}$ and r is its degree.

Proof. Let $j:\mathbb{A}^1\to\mathbb{P}^1$ be the inclusion. From the exact sequence

$$0\to\mathrm{Sym}^k\mathrm{Ai}_f\to j_\star\mathrm{Sym}^k\mathrm{Ai}_f\to(j_\star\mathrm{Sym}^k\mathrm{Ai}_f)_\infty\to 0$$

we get an exact sequence of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules

$$0 \to (j_\star \mathrm{Sym}^k \mathrm{Ai}_f)^{I_\infty} \to \mathrm{H}^1_c(\mathbb{A}^1, \mathrm{Sym}^k \mathrm{Ai}_f) \to \mathrm{H}^1(\mathbb{P}^1, j_\star \mathrm{Sym}^k \mathrm{Ai}_f) \to 0$$

and therefore a decomposition

$$\begin{aligned} M_k(f,T) &= \det(1 - \mathrm{Frob}\, T | \mathrm{H}^1_c(\mathbb{A}^1, \mathrm{Sym}^k \mathrm{Ai}_f)) \\ &= \det(1 - \mathrm{Frob}\, T | (j_\star \mathrm{Sym}^k \mathrm{Ai}_f)^{I_\infty}) \det(1 - \mathrm{Frob}\, T | \mathrm{H}^1(\mathbb{P}^1, j_\star \mathrm{Sym}^k \mathrm{Ai}_f)). \end{aligned}$$

The first factor is described by the previous corollary. On the other hand, by [6, Théorème 1.3] we have a perfect pairing

$$\mathrm{H}^1(\mathbb{P}^1, j_\star \mathrm{Sym}^k \mathrm{Ai}_f) \times \mathrm{H}^1(\mathbb{P}^1, j_\star \mathrm{Sym}^k \widehat{\mathrm{Ai}_f}) \to \overline{\mathbb{Q}}_\ell(-k-1)$$

where $\widehat{\mathrm{Ai}_f}$ is the dual of Ai_f, which is constructed in the same way as Ai_f using the complex conjugate character $\bar{\psi}$ instead of ψ. If the eigenvalues of the action of Frobenius on $\mathrm{H}^1(\mathbb{P}^1, j_\star \mathrm{Sym}^k \mathrm{Ai}_f)$ are $\alpha_1, \cdots, \alpha_r$, so that $P_k(f,T) = \prod(1-\alpha_i T)$, it follows that $\overline{P_k(f,T)} = \prod(1-(q^{k+1}/\alpha_i)T)$ and therefore the functional equation holds. Applying the functional equation twice we get $|c| = q^{r(k+1)/2}$. □

2.3 Some special cases

We will now see how the previous results apply to some special values of f. First, consider the case $f(t) = t^d$. In this case the equation $f'(t) + u(t)^{d-1} = 0$ gives $u(t) = r_0 t$, where $r_0^{d-1} = -d$. Then $v(t) = t/r_0$, and $g(t) = f(v(t)) + v(t)t^{d-1} = t^d(1/r_0^d + 1/r_0) = \frac{d-1}{dr_0} t^d$. By corollary 2.5, we get that the degree of $M_k(f;T)$ is the k-th coefficient in the power series expansion of

$$\frac{1}{d-1}\left(\frac{1}{(1-T)^{d-1}} - dF_{d-1}(\{1\};T)\right)$$

where

$$F_{d-1}(\{1\};T) = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\prod_{j=0}^{d-2}(1-\psi(\gamma\zeta^j)T)^{-1}.$$

Explicitly,

$$\deg M_k(f,T) = \frac{1}{d-1}\left(\binom{k+d-2}{d-2} - d \cdot \#S_{d-1}(k,\{1\})\right).$$

In particular, for $d = 3$

$$F_2(\{1\};T) = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}(1-\psi(\gamma)T)^{-1}(1-\psi(-\gamma)T)^{-1} = \frac{1}{p}\sum_{m=0}^{p-1}(1-\exp(\frac{2\pi i m}{p})T)^{-1}(1-\exp(\frac{-2\pi i m}{p})T)^{-1}.$$

It is easily checked that $S_2(k,\{1\}) := \{(a,b) | a+b=k, a \equiv b (\mod p)\}$ has $\lfloor\frac{k}{p}\rfloor + \delta$ elements, where $\delta = 0$ (resp. $\delta = 1$) if $k - \lfloor\frac{k}{p}\rfloor$ is odd (resp. even). So in this case we get an explicit formula for the degree:

$$\deg M_k(f(t)=t^3;T) = \frac{1}{2}\left(k+1-3\left(\left\lfloor\frac{k}{p}\right\rfloor + \delta\right)\right)$$

If $p > k$ this gives $(k+1)/2$ for k odd and $(k-2)/2$ for k even.

Corollary 2.9 states for $f(t) = t^d$ that the local L-function of $\mathrm{Sym}^k \mathrm{Ai}_f$ at infinity is $(1-\rho(d(d-1)/2)^k g(\psi,\rho)^k T)^{\#T_{d-1}(k,J)}$ if d is even, $(1-\rho(d(d-1)/2)^k g(\psi,\rho)^k T)^{\#U_{d-1}(k,J)}$ if d is odd and k is even and 1 if d and k are odd. For $d = 3$, we can again provide a more explicit expression.

Since 3 is odd, the local L-function is 1 for k odd. For k even, we can write $\#S_2(k,\{1\}) = \lfloor\frac{k}{p}\rfloor + \delta = 2\lfloor\frac{k}{2p}\rfloor + 1$. Every orbit of $\mathbb{Z}/2\mathbb{Z}$ acting on $S_2(k,\{1\})$ has two elements except for $\{(k/2,k/2)\}$, so $\#T_2(k,\{1\}) = \lfloor\frac{k}{2p}\rfloor + 1$. $U_2(k,\{1\})$ contains the orbits such that rk is a multiple of 4. If $k \equiv 0 (\mathrm{mod}\ 4)$ this includes all orbits. If $k \equiv 2 (\mathrm{mod}\ 4)$ the orbit $\{(\frac{k}{2},\frac{k}{2})\}$ must be excluded. So the trivial factor for k even is

$$\begin{aligned} &(1-g(\psi,\rho)^k T)^{\lfloor\frac{k}{2p}\rfloor} && \text{for } k \equiv 2 (\mathrm{mod}\ 4) \\ &(1-g(\psi,\rho)^k T)^{\lfloor\frac{k}{2p}\rfloor+1} && \text{for } k \equiv 0 (\mathrm{mod}\ 4) \end{aligned}$$

In particular, for $p > \frac{k}{2}$ the trivial factor of $M_k(t^3,T)$ is 1 if $k \equiv 2 (\mathrm{mod}\ 4)$ and $(1-g(\psi,\rho)^k T)$ if $k \equiv 0 (\mathrm{mod}\ 4)$.

We will now consider the case where $g(t) = \sum b_i t^i$ has $b_i \neq 0$ for $i = 1, \ldots, d-2$. This includes the generic case where all coefficients of $g(t)$ are non-zero as a special case. Suppose first that $b_{d-1} = 0$ (or, equivalently, that $c_{d-1} = 0$). $S_{d-1}(k, J)$ is the set of all $(a_0, \ldots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}$ such that $\sum a_i = k$ and $\sum a_i \zeta^{ji} = 0$ for all $j = 1, \ldots, d-2$. The system of equations $\{\sum_i \zeta^{ij} x_i = 0 | j = 1, \ldots, d-2\}$ has rank $d-2$ (since the $(d-2)\times(d-2)$ minors are Vandermonde determinants) and has $(1, 1, \ldots, 1)$ as a solution, so all solutions must be of the form $(a, a, \ldots, a)$ modulo p for some a. Therefore

$$
\begin{aligned}
F_{d-1}(J; T) :&= \sum_{k=0}^{\infty} \# S_{d-1}(k, J) T^k \\
&= \sum_{r=0}^{p-1} \sum_{a_0, \ldots, a_{d-2}=0}^{\infty} T^{(r+s_0 p)+\cdots+(r+s_{d-2}p)} \\
&= \sum_{r=0}^{p-1} T^{(d-1)r} \sum_{a_0, \ldots, a_{d-2}=0}^{\infty} T^{p(a_0+\cdots+a_{d-2})} \\
&= \frac{1 - T^{(d-1)p}}{(1-T^p)^{d-1}(1-T^{d-1})}
\end{aligned}
$$

Suppose now that $b_{d-1} \neq 0$ (or, equivalently, that $c_{d-1} \neq 0$). Making the change of variable $\hat{f}(t) = f(t - \frac{c_{n-1}}{nc_n})$ we eliminate the degree $d-1$ term. Moreover, $\mathrm{Ai}_{\hat{f}} = \mathrm{R}^1 \pi_{t!} \mathcal{L}_{\psi(f(x - \frac{c_{n-1}}{nc_n}) + tx)} = \mathrm{R}^1 \pi_{t!} \mathcal{L}_{\psi(f(x) + t(x + \frac{c_{n-1}}{nc_n}))} = \mathrm{Ai}_f \otimes \mathcal{L}_{\psi(\frac{c_{n-1}}{nc_n} t)}$ and thus $\mathrm{Sym}^k \mathrm{Ai}_f = (\mathrm{Sym}^k \mathrm{Ai}_{\hat{f}}) \otimes \mathcal{L}^{\otimes k}_{\psi(-\frac{c_{n-1}}{nc_n} t)}$. As a representation of D_∞, we have then $\mathrm{Ai}_f = [d-1]_\star(\mathcal{L}_{\psi(\hat{g}(t))} \otimes \mathcal{L}_{\rho^d(s_0 t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q} \otimes \mathcal{L}_{\psi(-\frac{c_{n-1}}{nc_n} t)} = [d-1]_\star(\mathcal{L}_{\psi(\hat{g}(t) - \frac{c_{n-1}}{nc_n} t^{d-1})} \otimes \mathcal{L}_{\rho^d(s_0 t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$. In other words, $g(t) = \hat{g}(t) - \frac{c_{n-1}}{nc_n} t^{d-1}$.

If p divides k, the condition $\sum_i a_i \zeta^{ij}$ for $j = d-1$ is void, so both the dimension of $M_k(f; T)$ and the trivial factor at infinity behave as in the $b_{d-1} = 0$ case. If p does not divide k, the condition $\sum_i a_i \zeta^{ij}$ does never hold for $j = d-1$, so $S_{d-1}(k, J_{\geq j}) = \emptyset$ for $j = 1, \ldots, d-1$. In particular, the trivial factor of $M_k(f; T)$ is 1. Furthermore, applying the formula for the degree, we get

$$
\deg M_k(f, T) = \frac{1}{d-1} \left(\binom{k+d-2}{d-2} - \# S_{d-1}(k, \{1\}) \right).
$$

As a final example, suppose that $d-1$ is prime and p is a multiplicative generator of $\mathbb{F}_{d-1}$. In this case, all non-trivial $(d-1)$-th roots of unity are conjugate over $\mathbb{F}_p$, so $a_0 + a_1\zeta + \cdots + a_{d-2}\zeta^{d-2} = 0$ if and only if $a_0 + a_1 \zeta^j + \cdots + a_{d-2}\zeta^{(d-2)j} = 0$ for any $j = 1, 2, \ldots, d-2$. Therefore $S_{d-1}(k, \{1\}) = S_{d-1}(k, J)$ for every $J \subset \mathbb{Z}$ such that $J \cap (d-1)\mathbb{Z} = \emptyset$. As in the previous example, we conclude that, if $c_{d-1} = 0$,

$$
F_{d-1}(J_{\geq j}; T) = \frac{1 - T^{(d-1)p}}{(1-T^p)^{d-1}(1-T^{d-1})}
$$

for *every* $j \in J$. By corollary 2.5, the degree of $M_k(f; T)$ is the k-th coefficient of the power series expansion of

$$
\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1-T^p)^{d-1}(1-T^{d-1})} \sum_{j \in J} h(j) = \frac{1}{(d-1)(1-T)^{d-1}} - \frac{d}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1-T^p)^{d-1}(1-T^{d-1})}.
$$

If $c_{d-1} \neq 0$ we have, as in the previous example, the same formula for the degree if k is a multiple of p, and the k-th coefficient in the power series expansion of

$$
\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1-T^p)^{d-1}(1-T^{d-1})}
$$

if k is prime to p.

3 The p-adic point-of-view

In the previous section, we considered the rational function

$$
M_k(f, T) = \frac{\det(1 - \mathrm{Frob}\, T | \mathrm{H}^1_c(\mathbb{A}^1_{\bar{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))}{\det(1 - \mathrm{Frob}\, T | \mathrm{H}^2_c(\mathbb{A}^1_{\bar{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f))}
$$

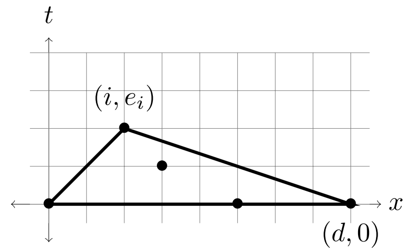

Figure 1: Newton polytope of $f(t, x)$

from the ℓ-adic point-of-view. In this section, we use techniques in Dwork theory which enable us to describe and analyze $M_k(f, T)$ p-adically. The motivation for this approach is to provide estimates for the p-adic absolute values of the zeros and poles of $M_k(f, T)$.

3.1 p-adic interpretation of $M_k(f, T)$

Let $\mathbb{F}_q$ be the finite field with $q = p^a$ elements. For future reference, and since the arguments are essentially identical, we will consider a slightly more general polynomial than considered in the previous section:

$$f(t, x) := x^d + P_{d-1}(t)x^{d-1} + \cdots + P_i(t)x^i,$$

with $(d, p) = 1$, $i \geq 1$, and each $P_j \in t\mathbb{F}_q[t]$ satisfying the following condition. For $i < j < d$, each $\delta_j := deg(P_j)$ satisfies

$$\frac{\delta_j}{d-j} < \frac{\delta_i}{d-i}. \tag{5}$$

This condition is equivalent to saying that the 2-dimensional Newton polytope of f, defined as the convex hull in $\mathbb{R}^2$ of the points $(0, 0)$ and (j, δ_j) for $j = i, \ldots, d$, is a triangle where every point (j, δ_j), with $j \neq i$ and $j \neq d$, lies strictly below the line joining (i, δ_i) and $(d, 0)$, as seen in Figure 3.1. Notice that the polynomial $f(x) + tx$ considered in the previous section satisfies these conditions.

Dwork's theory provides a way to represent the function $M_k(f, T)$ p-adic homologically. This is accomplished by defining "Frobenius" operators, both denoted by $\bar{\beta}$, which act on certain homology spaces $H_{1,k}$ and $H_{0,k}$, so that

$$M_k(f, T) = \frac{det(1 - \bar{\beta}T \mid H_{1,k})}{det(1 - q\bar{\beta}T \mid H_{0,k})}.$$

In this form, estimates for the p-adic absolute values of the zeros and poles of $M_k(f, T)$ may be derived from an analysis of $\bar{\beta}$ on the homology spaces $H_{0,k}$ and $H_{1,k}$. This analysis is carried out in section 3.2. For now, let us demonstrate how Dwork's theory may be used to derive the above homological description of $M_k(f, T)$.

p-adic Spaces. We begin by fixing some notation. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. The valuation on $\mathbb{Q}_p$ is normalized such that $ord_p(p) = 1$. Let $\mathbb{Q}_q$ denote the unramified extension of $\mathbb{Q}_p$ of degree a. Let $\pi \in \mathbb{C}_p$ be a root of the Artin-Hasse series $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}$ with $ord_p(\pi) = \frac{1}{p-1}$. Notice that $\mathbb{Q}_p(\pi)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $p-1$. Let $\mathbb{Q}_q(\pi)$ denote the compositum in $\mathbb{C}_p$ of the fields $\mathbb{Q}_q$ and $\mathbb{Q}_p(\pi)$, and denote its ring of integers by $\mathbb{Z}_q[\pi]$. Note that the residue class field of $\mathbb{Z}_q[\pi]$ is $\mathbb{F}_q$. Let $\tau \in Gal(\mathbb{Q}_q(\pi)/\mathbb{Q}_p(\pi))$ be a lifting of the Frobenius map $x \mapsto x^p$ in $Gal(\mathbb{F}_q/\mathbb{F}_p)$ such that $\tau(\pi) = \pi$.

Dwork's "splitting function" provides a bridge between the finite field $\mathbb{F}_q$ and the p-adic field $\mathbb{C}_p$, providing a way to describe exponential sums over finite fields p-adic analytically. Let $E(t) := \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right)$. Define $\pi_l := \sum_{i=0}^{l} \frac{\pi^{p^i}}{p^i}$ and note that $ord_p(\pi_l) \geq \frac{p^{l+1}}{p-1} - l - 1$. Dwork's infinite splitting function is defined as $\theta(t) := E(\pi t) = \sum_{i=0}^{\infty} \lambda_i t^i$. Observe that each coefficient satisfies $ord_p(\lambda_i) \geq \frac{i}{p-1}$.

Next, we need to describe the function spaces that our "Frobenius" operators will act upon. In order to obtain the best possible estimates for the zeros and poles of the L-function, the functions in these spaces will have specific growth conditions that fit our particular polynomial $f(t, x)$. These are described as follows. Let b and b' be two positive real numbers. Define functions $w_0, w_1 : \mathbb{Z}_{\geq 0} \to \frac{1}{d\delta_i}\mathbb{Z}$ by

$$w_0(n) := \frac{n}{d} \qquad \text{and} \qquad w_1(n) := \left(\frac{d-i}{d\delta_i}\right) n.$$

With $\rho \in \mathbb{R}$, define the p-adic spaces

$$\begin{aligned}
L(b;\rho) &:= \left\{ \sum_{n=0}^{\infty} B_n t^n \mid B_n \in \mathbb{Z}_q[\pi], ord_p(B_n) \geq bw_1(n) + \rho \text{ for all } n \geq 0 \right\} \\
L(b) &:= \bigcup_{\rho \in \mathbb{R}} L(b;\rho) \\
\mathcal{K}(b',b;\rho) &:= \left\{ \sum_{m=0}^{\infty} B_m x^m \mid B_m \in L(b'; bw_0(m) + \rho) \text{ for all } m \geq 0 \right\} \\
\mathcal{K}(b',b) &:= \bigcup_{\rho \in \mathbb{R}} \mathcal{K}(b',b;\rho) \\
\mathcal{K}(b',b)^{\bullet} &:= x\mathcal{K}(b',b).
\end{aligned}$$

Notice that the space $\mathcal{K}(b',b)$ is indeed suited for our polynomial as follows. Let $\hat{f}(t,x) \in \mathbb{Z}_q[t,x]$ denote the Teichmüller lifting of $f(t,x) \in \mathbb{F}_q[t,x]$. Write

$$\hat{f}(t,x) = \sum_{m,n \geq 0} a_{m,n} t^m x^n, \tag{6}$$

and note that the pair (n,m) satisfies $n \leq \left(\frac{\delta_i}{d-i}\right)(d-m)$ by (5). Thus $\left(\frac{d-i}{d\delta_i}\right) n + \frac{m}{d} - 1 \leq 0$. This means

$$ord_p(a_{m,n}) = 0 \geq \left(\frac{p}{p-1}\right)\left(\left(\frac{d-i}{d\delta_i}\right) m + \frac{n}{d} - 1\right). \tag{7}$$

Hence, $\hat{f} \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\frac{p}{p-1})$.

Relative homology. In order to define relative homology we need to define a twisted differential operator on $\mathcal{K}(b',b)$. The twisting is necessary since it allows the differential operator to commute with the Frobenius operator α, as seen below. Now, motivated by the observation $\theta(1)^{Tr(f(\bar{t},\bar{x}))} = \prod_{m,n \geq 0} \theta(a_{m,n} t^m x^n)$, where t and x are the Teichmüller representatives of $\bar{t}, \bar{x} \in \mathbb{F}_p$, using (6) define

$$\begin{aligned}
F(t,x) &:= \prod_{m,n \geq 0} \theta(a_{m,n} t^m x^n) \in \mathbb{Z}_q[\pi][[t,x]] \\
F_a(t,x) &:= \prod_{i=0}^{a-1} F^{\tau^i}(t^{p^i}, x^{p^i}) \in \mathbb{Z}_p[\pi][[t,x]].
\end{aligned}$$

Due to the shape of the Newton polytope of f, it follows that

$$F(t,x) \in \mathcal{K}(b'/p, b/p; 0) \qquad \text{and} \qquad F_a(t,x) \in \mathcal{K}(b'/q, b/q; 0)$$

for all real numbers $b' \leq b \leq p/(p-1)$. Next, we define a function $G(t,x)$ such that

$$F(t,x) = \frac{G(t,x)}{G^{\tau}(t^p, x^p)}.$$

Using this equation recursively, we see that $G(t,x)$ must be defined by

$$G(t,x) := \prod_{j=0}^{\infty} F^{\tau^j}(t^{p^j}, x^{p^j}) \in \mathbb{Z}_p[\pi][[t,x]].$$

Set $\hat{f}_x := x\frac{\partial}{\partial x}\hat{f}(t,x)$. Define a (twisted) differential operator on $\mathcal{K}(b',b)$ by

$$\begin{aligned}
D(t) :&= \frac{1}{G(t,x)} \circ x\frac{\partial}{\partial x} \circ G(t,x) \\
&= x\frac{\partial}{\partial x} + H(t,x)
\end{aligned} \tag{8}$$

where

$$H(t,x) := \sum_{j=0}^{\infty} \pi_j p^j \hat{f}_x^{\tau^j}(t^{p^j}, x^{p^j}).$$

Note, H acts by multiplication and $H \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -1)$, thus $D(t)$ is an endomorphism of $\mathcal{K}(b', b)$. Using this operator, we may define the relative homology spaces

$$\mathcal{H}_0 := ker(D(t) \mid \mathcal{K}(b', b)) \qquad \text{and} \qquad \mathcal{H}_1 := \mathcal{K}(b', b)^\bullet / D(t)\mathcal{K}(b', b).$$

In the introduction of this paper, the space $\mathcal{H}_1$ was denoted by Ai_f. Now, the above notation can be a bit ambiguous at times. That is, there are times when we will need to keep track of both t and the constants b' and b in the homology spaces $\mathcal{H}_1$ and $\mathcal{H}_0$. Thus, we will often denote $\mathcal{H}_1$ by $\mathcal{H}_{1,t}(b', b)$. More precisely, $\mathcal{H}_{1,t^n}(b_1, b_2)$ means $\mathcal{K}(b_1, b_2)^\bullet / D(t^n)\mathcal{K}(b_1, b_2)$ where $D(t^n) := x\frac{\partial}{\partial x} + H(t^n, x)$.

Symmetric powers of relative homology. The eigenvalues of the Frobenius operators which act on the relative homology spaces $\mathcal{H}_1$ and $\mathcal{H}_0$ consist of power series in the variable t that, when specialized to some Teichmüller representative of $\bar{t} \in \overline{\mathbb{F}}_p$, produce the reciprocal roots $\pi_1(\bar{t}), \ldots, \pi_{d-1}(\bar{t})$ of the L-function attached to f over $\mathbb{A}^1$, as mentioned in the introduction. Now, in the definition of $M_k(f, T)$, a type of symmetric product of these roots is presented. A homological description of this symmetric product is obtained by taking the symmetric power of relative homology. This we will now do. Let $\mathcal{H}_1^{(k)} := Sym^k \mathcal{H}_1$ denote the k-th symmetric power of $\mathcal{H}_1$ over $L(b')$. As we shall see in Theorem 6 below, $\mathcal{H}_1$ is a free $L(b')$-module with basis $\{x^i\}_{i=1}^{d-1}$. Thus, $\mathcal{H}_1^{(k)}$ is a free $L(b')$-module with basis $\{e_1^{i_1} \cdots e_{d-1}^{i_{d-1}}\}_{i_1 + \cdots + i_{d-1} = k}$ where $e_i := x^i$. For notational convenience, we will denote $e_1^{i_1} \cdots e_{d-1}^{i_{d-1}}$ by $e^{\mathbf{i}}$ with $\mathbf{i} := (i_1, \ldots, i_{d-1})$. For $\rho \in \mathbb{R}$, define

$$\mathcal{H}_1^{(k)}(b', b; \rho) := \{ \sum_{\substack{\mathbf{i} := (i_1, \ldots, i_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1} \\ i_1 + \cdots + i_{d-1} = k}} B_{\mathbf{i}} e^{\mathbf{i}} \mid B_{\mathbf{i}} \in L(b'; b w_0(\mathbf{i}) + \rho) \} \tag{9}$$

where $w_0(\mathbf{i})$ is defined by

$$w_0(\mathbf{i}) := w_0(i_1, \ldots, i_{d-1}) := \sum_{j=1}^{d-1} i_j w_0(j).$$

Note that

$$\mathcal{H}_1^{(k)} = \mathcal{H}_1^{(k)}(b', b) := \bigcup_{\rho \in \mathbb{R}} \mathcal{H}_1^{(k)}(b', b; \rho).$$

Next, we define a differential operator on $\mathcal{H}_1^{(k)}$ as follows. With $\hat{f}_t(t, x) := t\frac{\partial}{\partial t}\hat{f}(t, x)$, define

$$\begin{aligned} \partial :&= \frac{1}{G(t, x)} \circ t\frac{\partial}{\partial t} \circ G(t, x) \\ &= t\frac{\partial}{\partial t} + W(t, x) \end{aligned}$$

where

$$W(t, x) := \sum_{j=0}^{\infty} \pi_j p^j \hat{f}_t^{\tau^j}(t^{p^j}, x^{p^j}).$$

Next, $W(t, x) \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -1)$ and so ∂ is an endomorphism of $\mathcal{K}(b', b)$. Now, since ∂ commutes with $D(t)$ as endomorphisms of $\mathcal{K}(b', b)$, and its image lies in $x\mathcal{K}(b', b)$, it induces an operator on relative homology $\partial : \mathcal{H}_1 \to \mathcal{H}_1$. We may extend ∂ to $\mathcal{H}_1^{(k)}$ by extending linearly the following action: for the product $u_1 \cdots u_k \in \mathcal{H}_1^{(k)}$ define

$$\partial(u_1 \cdots u_k) := \sum_{j=1}^{k} u_1 \cdots \hat{u_j} \cdots u_k \partial(u_j).$$

We will sometimes denote ∂ by ∂_t to indicate that it is an endomorphism of $\mathcal{H}_{1,t}^{(k)}$. Define the homology spaces

$$H_{0,k} := ker(\partial \mid \mathcal{H}_1^{(k)}) \qquad \text{and} \qquad H_{1,k} := t\mathcal{H}_1^{(k)} / \partial \mathcal{H}_1^{(k)}.$$

In terms of ℓ-adic theory, these are the p-adic versions of $\mathrm{H}_c^2(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f)$ and $\mathrm{H}_c^1(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathrm{Sym}^k \mathrm{Ai}_f)$, respectively.

Frobenius map. Now that we have created the appropriate spaces, let us define the Frobenius operators. There will be two Frobenius operators involved, $\bar{\alpha}(t)$ which is defined on relative homology whose eigenvalues are power series in t and equal $\pi_j(\bar{t})$ when t is specialized, and another $\bar{\beta}$ which will give the homological description of $M_k(f, T)$. We begin with the definition of $\bar{\alpha}$.

Define the Cartier operator

$$\psi_x : \mathcal{K}(b', b/p) \to \mathcal{K}(b', b) \quad \text{by} \quad \sum_{m=0}^{\infty} B_m x^m \longmapsto \sum_{m=0}^{\infty} B_{pm} x^m.$$

Next, define the Dwork operator, which so far we have been calling the "Frobenius operator", by (recall, $q = p^a$)

$$\alpha(t) := \psi_x^a \circ F_a(t, x) : \mathcal{K}(b', b) \to \mathcal{K}(b'/q, b)$$

Since $\alpha(t) \circ D(t) = qD(t^q) \circ \alpha(t)$, $\alpha(t)$ induces a map on relative homology

$$\bar{\alpha}(t) : \mathcal{H}_{1,t}(b', b) \to \mathcal{H}_{1,t^q}(b'/q, b).$$

We may extend this map to the symmetric powers of relative homology, which we will denote by $\bar{\alpha}^{(k)}(t) : \mathcal{H}_{1,t}^{(k)}(b', b) \to \mathcal{H}_{1,t^q}^{(k)}(b'/q, b)$. Next, define the Cartier operator $\psi_t : \mathcal{H}_{1,t^p}^{(k)}(b'/p, b) \to \mathcal{H}_{1,t}^{(k)}(b', b)$ by

$$\left(\sum_{n=0}^{\infty} B_{\mathbf{i},n} t^n \right) e^{\mathbf{i}} \mapsto \left(\sum_{n=0}^{\infty} B_{\mathbf{i},pn} t^n \right) e^{\mathbf{i}},$$

where we have used the fixed basis in (9) to describe the elements of $\mathcal{H}_1^{(k)}$. Finally, define the Dwork operator

$$\beta := \psi_t^a \circ \bar{\alpha}^{(k)}(t) : \mathcal{H}_{1,t}^{(k)}(b', b) \to \mathcal{H}_{1,t}^{(k)}(b', b).$$

Since $\beta \circ \partial = q\partial \circ \beta$, β induces linear maps

$$\bar{\beta} : H_{0,k} \to H_{0,k} \qquad \text{and} \qquad \bar{\beta} : H_{1,k} \to H_{1,k}.$$

Using Dwork's trace formula twice, precisely as in [12], we have

$$M_k(f, T) = \frac{det(1 - \bar{\beta}T \mid H_{1,k})}{det(1 - q\bar{\beta}T \mid H_{0,k})}.$$

3.2 Estimates for p-adic absolute values of $M_k(f, T)$

We now have a p-adic homological representation of the L-function $M_k(f, T)$ as a quotient of characteristic polynomials of the Frobenius $\bar{\beta}$ acting on $H_{1,k}$ and $H_{0,k}$. In this section, we will focus our attention on understanding the roots of these polynomials, and since $H_{0,k}$ is often trivial, we will on $\bar{\beta}$ acting on $H_{1,k}$. The approach we will take in finding estimates for the p-adic absolute values of the roots lies in the following key idea. On the chain level, β acts on $\mathcal{H}_1^{(k)}$, a space consisting of functions with specific growth conditions. Since these growth conditions are estimates on the p-adic absolute values of the coefficients of the power series of these functions, if one knows how functions in $\mathcal{H}_1^{(k)}$ reduce to elements in homology $H_{1,k}$, perhaps the growth rates of these functions transcends to p-adic estimates of the reduced elements in homology. This is Dwork's decomposition theory, and it is demonstrated in equation (10) below. As we will see in this section, once this is assumed we may apply the theory to obtain estimates for the p-adic absolute values of $M_k(f, T)$.

Define $W : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{d-1} \to \frac{1}{d\delta_i}\mathbb{Z}_{\geq 0}$ by

$$W(n, \mathbf{i}) := w_1(n) + w_0(\mathbf{i})$$

where w_1 and w_0 were defined in the previous section. For notational convenience, define $\mathcal{M} := \mathcal{H}_{1,t}^{(k)}$ and $\mathcal{M}(b', b; \rho) := \mathcal{H}_{1,t}^{(k)}(b', b; \rho)$, and $\mathcal{N} := t\mathcal{H}_{1,t}^{(k)}$ and $\mathcal{N}(b', b; \rho) := (t\mathcal{H}_{1,t}^{(k)}) \cap \mathcal{H}_{1,t}^{(k)}(b', b; \rho)$. In general, if V is a subset of $\mathcal{M}$ define $V(b', b; \rho) := V \cap \mathcal{M}(b', b; \rho)$.

Theorem 5 below, the main theorem of this section, provides estimates for the p-adic absolute values of the roots of $M_k(f, T)$. Define the q-adic valuation $ord_q(\cdot) := \frac{1}{a} ord_p(\cdot)$.

Theorem 5. *Let p be a prime number such that $(p, d) = 1$. Suppose there exists a free, finite rank $\mathbb{Z}_q[\pi]$-submodule V of $\mathcal{N}$ with basis $\Gamma := \{t^n e^{\mathbf{i}}\}_{(n,\mathbf{i}) \in A}$ for some index A such that, with $b := p/(p-1)$,*

$$\mathcal{N}(b, b; 0) \subset V(b, b; 0) \oplus \partial\mathcal{M}(b, b; \epsilon) \tag{10}$$

for some $\epsilon \in \mathbb{R}$. Then, writing $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k}) = \sum_{m=0}^{dim_{\mathbb{Q}_q(\pi)}(H_{1,k})} c_m T^m$, we have

$$ord_q(c_m) \geq \min\{\sum_{j=1}^{m} W(n^{(j)}, \mathbf{i}^{(j)})\}$$

where the minimum runs over all sets consisting of m distinct elements of the form $(n, \mathbf{i}) \in A$. Equivalently, the q-adic Newton polygon of $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k})$ lies on or above the lower convex hull of the points

$$\left(\sum_{N=0}^{R} r_N, \frac{1}{d\delta_i} \sum_{N=0}^{R} N r_N \right) \qquad R = 0, 1, \ldots, dim_{\mathbb{Q}_q(\pi)}(H_{1,k}),$$

where $r_N := \#\{(n, \mathbf{i}) \in A \mid W(n, \mathbf{i}) = N/(d\delta_i)\}$.

Before proceeding to the proof of this theorem, let us give an example illustrating the theorem. In [12], the function $M_k(f, T)$ was studied for the cubic family $f(t, x) = x^3 + tx$. In particular, $\mathcal{N} = t\mathcal{H}_1^{(k)}$ was essentially shown to satisfy (10) for some V. Consequently, using Theorem 5, we have the following result, which was conjectured in *loc.cit.*.

Corollary 1. *Let $p \geq 5$ be a prime number. For the cubic family $f(t, x) := x^3 + tx$, when the symmetric power k is odd and $k < p$, then $M_k(x^3, T)$ is a polynomial. Furthermore, writing $M_k(x^3, T) = det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k}) = \sum_{m=0}^{dim_{\mathbb{Q}_q(\pi)}(H_{1,k})} c_m T^m$, we have*

$$ord_q(c_m) \geq \frac{1}{3}(m^2 + m + km).$$

Proof of Corollary. By Corollary 3 and [12, Section 6.2], the $\mathbb{Z}_q[\pi]$-submodule V defined by the $\mathbb{Z}_q[\pi]$-span of the set $\{te_1^{k-2i} e_2^{2i}\}_{i=0}^{\frac{k-1}{2}} \subset \mathcal{N}$ satisfies (10) when k is odd and $k < p$. Thus, $A = \{(1, (k-2i, 2i))\}_{i=0}^{\frac{k-1}{2}}$, and so by Theorem 5,

$$\begin{aligned} ord_q(c_m) &\geq \sum_{j=0}^{m-1} (w_1(1) + w_0(k-2i, 2i)) \\ &= m w_1(1) + km w_0(1) + (w_0(2) - w_0(1)) m(m-1). \end{aligned}$$

The result follows since $w_1(n) := 2n/3$ and $w_0(n) := n/3$. □

The main result in section 3.4 below will be to reduce the hypothesis (10) to a similar, but weaker, hypothesis. It is expected that this weaker hypothesis will hold under certain conditions on the prime p, the degree d, and the symmetric power k. However, it is also expected that this weaker hypothesis will fail just as often, yet (10) will still hold. The conditions under which the weaker hypothesis is valid is currently under investigation.

The rest of this section is devoted to the proof of Theorem 5, whose argument closely follows that of Dwork's [7][Section 7] and Adolphson-Sperber's [2]. The proof rests on relating the Newton polygon of $\bar{\beta}$ to another operator, $\bar{\beta}_1$, whose Newton polygon is much easier to estimate due to the Dwork decomposition of $\mathcal{N}$ given in (10). The reason is that Dwork decomposition allows us to work on the chain level, where the operator β_1 acts in an easily understood way. Once estimates on β_1 are found on the chain level Dwork decomposition provides estimates in homology of $\bar{\beta}_1$.

For $b' \leq b$ and $b \leq p/(p-1)$, define $\alpha_1 : \mathcal{K}(b', b) \to \mathcal{K}(b'/p, b)$ by

$$\begin{aligned} \alpha_1 :&= \tau^{-1} \circ \psi_x \circ F(t, x) \\ &= \frac{1}{G(t^p, x)} \circ \tau^{-1} \circ \psi_x \circ G(t, x). \end{aligned}$$

Notice that $\alpha_1 \circ D(t) = pD(t^p) \circ \alpha_1$, and so α_1 induces a map

$$\bar{\alpha}_1(t) : \mathcal{H}_{1,t}(b', b) \to \mathcal{H}_{1,t^p}(b'/p, b).$$

On $\mathcal{K}(b', b)$, since

$$\begin{aligned} \alpha(t) &= \psi_x^a \circ F_a(t, x) \\ &= \psi_x^a \circ F^{\tau^{a-1}}(t^{p^{a-1}}, x^{p^{a-1}}) \cdots F^{\tau}(t^p, x^p) F(t, x) \\ &= \left(\tau^{-1} \circ \psi_x \circ F(t^{p^{a-1}}, x) \right) \circ \cdots \circ \left(\tau^{-1} \circ \psi_x \circ F(t^p, x) \right) \circ \left(\tau^{-1} \circ \psi_x \circ F(t, x) \right) \\ &= \alpha_1(t^{p^{a-1}}) \circ \cdots \circ \alpha_1(t^p) \circ \alpha_1(t), \end{aligned}$$

it follows that

$$\bar{\alpha}(t) = \bar{\alpha}_1(t^{p^{a-1}}) \circ \cdots \circ \bar{\alpha}_1(t^p) \circ \bar{\alpha}_1(t). \tag{11}$$

We also have the property that

$$\psi_t \circ \bar{\alpha}_1(t^p) = \bar{\alpha}_1(t) \circ \psi_t. \tag{12}$$

Consequently, with $\beta_1 := \psi_t \circ Sym^k(\bar{\alpha}_1(t)) : \mathcal{H}_{1,t}^{(k)}(b', b) \to \mathcal{H}_{1,t}^{(k)}(b', b)$, we have the relation

$$\begin{aligned}\beta_1^a &= \psi_t^a \circ Sym^k\left(\bar{\alpha}_1(t^{p^{a-1}}) \circ \cdots \circ \bar{\alpha}_1(t^p) \circ \bar{\alpha}_1(t)\right) \\ &= \psi_t^a \circ Sym^k(\bar{\alpha}(t)) \\ &= \beta\end{aligned}$$

where we have used (12) for the first equality and (11) for the second. Since $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k}) \in \mathbb{Q}_p(\pi)$, we have that

$$\begin{aligned}\left(det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k})\right)^a &= Norm_{\mathbb{Q}_q(\pi)/\mathbb{Q}_p(\pi)} det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k}) \\ &= det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}T \mid H_{1,k}).\end{aligned}$$

Thus,

$$\begin{aligned}\left(det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T^a \mid H_{1,k})\right)^a &= det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}T^a \mid H_{1,k}) \\ &= det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1^a T^a \mid H_{1,k}) \\ &= \prod_{\zeta^a = 1} det_{\mathbb{Q}_p(\pi)}(1 - \zeta\bar{\beta}_1 T \mid H_{1,k}).\end{aligned} \tag{13}$$

Counting multiplicities, let m_i denote the number of reciprocal roots of $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k})$ which have slope s_i; note, we say λ has slope s_i if $ord_p(\lambda) = s_i$. Then, from (13), $\left(det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T^a \mid H_{1,k})\right)^a$ has am_i reciprocal roots of slope s_i, and so $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T^a \mid H_{1,k})$ has m_i reciprocal roots with slope s_i. We conclude that $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k})$ has m_i/a reciprocal roots of slope as_i.

Next, for the q-adic valuation $ord_q(\cdot) := \frac{1}{a} ord_p(\cdot)$, we will say a root λ has q-adic slope s_i if $ord_q(\lambda) = s_i$. Observe that the above paragraph has demonstrated that $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k})$ has m_i reciprocal roots with *q-adic* slope s_i if and only if $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k})$ has am_i reciprocal roots with *p-adic* slope s_i. In terms of Newton polygons, this means the vertices of the *q-adic* Newton polygon of $det_{\mathbb{Q}_q(\pi)}(1 - \bar{\beta}T \mid H_{1,k})$ are

$$(0,0) \quad \text{and} \quad \left(\sum_{i=1}^n m_i, \sum_{i=1}^n m_i s_i\right) \qquad n = 1, 2, \ldots, dim_{\mathbb{Q}_q(\pi)}(H_{1,k})$$

if and only if the vertices of the *p-adic* Newton polygon of $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k})$ are

$$(0,0) \quad \text{and} \quad \left(\sum_{i=1}^n am_i, \sum_{i=1}^n am_i s_i\right) \qquad n = 1, 2, \ldots, dim_{\mathbb{Q}_p(\pi)}(H_{1,k}).$$

Using this relation, any lower bound for the p-adic Newton polygon of the latter may be transformed to a lower bound of the q-adic Newton polygon of the former by dividing the coordinates of the vertices by a. Let us now concentrate on a lower bound for the p-adic Newton polygon of $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k})$.

Define $\mathcal{K}(b', b)^\bullet := x\mathcal{K}(b', b)$, and let $\mathcal{V}$ be the $L(b')$-span of the set $\{x, x^2, \ldots, x^{d-1}\}$ in $\mathcal{K}(b', b)^\bullet$. Define $\mathcal{K}(b', b; \rho)^\bullet := \mathcal{K}(b', b)^\bullet \cap \mathcal{K}(b', b; \rho)$ and $\mathcal{V}(b', b; \rho) := \mathcal{V} \cap \mathcal{K}(b', b; \rho)$. By our hypothesis on the prime p, we will prove in the following section the Dwork decomposition

$$\mathcal{K}(b', b; 0)^\bullet \subset \mathcal{V}(b', b; 0) \oplus D(t)\mathcal{K}(b', b; e), \tag{14}$$

where $e := b - \frac{1}{p-1}$. Consequently, $\mathcal{K}(b', b)^\bullet = \mathcal{V} \oplus D(t)\mathcal{K}(b', b)$, so we may identify $\mathcal{H}_1$ with $\mathcal{V}$, a free $L(b')$-module of rank $d-1$ with basis $\{x, x^2, \ldots, x^{d-1}\}$. Consequently, $\mathcal{H}_1^{(k)}$ is a free $L(b')$-module with basis $\{e_1^{i_1} \cdots e_{d-1}^{i_{d-1}} := e^{\mathbf{i}}\}$, where $\mathbf{i} = (i_1, \ldots, i_{d-1})$, each i_j is a nonnegative integer satisfying $i_1 + \cdots + i_{d-1} = k$, and $e_j := x^j$.

Set $b = \frac{p}{p-1}$. For $i = 1, \ldots, d-1$, $x^i \in \mathcal{K}(\frac{b}{p}, \frac{b}{p}; -\frac{b}{p} w_0(i))^\bullet$ and so $F(t,x)x^i \in \mathcal{K}(\frac{b}{p}, \frac{b}{p}; -\frac{b}{p} w_0(i))^\bullet$. Thus, $\alpha_1(x^i) \in \mathcal{K}(\frac{b}{p}, b; -\frac{b}{p} w_0(i))^\bullet$. By (14) we may write this as

$$\alpha_1(x^i) = A_{i,1}x + \cdots + A_{i,d-1}x^{d-1} \mod(D(t^p)\mathcal{K}(\frac{b}{p}, b)) \tag{15}$$

where $A_{i,j} \in L(\frac{b}{p}; \frac{b}{p}(pw_0(j) - w_0(i)))$.

Let $S(i_1, \ldots, i_{d-1})$ denote the set of nonnegative integers $(l_s^{(r)})_{1 \le s,r \le d-1}$ that satisfy the system

$$\begin{aligned} l_1^{(1)} + \cdots + l_{d-1}^{(1)} &= i_1 \\ \vdots \qquad & \quad \vdots \\ l_1^{(d-1)} + \cdots + l_{d-1}^{(d-1)} &= i_{d-1}. \end{aligned}$$

and $T(j_1, \ldots, j_{d-1})$ denote the set of nonnegative integers $(l_s^{(r)})_{1 \le s,r \le d-1}$ that satisfy the system

$$\begin{aligned} l_1^{(1)} + \cdots + l_1^{(d-1)} &= j_1 \\ \vdots \qquad & \quad \vdots \\ l_{d-1}^{(1)} + \cdots + l_{d-1}^{(d-1)} &= j_{d-1}. \end{aligned}$$

The k-th symmetric power of $\bar{\alpha}_1$ acts on the basis $\{e^{\mathbf{i}}\}$ as follows:

$$\begin{aligned} & Sym^k(\bar{\alpha}_1(t)) e_1^{i_1} \cdots e_{d-1}^{i_{d-1}} \\ & \quad = (\bar{\alpha}_1(t) e_1)^{i_1} \cdots (\bar{\alpha}_1(t) e_{d-1})^{i_{d-1}} \\ & \quad = \left(\sum_{j=1}^{r} A_{1,j} e_j \right)^{i_1} \cdots \left(\sum_{j=1}^{r} A_{d-1,j} e_j \right)^{i_{d-1}} \\ & \quad = \sum_{(l_s^{(r)}) \in S(i_1, \ldots, i_{d-1})} \mathbb{Z}_{>0} \left(A_{1,1}^{l_1^{(1)}} \cdots A_{1,d-1}^{l_{d-1}^{(1)}} \right) \cdots \left(A_{d-1,1}^{l_1^{(d-1)}} \cdots A_{d-1,d-1}^{l_{d-1}^{(d-1)}} \right) e_1^{l_1^{(1)} + \cdots + l_1^{(d-1)}} \cdots e_{d-1}^{l_{d-1}^{(1)} + \cdots + l_{d-1}^{(d-1)}} \\ & \quad = \sum_{\substack{\mathbf{j} := (j_1, \ldots, j_{d-1}) \in \mathbb{Z}_{\ge 0}^{d-1} \\ j_1 + \cdots + j_{d-1} = k}} B(\mathbf{i}, \mathbf{j}) e_1^{j_1} \cdots e_{d-1}^{j_{d-1}}, \end{aligned}$$

where

$$B(\mathbf{i}; \mathbf{j}) := \sum_{(l_s^{(r)}) \in S(i_1, \ldots, i_{d-1}) \cap T(j_1, \ldots, j_{d-1})} (\mathbb{Z}_{>0}) \left(A_{1,1}^{l_1^{(1)}} \cdots A_{1,d-1}^{l_{d-1}^{(1)}} \right) \cdots \left(A_{d-1,1}^{l_1^{(d-1)}} \cdots A_{d-1,d-1}^{l_{d-1}^{(d-1)}} \right)$$

and "$\mathbb{Z}_{>0}$" is some determinable nonzero positive integer. It follows that $B(\mathbf{i}; \mathbf{j}) \in L(\frac{b}{p}; \frac{b}{p}(pw_0(\mathbf{j}) - w_0(\mathbf{i})))$, and so

$$Sym^k(\bar{\alpha}_1(t)) : \mathcal{H}_1^{(k)}(\frac{b}{p}, \frac{b}{p}; 0) \to \mathcal{H}_1^{(k)}(\frac{b}{p}, b; 0). \tag{16}$$

Recall, $\mathcal{M} := \mathcal{H}_{1,t}^{(k)}$ and $\mathcal{N} := t\mathcal{H}_{1,t}^{(k)}$. We are supposing that there exists a free $\mathbb{Z}_q[\pi]$-submodule $\mathcal{V}$ of $\mathcal{N}$ with basis $\Gamma := \{t^n e^{\mathbf{i}} \mid (n; \mathbf{i}) \in A\}$ such that

$$\mathcal{N}(b, b; 0) \subset \mathcal{V}(b, b; 0) \oplus \partial \mathcal{M}(b, b; \epsilon) \tag{17}$$

for some $\epsilon \in \mathbb{R}$. Now, Γ represents a basis of $H_{1,k}$ over $\mathbb{Q}_q(\pi)$, but we need to understand the Fredholm determinant of $\bar{\beta}_1$ on $H_{1,k}$ viewed as a vector space over $\mathbb{Q}_p(\pi)$. To do this recall that $\mathbb{Q}_q(\pi)$ is an unramified extension field of $\mathbb{Q}_p(\pi)$. We have denoted by $\mathbb{Z}_q[\pi]$ the ring of integers of $\mathbb{Q}_q(\pi)$ with uniformizer π and residue field $\mathbb{F}_q$, and $\mathbb{Z}_p[\pi]$ the ring of integers of $\mathbb{Q}_p(\pi)$ with uniformizer π and residue field $\mathbb{F}_p$. Let $\{\bar{\eta}_1, \ldots, \bar{\eta}_a\}$ be a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, and let $\{\eta_1, \ldots, \eta_a\}$ be a lifting of this basis to an integral basis of $\mathbb{Q}_q(\pi)$ over $\mathbb{Q}_p(\pi)$.

Lemma 1. *The basis $\{\eta_i\}$ has the property of p-adic directness; that is, for any $g \in \mathbb{Q}_q(\pi)$, writing $g = h_1\eta_1 + \cdots + h_a\eta_a$ with $h_i \in \mathbb{Q}_p(\pi)$, then*

$$ord_p(g) = \min_{i=1,\ldots,a} \{ord_p(h_i)\}.$$

Proof. Without loss of generality, we may assume that $ord_p(g) = 0$. Set $-c := (p-1)\min\{ord_p(h_i)\} \in \mathbb{Z}$. Suppose $c > 0$. For any $\xi \in \mathbb{Z}_q[\pi]$, denote by $\bar{\xi}$ its image in the residue field $\mathbb{F}_q$. Using this notation, we see that

$$0 = (\overline{\pi^c g}) = (\overline{\pi^c h_1})\bar{\eta}_1 + \cdots + (\overline{\pi^c h_a})\bar{\eta}_a \mod(\pi).$$

Since $\{\bar{\eta}_i\}$ is a basis of $\mathbb{F}_q$, we must have $\overline{\pi^c h_i} = 0$ in $\mathbb{F}_q$ for every i. Hence, $\pi^c h_i \in \pi \mathbb{Z}_q[\pi]$, and so $h_i \in \pi^{1-c}\mathbb{Z}_q[\pi]$ for every i. Thus, for each i we have

$$ord_p(h_i) \ge \frac{1-c}{p-1} = \frac{1}{p-1} + \min_{j=1,\ldots,a} \{ord_p(h_j)\}.$$

However, since this is not possible we must have c nonnegative. Thus, $-c \geq 0$ which means $\min\{ord_p(h_i)\} \geq 0 = ord_p(g)$. Since we easily have $ord_p(g) \geq \min\{ord_p(h_i)\}$, we must have equality, proving the lemma. $\square$

Since $t^n e^{\mathbf{i}} \in \mathcal{M}(\frac{b}{p}, \frac{b}{p}; -\frac{b}{p}W(n, \mathbf{i}))$, we have by (16) that $Sym^k(\alpha_1(t))(t^n e^{\mathbf{i}} \in \mathcal{N}(\frac{b}{p}, b; -\frac{b}{p}W(n, \mathbf{i}))$ and so $\beta_1(t^n e^{\mathbf{i}}) \in \mathcal{N}(b, b; -\frac{b}{p}W(n, \mathbf{i}))$. By Dwork decomposition (17), this means

$$\beta_1(t^n e^{\mathbf{i}}) = \sum_{(m,\mathbf{j}) \in \Gamma} C(n, \mathbf{i}; m, \mathbf{j}) t^m e^{\mathbf{j}} \quad mod(\partial \mathcal{M})$$

with

$$ord_p(C(n, \mathbf{i}; m, \mathbf{j})) \geq \frac{b}{p}(pW(m, \mathbf{j}) - W(n, \mathbf{i})).$$

From the lemma above, if $B \in \mathbb{Z}_q[\pi]$ satisfies $ord_p(B) \geq \rho$, then writing $B = B_1\eta_1 + \cdots + B_a\eta_a$ the coefficients satisfy $ord_p(B_i) \geq \rho$. Thus, we may write

$$C(n, \mathbf{i}; m, \mathbf{j}) = \sum_{r=1}^{a} C(n, \mathbf{i}; m, \mathbf{j})_r \eta_r$$

with $C(n, \mathbf{i}; m, \mathbf{j})_r \in \mathbb{Z}_p[\pi]$ and $ord_p(C(n, \mathbf{i}; m, \mathbf{j})_r) \geq \frac{b}{p}(pW(m, \mathbf{j}) - W(n, \mathbf{i}))$. Now, a basis of $H_{1,k}$ over the field $\mathbb{Q}_p(\pi)$ is given by $\Gamma' := \{\eta_j t^n e^{\mathbf{i}} \mid j = 1, \ldots, a, (n, \mathbf{i}) \in A\}$. Thus, for $\eta_j t^n e^{\mathbf{i}} \in \Gamma'$, we have

$$\begin{aligned} \beta_1(\eta_j t^n e^{\mathbf{i}}) &= \tau^{-1}(\eta_j) \sum_{(m,\mathbf{j}) \in \Gamma} C(n, \mathbf{i}; m, \mathbf{j}) t^m e^{\mathbf{j}} \quad mod(\partial \mathcal{M}) \\ &= \tau^{-1}(\eta_j) \sum_{(m,\mathbf{j}) \in \Gamma} \sum_{r=1}^{a} C(n, \mathbf{i}; m, \mathbf{j})_r \eta_r e^{\mathbf{j}} \end{aligned} \tag{18}$$

Writing

$$\tau^{-1}(\eta_j)\eta_r = \sum_{s=1}^{a} b_{s,j,r}\eta_s \quad \text{with } b_{s,j,r} \in \mathbb{Z}_p[\pi],$$

then (18) becomes

$$\bar{\beta}_1(\eta_j t^n e^{\mathbf{i}}) = \sum_{(m,\mathbf{j}) \in \Gamma} \sum_{s=1}^{a} D(j, n, \mathbf{i}; s, m, \mathbf{j}) \eta_s t^m e^{\mathbf{j}}$$

where

$$D(j, n, \mathbf{i}; s, m, \mathbf{j}) := \sum_{r=1}^{a} C(n, \mathbf{i}; m, \mathbf{j})_r b_{s,j,r}.$$

It follows that

$$ord_p(D(j, n, \mathbf{i}; s, m, \mathbf{j})) \geq \frac{b}{p}(pW(m, \mathbf{j}) - W(n, \mathbf{i})).$$

Writing $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k}) = \sum_{m=0}^{\infty} c_m T^m$ then

$$c_m = (-1)^m \sum \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma) \prod_{l=1}^{m} D(j_l, n_l, \mathbf{i}^{(l)}; j_{\sigma(l)}, n_{\sigma(l)}, \mathbf{i}^{(\sigma(l))}),$$

where $\mathcal{S}_m$ is the permutation group on $\{1, \ldots, m\}$ and the outer summation runs over all sets consisting of m distinct elements of the form $(j, n, \mathbf{i})$ where $\eta_j t^n e^{\mathbf{i}} \in \Gamma'$. It follows that

$$ord_p(c_m) \geq \min \left\{ \sum_{l=1}^{m} W(n_l, \mathbf{i}^{(l)}) \right\}$$

where the minimum runs over all sets consisting of m distinct elements of the form $(j, n, \mathbf{i})$ where $\eta_j t^n e^{\mathbf{i}} \in \Gamma'$.

Let $r_N := \#\{t^n e^{\mathbf{i}} \in \Gamma \mid W(n, \mathbf{i}) = N/(d\delta_i)\}$. Then there are ar_N number of elements $\eta_j t^n e^{\mathbf{i}} \in \Gamma'$ with weight $W(n, \mathbf{i}) = N/(d\delta_i)$. Thus, if

$$m = \sum_{N=0}^{R} ar_N$$

then

$$ord_p(c_m) \geq \sum_{N=0}^{R} ar_N \left(\frac{N}{d\delta_i}\right).$$

In other words, the p-adic Newton polygon of $det_{\mathbb{Q}_p(\pi)}(1 - \bar{\beta}_1 T \mid H_{1,k})$ lies on or above the lower convex hull of the points

$$\left(\sum_{N=0}^{R} ar_N, \sum_{N=0}^{R} ar_N \frac{N}{d\delta_i}\right) \qquad R = 0, 1, \ldots, dim_{\mathbb{Q}_p(\pi)} H_{1,k}$$

Thus, the q-adic Newton polygon of $det_{\mathbb{Q}_q(\pi)}(1 - \beta T \mid H_{1,k})$ lies on or above the lower convex hull of the points

$$\left(\sum_{N=0}^{R} r_N, \sum_{N=0}^{R} r_N \frac{N}{d\delta_i}\right) \qquad R = 0, 1, \ldots, dim_{\mathbb{Q}_q(\pi)} H_{1,k}.$$

This finishes the proof of Theorem 5.

3.3 Relative Dwork homology

Integral to the proof of Theorem 5 was the Dwork decomposition of relative homology given on (14). The main result of this section is to provide a proof of this result. We begin by recalling that

$$\begin{aligned}
\mathcal{V}(b', b; \rho) &:= \left(L(b')x \oplus \cdots \oplus L(b')x^{d-1}\right) \cap \mathcal{K}(b', b; \rho) \\
\mathcal{V}(b', b) &:= \bigcup_{\rho \in \mathbb{R}} \mathcal{V}(b', b; \rho).
\end{aligned}$$

Theorem 6. *Suppose $(p, d) = 1$. Let b and b' be real numbers satisfying $b' \leq b$ and $\frac{1}{p-1} \leq b \leq \frac{p}{p-1}$. Set $e := b - \frac{1}{p-1}$. Then*

1. *$\mathcal{K}(b', b)^{\bullet} = \mathcal{V}(b', b) \oplus D(t)\mathcal{K}(b', b)$,*
2. *$\mathcal{K}(b', b; 0)^{\bullet} \subset \mathcal{V}(b', b; 0) \oplus D(t)\mathcal{K}(b', b; e)$,*
3. *$D(t)$ is injective if $b > \frac{1}{p-1}$,*
4. *if $g \in \mathcal{K}(b', b)^{\bullet}$ is divisible by t^n and we write $g = \xi + D(t)\zeta$ with $\xi \in \mathcal{V}(b', b)$ and $\zeta \in \mathcal{K}(b', b)$, then t^n divides ξ and ζ.*

The proof of this theorem will consist of a series of lemmas which will comprise the rest of this section.

Lemma 2. *Suppose $b' \leq b$. Then*

$$\mathcal{K}(b', b; 0)^{\bullet} \subset \mathcal{V}(b', b; 0) + \pi \hat{f}_x \mathcal{K}(b', b; e).$$

Furthermore, if $\xi \in \mathcal{K}(b', b)^{\bullet}$ is divisible by t^n then when we write $\xi = \zeta + (\pi \hat{f}_x)\nu$ with $\zeta \in \mathcal{V}(b', b)$ and $\nu \in \mathcal{K}(b', b)$, then $t^{n+1} \mid \zeta$ and $t^n \mid \nu$.

Proof. Let $\hat{P}_j \in \mathbb{Z}_q$ be the Teichmüller lifting of the polynomials P_j, where $f(t, x) = x^d + \sum_{j=i}^{d-1} P_j(t)x^j$. Now, with

$$\pi \hat{f}_x(t, x) = \pi(dx^d + \sum_{l=i}^{d-1} l\hat{P}_l(t)x^l)$$

we may write, for $m \geq d$,

$$\begin{aligned}
x^m &= \pi d x^d \left(\frac{1}{\pi d} x^{m-d}\right) + \pi \sum_{l=i}^{d-1} l\hat{P}_l(t)x^l \left(\frac{1}{\pi d} x^{m-d} - \frac{1}{\pi d} x^{m-d}\right) \\
&= \pi \hat{f}_x(t, x) \left(\frac{1}{\pi d} x^{m-d}\right) - \sum_{l=i}^{d-1} \frac{l}{d} \hat{P}_l(t) x^{m+l-d}.
\end{aligned}$$

Notice that the last right-hand sum consists of terms in x of degree strictly smaller than x^m. This is our reduction formula for x^m, reducing all monomials x^m to some linear combination of $\{x, x^2, \ldots, x^{d-1}\}$.

Next, consider $B_{nm}t^n x^m \in \mathcal{K}(b', b; 0)$ with $B_{nm} \in \mathbb{Z}_q[\pi]$. The reduction formula takes the form

$$B_{nm}t^n x^m = \pi \hat{f}_x(t,x)\left(\frac{B_{nm}}{\pi d}t^n x^{m-d}\right) - \sum_{l=i}^{d-1} \frac{lB_{nm}}{d}\hat{P}_l(t)t^n x^{m+l-d}. \tag{19}$$

With $m \geq d$, it is immediate that $\frac{B_{nm}}{\pi d}t^n x^{m-d} \in \mathcal{K}(b', b; e)$. Next, consider the term $\frac{lB_{nm}}{d}\hat{P}_l(t)t^n x^{m+l-d}$. Notice that it is divisible by t^{n+1}; this demonstrates the second part of the lemma. Writing $\hat{P}_l(t) = \sum_{j=0}^{\delta_l} a_j t^j$, where the a_j are p-adic units, then we are reduced to considering terms of the form $\frac{lB_{nm}}{d}a_j t^{n+j}x^{m+l-d}$. Then

$$\begin{aligned}\mathrm{ord}_p(\frac{lB_{nm}}{d}a_j) &\geq b'w_1(n) + bw_0(m)\\ &= b'w_1(n+j) + bw_0(m+l-d) + \underbrace{(-b'w_1(j) - bw_0(l-d))}_{=:\rho}.\end{aligned}$$

Using that $b \geq b'$ and (7), we have

$$\rho \geq -b\left(w_1(j) + w_0(l) - 1\right) \geq 0.$$

Hence, iterating the recursive equation (19), we obtain

$$B_{nm}t^n x^m \in \mathcal{V}(b', b; 0) + \pi\hat{f}_x(t,x)\mathcal{K}(b', b; e). \tag{20}$$

Next, let $\xi = \sum_{n,m\geq 0} B_{nm}t^n x^m \in \mathcal{K}(b', b; 0)^{\bullet}$. For each $N \in \mathbb{Z}_{\geq 0}$ we may write $\xi = \zeta^{(N)} + \sum_{n\geq 0}\eta^{(n,N)}$ where

$$\eta^{(n,N)} := \sum_{m=0}^{N} B_{nm}T^n x^m \qquad \text{and} \qquad \zeta^{(N)} := \sum_{n\geq 0}\sum_{m\geq N+1} B_{nm}t^n x^m.$$

Observe that $x^{N+1} \mid \zeta^{(N)}$ for every N, and $t^{n+1} \mid \eta^{(n,N)}$. By (20), we may write $\eta^{(n,N)} = \nu_1^{(n,N)} + (\pi\hat{f}_x)\nu_2^{(n,N)}$ with $\nu_1^{(n,N)} \in V(b', b; 0)$ and $\nu_2^{(n,N)} \in \mathcal{K}(b', b; e)$, both with the property that they are divisible by t^n. Hence, $\sum_{n\geq 0}\nu_1^{(n,N)}$ and $\sum_{n\geq 0}\nu_2^{(n,N)}$ produce well-defined elements of $V(b', b; 0)$ and $\mathcal{K}(b', b; e)$, respectively. Let us denote these elements by $\nu_1^{(N)}$ and $\nu_2^{(N)}$. We have thus constructed sequences of elements $\{\nu_1^{(N)}\}_{N\geq 1}$ in $V(b', b; 0)$ and $\{\nu_2^{(N)}\}_{N\geq 1}$ in $\mathcal{K}(b', b; e)$ which satisfy

$$\xi = \zeta^{(N)} + \nu_1^{(N)} + (\pi\hat{f}_x)\nu_2^{(N)}$$

Now, in the topology of coefficient-wise convergence (i.e. the (π, t, x)-adic topology), $\mathbb{Z}_q[\pi][[t, x]]$ is compact. Thus, $\mathcal{K}(b', b; \rho)$ is compact for each ρ in the induced topology. Hence, we may restrict ourselves to convergent subsequences of $\{\nu_1^{(N)}\}_{N\geq 1}$ and $\{\nu_2^{(N)}\}_{N\geq 1}$ with limits $\nu_1 \in V(b', b; 0)$ and $\nu_2 \in \mathcal{K}(b', b; e)$, respectively. Thus,

$$\xi = \lim_{N\to\infty}\left(\zeta^{(N)} + \nu_1^{(N)} + (\pi\hat{f}_x)\nu_2^{(N)}\right) = \nu_1 + (\pi\hat{f}_x)\nu_2,$$

where $\lim_{N\to\infty}\zeta^{(N)} = 0$ since $x^{N+1} \mid \zeta^{(N)}$ for each N. This proves the lemma. □

Lemma 3. *Let b and b' be real numbers. Then $\mathcal{V}(b', b) \cap \pi\hat{f}_x\mathcal{K}(b', b) = \{0\}$.*

Proof. Suppose $\eta := c_1 x + \cdots + c_{d-1}x^{d-1} \in \mathcal{V}(b', b) \cap \pi\hat{f}_x\mathcal{K}(b', b)$. Let $\zeta := \sum_{j=0}^{\infty} B_j x^j \in \mathcal{K}(b', b)$ satisfy $\eta = \pi\hat{f}_x\zeta$. Now,

$$\pi\hat{f}_x\sum_{j=0}^{\infty} B_j x^j = c_1 x + c_2 x^2 + \cdots + c_{d-1}x^{d-1}.$$

Writing $\pi\hat{f}_x(t,x) = \pi(dx^d + \sum_{l=i}^{d-1} l\hat{P}_l(t)x^l)$, we have

$$\sum_{j=0}^{\infty}\pi dB_j x^{d+j} + \sum_{r=0}^{\infty}\sum_{l=i}^{d-1}\pi B_r\hat{P}_l x^{l+r} = c_1 x + \cdots + c_{d-1}x^{d-1}.$$

For $j \geq 0$, since the coefficient of x^{d+j} in this equation must vanish, we have

$$\pi dB_j + \sum_{r=j+1}^{d+j-i}\pi B_r\hat{P}_{d+j-r} = 0$$

and so

$$B_j = -\frac{1}{d} \sum_{r=j+1}^{d+j-i} B_r \hat{P}_{d+j-r}. \tag{21}$$

Since $t \mid P_l$ for each l, we have $t \mid B_j$ for every j. Using (21) recursively, we see that $t^n \mid B_j$ for every positive integer n. Hence, $B_j = 0$ for all $j \geq 0$ as desired. $\square$

Lemma 4. *Let $\xi \in \mathcal{K}(b', b)$ and suppose $\pi \hat{f}_x \xi \in \mathcal{K}(b', b; \rho)$. Then $\xi \in \mathcal{K}(b', b; \rho + e)$.*

Proof. Let $\xi = \sum_{j=0}^{\infty} B_j x^j \in \mathcal{K}(b', b)$ and $\pi \hat{f}_x \xi = \sum_{j=0}^{\infty} C_j x^j \in \mathcal{K}(b', b; \rho)$. Writing $\pi \hat{f}_x(t, x) = \pi(dx^d + \sum_{l=i}^{d-1} l \hat{P}_l(t) x^l)$ we have

$$\sum_{j=0}^{\infty} \pi d B_j x^{d+j} + \sum_{r=0}^{\infty} \sum_{l=i}^{d-1} \pi B_r \hat{P}_l x^{l+r} = \sum_{j=0}^{\infty} C_j x^j.$$

From this, the coefficient of x^{d+j} satisfies

$$\pi d B_j + \sum_{r=j+1}^{d+j-i} \pi B_r \hat{P}_{d+j-r} = C_{d+j}$$

for all $j \geq 0$. Rewriting this, we have

$$B_j = \frac{1}{\pi d} C_{d+j} + \frac{1}{d} \sum_{r=j+1}^{d+j-i} B_r \hat{P}_{d+j-r}.$$

Iterating this n-times produces

$$B_j = \zeta_n^{(j)} + \left(\xi_1^{(j)} + \xi_2^{(j)} + \cdots + \xi_n^{(j)} \right)$$

where

$$\zeta_n^{(j)} := \sum_{r_1=j+1}^{d+j-i} \sum_{r_2=r_1+1}^{d+r_1-i} \cdots \sum_{r_n=r_{n-1}+1}^{d+r_{n-1}-i} \frac{1}{d^n} B_{r_n} \hat{P}_{d+j-r_1} \hat{P}_{d+r_1-r_2} \cdots \hat{P}_{d+r_{n-1}-r_n}$$

and

$$\begin{aligned}
\xi_1^{(j)} &:= \frac{1}{\pi d} C_{d+j} \\
\xi_2^{(j)} &:= \frac{1}{d^2 \pi} \sum_{r_1=j+1}^{d+j-i} C_{d+r_1} \hat{P}_{d+j-r_1} \\
\xi_3^{(j)} &:= \frac{1}{d^3 \pi} \sum_{r_1=j+1}^{d+j-i} \sum_{r_2=r_1+1}^{d+r_1-i} C_{d+r_2} \hat{P}_{d+j-r_1} \hat{P}_{d+r_1-r_2} \\
&\vdots \\
\xi_n^{(j)} &:= \frac{1}{d^n \pi} \sum_{r_1=j+1}^{d+j-i} \sum_{r_2=r_1+1}^{d+r_1-i} \cdots \sum_{r_{n-1}=r_{n-2}+1}^{d+r_{n-2}-i} C_{d+r_{n-1}} \hat{P}_{d+j-r_1} \hat{P}_{d+r_1-r_2} \cdots \hat{P}_{d+r_{n-2}-r_{n-1}}.
\end{aligned}$$

Now, we have assumed that each $\hat{P}_l$ is divisible by t, which shows $t^n \mid \zeta_n^{(j)}$. Hence, t-adically $\zeta_n^{(j)} \to 0$ as n tends to infinity. Next, to complete the lemma, let us show $\sum_{n=1}^{\infty} \xi_n^{(j)} \in L(b'; bw_0(j) + e)$.

We know $C_{d+r_{n-1}} \in L(b'; bw_0(d + r_{n-1}) + \rho)$. Thus

$$\frac{1}{d^{n-1} \pi} C_{d+r_{n-1}} \hat{P}_{d+j-r_1} \hat{P}_{d+r_1-r_2} \cdots \hat{P}_{d+r_{n-2}-r_{n-1}} \in L(b'; \epsilon(j, r_1, \ldots, r_{n-1}) + \rho)$$

where

$$\begin{aligned}
\epsilon(j, r_1, \ldots, r_{n-1}) :&= bw_0(d + r_{n-1}) - \frac{1}{p-1} - b' \left(w_1(deg(\hat{P}_{d+j-r_1})) + \cdots + w_1(deg(\hat{P}_{d+r_{n-2}-r_{n-1}})) \right) \\
&\geq e + \frac{b}{d} \left(r_{n-1} - (r_1 - j) - (r_2 - r_1) - \cdots - (r_{n-2} - r_{n-1}) \right) \\
&= e + bw_0(j).
\end{aligned}$$

Note, for the inequality in the above we used that, from (5), the degree of the polynomial $\hat{P}_{d+j-r}$ satisfies

$$deg(\hat{P}_{d+j-r}) =: \delta_{d+j-r} \le \delta_i \left(\frac{d-(d+j-r)}{d-i}\right) = \delta_i \left(\frac{r-j}{d-i}\right).$$

Next, since $t \mid \hat{P}_l$ for every l, $t^{n-1} \mid \xi_n^{(j)}$. It follows that

$$B_j = \lim_{n\to\infty}\left(\zeta_n^{(j)} + \sum_{l=1}^{n} \xi_l^{(j)}\right) = \sum_{n=1}^{\infty} \xi_n^{(j)} \in L(b'; bw_0(j) + e + \rho)$$

which concludes the lemma. □

Lemma 5. *Let $b' \le b$ and $\frac{1}{p-1} \le b \le \frac{p}{p-1}$. With $e := b - \frac{1}{p-1}$ we have*

$$\mathcal{K}(b', b; 0)^{\bullet} \subset \mathcal{V}(b', b; 0) + D(t)\mathcal{K}(b', b; e).$$

Furthermore, if $\xi \in \mathcal{K}(b', b)^{\bullet}$ is divisible by t^n, then when we write $\xi = \eta + D(t)\zeta$ with $\eta \in \mathcal{V}(b', b)$ and $\zeta \in \mathcal{K}(b', b)$, then η and ζ are also divisible by t^n.

Proof. Recall, $D(t) = x\frac{\partial}{\partial x} + H(t, x)$ with $H(t, x) := \sum_{j=0}^{\infty} \pi_j p^j \hat{f}_x^{\tau^j}(t^{p^j}, x^{p^j})$. Now, observe that

$$\hat{f}_x^{\tau^j}(t^{p^j}, x^{p^j}) = \hat{f}_x(t, x)^{p^j} + ph_j(t, x)$$

for some polynomial h_j with coefficients in $\mathbb{Z}_q[\pi]$. Write

$$H(t, x) = \pi\hat{f}_x(t, x)Q_1(t, x) + K_1(t, x)$$

where

$$\begin{aligned} Q_1(t, x) &:= \sum_{j=0}^{\infty} \pi_j \pi^{-1} p^j \hat{f}_x(t, x)^{p^j - 1} \\ K_1(t, x) &:= \sum_{j=1}^{\infty} \pi_j p^{j+1} h_j(t, x). \end{aligned}$$

We claim that $Q_1, \frac{1}{Q_1}, K_1 \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; 0)$. To see this, note that since $\hat{f} \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\frac{p}{p-1})$, we have

$$\hat{f}_x^{p^j - 1} \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right)(p^j - 1)).$$

Thus,

$$\pi_j \pi^{-1} p^j \hat{f}_x^{p^j - 1} \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; 0)$$

proving the result for Q_1 and $1/Q_1$. Next, since h_j consists of terms coming from the expansion of $\hat{f}_x^{p^j}$, we see that $h_j \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right)p^j)$. Thus $\pi_j p^{j+1} h_j \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; 0)$ proving the result for K_1.

We will first suppose $b > \frac{1}{p-1}$. Let $\xi \in \mathcal{K}(b', b; 0)^{\bullet}$. By Lemma 2, there exists $\eta_1 \in \mathcal{V}(b', b; 0)$ and $\zeta_1 \in \mathcal{K}(b', b; e)$ such that $\xi = \eta_1 + (\pi\hat{f}_x)\zeta_1$. Thus,

$$\begin{aligned} \xi &= \eta_1 + (H - K_1)Q_1^{-1}\zeta_1 \\ &= \eta_1 + \underbrace{(Q_1^{-1}K_1\zeta_1 - x\frac{\partial}{\partial x}Q_1^{-1}\zeta_1)}_{=:\nu_1} + D(t)(\underbrace{Q_1^{-1}\zeta_1}_{=:\zeta_1'}). \end{aligned}$$

Notice that $\nu_1 \in \mathcal{K}(b', b; e)^{\bullet}$ and $\zeta_1' \in \mathcal{K}(b', b; e)$. Continuing this same process, but now with ν_1 instead of ξ, we are lead to

$$\xi = (\eta_1 + \cdots + \eta_N) + \nu_N + D(t)(\zeta_1' + \cdots + \zeta_N')$$

where $\eta_i \in \mathcal{V}(b', b; (i-1)e)$, $\nu_N \in \mathcal{K}(b', b; Ne)^{\bullet}$, and $\zeta_i' \in \mathcal{K}(b', b; ie)$. Thus, $\eta := \sum_{i=1}^{\infty} \eta_i \in \mathcal{V}(b', b; 0)$ and $\zeta' := \sum_{i=1}^{\infty} \zeta_i' \in \mathcal{K}(b', b; e)$. Upon taking the limit in the coefficientwise convergence topology we see that

$$\xi = \eta + D(t)\zeta'$$

as desired.

We now consider the case when $b = \frac{1}{p-1}$. Let $\xi \in \mathcal{K}(b', b; 0)^{\bullet}$. For each $N \in \mathbb{Z}_{\geq 0}$, we may write

$$\xi = \sum_{n=1}^{N} B_n x^n + \sum_{n \geq N+1} B_n x^n.$$

Let $\epsilon > 0$. For $1 \leq n \leq N$, since $bw_0(n) \geq (b + \frac{\epsilon}{w_0(N)})w_0(n) - \epsilon$ we see that

$$\sum_{n=1}^{N} B_n x^n \in \mathcal{K}(b', b + \frac{\epsilon}{w_0(N)}; -\epsilon)^{\bullet}.$$

Since $b + \frac{\epsilon}{w_0(N)} > \frac{1}{p-1}$, there exists $\eta^{(\epsilon,N)} \in \mathcal{V}(b', b + \frac{\epsilon}{w_0(N)}; -\epsilon)$ and $\zeta^{(\epsilon,N)} \in \mathcal{K}(b', b + \frac{\epsilon}{w_0(N)}; -\epsilon + \frac{\epsilon}{w_0(N)})$ such that

$$\sum_{n=1}^{N} B_n x^n = \eta^{(\epsilon,N)} + D(t)\zeta^{(\epsilon,N)}.$$

We have just constructed sequences $\{\eta^{(\epsilon,N)}\}_{N=1}^{\infty} \subset \mathcal{V}(b', b; -\epsilon)$ and $\{\zeta^{(\epsilon,N)}\}_{N=1}^{\infty} \subset \mathcal{K}(b', b; -\epsilon)$. Since $\mathcal{K}(b', b; \rho)$ and $\mathcal{V}(b', b; \rho)$ are compact in the coefficientwise convergence topology for each ρ, we may restrict ourselves to convergent subsequences with limits $\eta^{(\epsilon)} \in \mathcal{V}(b', b; -\epsilon)$ and $\zeta^{(\epsilon)} \in \mathcal{K}(b', b; -\epsilon)$ which satisfy

$$\xi = \eta^{(\epsilon)} + D(t)\zeta^{(\epsilon)}.$$

In the coefficientwise convergence topology, letting $\epsilon \to 0^+$, there exists $\eta \in \mathcal{V}(b', b; 0)$ and $\zeta \in \mathcal{K}(b', b; 0)$ such that

$$\xi = \lim_{\epsilon \to 0^+} \eta^{(\epsilon)} + D(t)\zeta^{(\epsilon)} = \eta + D(t)\zeta.$$

This proves the first part of the lemma.

The second part follows from the divisibility result in Lemma 2 and running through the above argument. $\square$

Lemma 6. *Let $b' \leq b$ and $\frac{1}{p-1} \leq b \leq \frac{p}{p-1}$. Then*

$$\mathcal{V}(b', b) \cap D(t)\mathcal{K}(b', b).$$

Proof. Let us first assume $b > \frac{1}{p-1}$. Let $\eta \in V(b', b) \cap D(t)\mathcal{K}(b', b)$ be non-zero, and let $\xi \in \mathcal{K}(b', b)$ be such that $D(t)\xi = \eta$. Find a real number c such that $\xi \in \mathcal{K}(b', b; c)$ but $\xi \notin \mathcal{K}(b', b; c + e)$. We will prove that no such c exists, contradicting the existence of η.

Since $D(t) = x\frac{\partial}{\partial x} + (\pi \hat{f}_x)Q_1 + K_1$, we have

$$\eta = D(t)\xi = (\pi \hat{f}_x)Q_1\xi + \left(x\frac{\partial}{\partial x}\xi + K_1\xi\right).$$

Since $x\frac{\partial}{\partial x}\xi + K_1\xi \in \mathcal{K}(b', b; c)^{\bullet}$, by Lemma 5 there exists $\nu_1 \in \mathcal{V}(b', b; c)$ and $\zeta_1 \in \mathcal{K}(b', b; c + e)$ such that

$$x\frac{\partial}{\partial x}\xi + K_1\xi = \nu_1 + D(t)\zeta_1.$$

Hence,

$$\begin{aligned} \eta &= (\pi \hat{f}_x)Q_1\xi + \nu_1 + D(t)\zeta_1 \\ &= (\pi \hat{f}_x)Q_1(\xi + \zeta_1) + \nu_1 + K_1\zeta_1. \end{aligned}$$

Since $K_1\zeta_1 \in \mathcal{K}(b', b; c + e)$, applying Lemma 5 again produces $\nu_2 \in \mathcal{V}(b', b; c + e)$ and $\zeta_2 \in \mathcal{K}(b', b; c + 2e)$ such that $K_1\zeta_1 = \nu_2 + D(t)\zeta_2$. Thus,

$$\eta = (\pi \hat{f}_x)Q_1(\xi + \zeta_1 + \zeta_2) + (\nu_1 + \nu_2) + K_1\zeta_2.$$

Iterating this via induction, and taking the limit in the coefficient-wise convergence topology, we obtain

$$\eta = (\pi \hat{f}_x)Q_1(\xi + \sum_{i=1}^{\infty} \zeta_i) + \sum_{i=1}^{\infty} \nu_i.$$

This means $(\pi \hat{f}_x)Q_1(\xi + \sum_{i=1}^{\infty} \zeta_i) \in \mathcal{V}(b', b)$, and so by Lemma 3, $\xi = -\sum_{i=1}^{\infty} \zeta_i \in \mathcal{K}(b', b; c)$ which is impossible.

Suppose now that $b = \frac{1}{p-1}$ and $\eta \in \mathcal{V}(b', b) \cap D(t)\mathcal{K}(b', b)$. With this choice of b, $\alpha_1 := \tau^{-1} \circ \psi_x \circ F(t, x)$ is a map from $\mathcal{K}(b', b)$ to $\mathcal{K}(b', pb)$. Therefore, since $\alpha_1 \circ D(t) = pD(t^p) \circ \alpha_1$, we see that

$$\alpha_1(\eta) \in D(t^p)\mathcal{K}(b', pb).$$

Thus, the reduction of $\alpha_1(\eta)$ equals zero in $\mathcal{H}_{1,t^p}(b', pb)$. Now, we may also view α_1 as an endomorphism of $\mathcal{K}(b', pb)$ and so, abusing notation, we obtain a map on homology $\bar{\alpha}_1 : \mathcal{H}_{1,t}(b', pb) \to \mathcal{H}_{1,t^p}(b', pb)$. By Lemma 7 below, this map $\bar{\alpha}_1$ is invertible. Therefore, since the reduction of $\alpha_1(\eta)$ is zero in $\mathcal{H}_{1,t^p}(b', pb)$, we must have the reduction of η in $\mathcal{H}_{1,t}(b', pb)$ equal to zero as well. Hence, $\eta \in D(t)\mathcal{K}(b', pb)$. Since $D(t)\mathcal{K}(b', pb) \subset \mathcal{K}(b', pb)$, we see that $\eta \in V(b', pb) \cap D(t)\mathcal{K}(b', pb)$. However, this intersection equals $\{0\}$ by the argument above since $bp > 1/(p-1)$, proving $\eta = 0$ as desired. ☐

Lemma 7. *Let $b' \leq 1/(p-1)$. Then $\bar{\alpha}_1 : \mathcal{H}_{1,t}(b', p/(p-1)) \to \mathcal{H}_{1,t^p}(b', p/(p-1))$ is an isomorphism.*

Proof. Since $b' \leq 1/(p-1)$, it follows from definition that α_1 is a map from $\mathcal{K}(b', 1/(p-1))$ to $\mathcal{K}(b', p/(p-1))$. Now, define a map $\alpha_1' : \mathcal{K}(b', p/(p-1)) \to \mathcal{K}(b', 1/(p-1))$ by

$$\alpha_1' := F(t, x)^{-1} \circ \Phi_x \circ \tau$$

where Φ_x is the map $x \mapsto x^p$. Clearly, $\alpha_1 \circ \alpha_1' = id$, the identity map on $\mathcal{K}(b', p/(p-1))$. Hence, we have

$$\mathcal{K}(b', p/(p-1)) = \alpha_1 \alpha_1' \mathcal{K}(b', p/(p-1)) \subset \alpha_1 \mathcal{K}(b', 1/(p-1)) \subset \mathcal{K}(b', p/(p-1)).$$

Hence, α_1 maps $\mathcal{K}(b', 1/(p-1))$ isomorphically onto $\mathcal{K}(b', p/(p-1))$. A similar argument shows α_1 maps $\mathcal{K}(b', 1/(p-1))^\bullet$ isomorphically onto $\mathcal{K}(b', p/(p-1))^\bullet$.

By Lemma 5, we know

$$\mathcal{K}(b', 1/(p-1))^\bullet \subset \mathcal{V}(b', 1/(p-1)) + D(t)\mathcal{K}(b', 1/(p-1)).$$

Applying α_1 to this we obtain

$$\mathcal{K}(b', p/(p-1))^\bullet = \alpha_1 \mathcal{K}(b', 1/(p-1)) \subset \alpha_1 \mathcal{V}(b', 1/(p-1)) + D(t^p)\mathcal{K}(b', p/(p-1)).$$

Since $\mathcal{V}(b', p/(p-1)) \subset \mathcal{K}(b', p/(p-1))^\bullet$, we have

$$\mathcal{V}(b', p/(p-1)) \subset \alpha_1 \mathcal{V}(b', 1/(p-1)) + D(t^p)\mathcal{K}(b', p/(p-1)).$$

Now, it follows from the definition that $\mathcal{V}(b', b_1) = \mathcal{V}(b', b_2)$ for any positive real numbers b_1 and b_2. Thus,

$$\mathcal{V}(b', p/(p-1)) \subset \alpha_1 \mathcal{V}(b', p/(p-1)) + D(t^p)\mathcal{K}(b', p/(p-1)).$$

Viewing α_1 as an endomorphism of $\mathcal{K}(b', p/(p-1))^\bullet$, this shows $\bar{\alpha}_1 : \mathcal{H}_{1,t}(b', p/(p-1)) \to \mathcal{H}_{1,t^p}(b', p/(p-1))$ is surjective. Since both of these spaces are free $L(b')$-modules of finite rank, $\bar{\alpha}_1$ must also be injective. This finishes the lemma. ☐

Lemma 8. *Suppose $b > \frac{1}{p-1}$. If $\xi \in \mathcal{K}(b', b)$ and $D(t)\xi \in \mathcal{K}(b', b; \rho)$, then $\xi \in \mathcal{K}(b', b; \rho + e)$.*

Proof. Suppose $\xi \neq 0$. Choose $c \in \mathbb{R}$ such that $\xi \in \mathcal{K}(b', b; c)$ but $\xi \notin \mathcal{K}(b', b; c + e)$. Then

$$(\pi \hat{f}_x)Q_1\xi = D(t)\xi - x\frac{\partial}{\partial x}\xi - K_1\xi \in \mathcal{K}(b', b; \min\{\rho, c\}).$$

Thus, $(\pi \hat{f}_x)\xi \in \mathcal{K}(b', b; \min\{\rho, c\})$ which, by Lemma 4, implies $\xi \in \mathcal{K}(b', b; \min\{\rho, c\} + e)$. By our choice of c the lemma follows. ☐

Corollary 2. *Suppose $b > \frac{1}{p-1}$. Then $D(t)$ is injective.*

Proof. Suppose there exists nonzero $\xi \in \mathcal{K}(b', b)$ such that $D(t)\xi = 0$. Then, by Lemma 8, since $0 \in \mathcal{K}(b', b; \rho)$ for every ρ, we have $\xi \in \mathcal{K}(b', b; \rho + e)$ for every ρ. Hence, ξ must be zero. ☐

3.4 Dwork decomposition for the symmetric powers of relative homology

Theorem 5 demonstrated one consequence of Dwork decomposition with the operator ∂. In this section, we wish to take the hypothesis of Dwork decomposition in that theorem and reduce it to a weaker hypothesis. We expect that this "weaker hypothesis" may be demonstrated for a large class of $f(t,x)$. However, we also expect this weaker hypothesis to fail just as often. New ideas will be required to handle the latter case.

It is easiest to see the main obstructions to the theory if we generalize a bit. Let d be a positive integer. We call a function $w : \mathbb{Z}_{\geq 0}^s \to \frac{1}{d}\mathbb{Z}_{\geq 0}$ a *weight function* if it satisfies the following three properties:

1. $w(0) = 0$,
2. $w(cu) = cw(u)$ for every $c \in \mathbb{Q}_{\geq 0}$, and
3. $w(u+v) \leq w(u) + w(v)$ for all $u, v \in \mathbb{Z}_{\geq 0}^s$.

Let $w_1 : \mathbb{Z}_{\geq 0}^s \to \frac{1}{d_1}\mathbb{Z}_{\geq 0}$ be a weight function. Let b and b' be positive real numbers. For each $\rho \in \mathbb{R}$ define

$$L(b';\rho) := \left\{ \sum_{\mathbf{j} \in \mathbb{Z}_{\geq 0}^s} A_{\mathbf{j}} t^{\mathbf{j}} \mid A_{\mathbf{j}} \in \mathbb{Z}_q[\pi], ord_p(A_{\mathbf{j}}) \geq b' w_1(\mathbf{j}) + \rho \right\}.$$

This is a p-adic Banach space with norm given by $\| \sum A_{\mathbf{j}} t^{\mathbf{j}} \| := \min_{\mathbf{j}} ord_p(A_{\mathbf{j}})$. Define

$$L(b') := \bigcup_{\rho \in \mathbb{R}} L(b';\rho).$$

Let $\mathcal{M}$ be a free $L(b')$-module with basis $\{e_1, \ldots, e_r\}$. We place a weight on each basis element e_i as follows: fix a positive integer d_0 and let $w_0 : \{1, \ldots, r\} \to \frac{1}{d_0}\mathbb{Z}_{\geq 0}$ be any function. Note, w_0 is not a weight function since the set $\{1, \ldots, r\}$ is finite. Define

$$\mathcal{M}(b', b; \rho) := \left\{ \sum_{i=1}^r B_i e_i \mid B_i \in L(b'; b w_0(i) + \rho) \right\} \tag{22}$$

and

$$\mathcal{M}(b', b) := \mathcal{M} = \bigcup_{\rho \in \mathbb{R}} \mathcal{M}(b', b; \rho).$$

For any subset U of $\mathcal{M}$, we may define $U(b', b; \rho) := U \cap \mathcal{M}(b', b; \rho)$ and $U(b', b) := U \cap \mathcal{M}(b', b)$.

Denote by $\mathcal{M}^{(k)} := Sym^k \mathcal{M}$ the k-th symmetric power of $\mathcal{M}$ over $L(b')$. Similar to (22) define

$$\mathcal{M}^{(k)}(b', b; \rho) := \left\{ \sum_{\substack{\mathbf{i} := (i_1, \ldots, i_r) \in \mathbb{Z}_{\geq 0}^r \\ i_1 + \cdots + i_r = k}} B_{\mathbf{i}} e^{\mathbf{i}} | B_{\mathbf{i}} \in L(b'; b w_0(\mathbf{i}) + \rho) \right\}$$

where $e^{\mathbf{i}} := e_1^{i_1} \cdots e_r^{i_r}$ and

$$w_0(\mathbf{i}) := w_0(i_1, \ldots, i_r) := \sum_{j=1}^r i_j w_0(j).$$

Let $\mathcal{N}$ be a free, finite rank $L(b')$-module, and denote by $\mathcal{N}^{(k)}$ the k-th symmetric power of $\mathcal{N}$ over $L(b')$. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be an $L(b')$-module morphism. Define the Leibnitz operator of Φ to be the operator $\mathcal{L}_\Phi : \mathcal{M}^{(k)} \to \mathcal{N}^{(k)}$ defined by

$$\mathcal{L}_\Phi(e_1^{i_1} \cdots e_r^{i_r}) := \sum_{m=1}^r i_m e_1^{i_1} \cdots e_m^{i_m - 1} \cdots e_r^{i_r} \Phi(e_m).$$

Next, we mention a short technical lemma which will be useful.

Lemma 9. *Suppose $\Phi : \mathcal{M}(b', b; 0) \to \mathcal{N}(b', b; \rho)$. Then $\mathcal{L}_\Phi : \mathcal{M}^{(k)}(b', b; 0) \to \mathcal{N}^{(k)}(b', b; \rho)$.*

Proof. Since $e_i \in \mathcal{M}(b', b; -bw_0(i))$ we have $\Phi(e_i) = a_{i,1}e_1 + \cdots + a_{i,r}e_r \in \mathcal{N}(b', b; \rho - bw_0(i))$. Thus, $a_{i,j} \in L(b'; \rho + bw_0(j) - bw_0(i))$. Similarly, since $e_1^{i_1} \cdots e_r^{i_r} \in \mathcal{M}(b', b; -bw_0(i_1, \ldots, i_r))$ we see that

$$\begin{aligned}
\mathcal{L}_\Phi(e_1^{i_1} \cdots e_r^{i_r}) &= \sum_{l=1}^{r} i_l e_1^{i_1} \cdots e_l^{i_l - 1} \cdots e_r^{i_r} \Phi(e_{i_l}) \\
&= \sum_{l=1}^{r} \Bigg(\sum_{j<l} i_l a_{l,j} e_1^{i_1} \cdots e_j^{i_j+1} \cdots e_l^{i_l-1} \cdots e_r^{i_r} \\
&\qquad\qquad + \sum_{j>l} i_l a_{l,j} e_1^{i_1} \cdots e_j^{i_j-1} \cdots e_l^{i_l+1} \cdots e_r^{i_r} \\
&\qquad\qquad\qquad\qquad + i_l a_{l,l} e_1^{i_1} \cdots e_r^{i_r} \Bigg).
\end{aligned}$$

Let us consider one of these terms, say $i_l a_{l,j} e_1^{i_1} \cdots e_j^{i_j+1} \cdots e_l^{i_l-1} \cdots e_r^{i_r}$ with $j < l$. Observe

$$\begin{aligned}
ord_p(a_{l,j}) &\geq \rho + bw_0(j) - bw_0(l) \\
&= \rho + bw_0(i_1, \ldots, i_j + 1, \ldots, i_l - 1, \ldots, i_r) - bw_0(i_1, \ldots, i_r).
\end{aligned}$$

The other terms are similar, proving $\mathcal{L}_\Phi(e^{\mathbf{i}}) \in \mathcal{N}^{(k)}(b', b; \rho - bw_0(\mathbf{i}))$ as desired. $\square$

Let us now return to our particular family. With $\hat{f}_t := t\frac{\partial}{\partial t}\hat{f}(t,x)$, since $\pi \hat{f}_t \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -1)$, by Theorem 6 we have for each $i = 1, \ldots, d-1$,

$$\pi \hat{f}_t(t,x) x^i = A_{i,1}x + \cdots + A_{i,d-1}x^{d-1} + D(t)(h_i(t,x))$$

for some $A_{i,j} \in L(\frac{p}{p-1})$ and $h_i \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1})$. This defines a morphism

$$\pi \hat{f}_t : \mathcal{H}_{1,t}(\frac{p}{p-1}, \frac{p}{p-1}; 0) \to \mathcal{H}_{1,t}(\frac{p}{p-1}, \frac{p}{p-1}; -1)$$

via the matrix $(A_{i,j})_{1 \leq i,j \leq d-1}$ which acts on the right. With $b' \leq b$ and $b \leq p/(p-1)$, this defines an endomorphism $\mathcal{L}_{\pi \hat{f}_t}$ on $\mathcal{H}_{1,t}^{(k)}(b', b)$. Set $\mathcal{N}(b', b) := t\mathcal{H}_{1,t}^{(k)}(b', b)$ and $\mathcal{M}(b', b) := \mathcal{H}_{1,t}^{(k)}(b', b)$.

Theorem 7. *Let p be a prime number such that $p > \delta_i$ and $(p, d) = 1$. Suppose $\mathcal{L}_{\pi \hat{f}_t}$ satisfies the following: there exists a free, finite rank $\mathbb{Z}_q[\pi]$-submodule V_k of $\mathcal{N}(p/(p-1), p/(p-1))$ such that for any $\frac{p}{p-1} \geq b \geq b' \geq \frac{1}{p-1}$, with $e' := b' - \frac{1}{p-1}$,*

1. *$\mathcal{N}(b', b; 0) \subset V_k(b', b; 0) + \mathcal{L}_{\pi \hat{f}_t}\mathcal{M}(b', b; e')$*
2. *$V_k(b', b) \cap \mathcal{L}_{\pi \hat{f}_t}\mathcal{M}(b', b) = \{0\}$*

Then ∂ satisfies:

1. *$\mathcal{N}(b', b; 0) \subset V_k(b', b; 0) + \partial\mathcal{M}(b', b; e')$*
2. *$V_k(b', b) \cap \partial\mathcal{M}(b', b) = \{0\}$ for $\frac{1}{p-1} < b' \leq \frac{p}{p-1}$ and $\frac{1}{p-1} \leq b \leq \frac{p}{p-1}$.*

As an example of the theorem, consider the cubic family $f(t,x) = x^3 + tx$. In [12] it was shown that $\mathcal{L}_{\pi \hat{f}_t}$ satisfied the hypothesis in the theorem when k is odd and $k < p$. Consequently, we have the following result.

Corollary 3. *Let $f(t,x) = x^3 + tx$ and $p \geq 5$ be a prime number. Suppose k is an odd positive integer satisfying $k < p$. Let V_k denote the $\mathbb{Z}_q[\pi]$-span of the set $\{te_1^{k-2i}e_2^{2i}\}_{i=0}^{\frac{k-1}{2}} \subset \mathcal{N}$ Then*

$$\mathcal{N}(b', b; 0) \subset V_k(b', b; 0) + \partial\mathcal{M}(b', b; e').$$

The proof of Theorem 7 will consist of a series of lemmas which will comprise the rest of this section.

Lemma 10. *Suppose $p > \delta_i$, $(p, d) = 1$, and $1/(p-1) \leq b' \leq b \leq p/(p-1)$. Set $e' := b' - \frac{1}{p-1}$. If*

$$\mathcal{N}(b', b; 0) \subset V(b', b; 0) + \mathcal{L}_{\pi \hat{f}_t}\mathcal{M}(b', b; e')$$

then

$$\mathcal{N}(b', b; 0) \subset V(b', b; 0) + \partial\mathcal{M}(b', b; e').$$

Proof. On $\mathcal{K}(b', b)$ we may write $\partial := t\frac{\partial}{\partial t} + W(t,x)$ where

$$W(t,x) := \sum_{j=0}^{\infty} \pi_j p^j \hat{f}_t^{\tau^j}(t^{p^j}, x^{p^j}).$$

For each $j \in \mathbb{Z}_{\geq 0}$ write

$$\hat{f}_t^{\tau^j}(t^{p^j}, x^{p^j}) = \hat{f}_t(t,x)^{p^j} + p g_j(t,x)$$

for some polynomial g_j whose coefficients lie in $\mathbb{Z}_q[\pi]$. This allows us to write

$$W(t,x) = \pi \hat{f}_t(t,x) Q_2(t,x) + K_2(t,x) \tag{23}$$

where

$$Q_2(t,x) := \sum_{j=0}^{\infty} \pi_j \pi^{-1} p^j \hat{f}_t(t,x)^{p^j - 1}$$
$$K_2(t,x) := \sum_{j=1}^{\infty} \pi_j p^{j+1} g_j(t,x).$$

Using a similar argument as that in Lemma 2, one may show $Q_2, \frac{1}{Q_2}, K_2 \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; 0)$.

Since $tx \mid \hat{f}_t$, we may write $Q_2(t,x) = 1 + (tx)^{p-1}\nu(t,x)$ for some ν, making $Q_2(t,x)x = x + t^{p-1}x^p\nu(t,x) \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right) w_0(1))$. By Lemma 2, we may write

$$t^{p-1}x^p\nu(t,x) = B_1 x + \cdots + B_{d-1}x^{d-1} + \pi \hat{f}_x(t,x) h(t,x),$$

where $B_j x^j \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right) w_0(1))$ and $h \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right) w_0(1) + 1)$. Furthermore, since t^{p-1} divides the left-hand side, by Lemma 2 each B_j for $j = 1, \ldots, d-1$ is also divisible by t^{p-1}, as well as h.

From (23), since

$$Q_2(t,x)x = (1 + B_1)x + B_2 x^2 + \cdots + B_{d-1}x^{d-1} + \pi \hat{f}_x(t,x) h(t,x),$$

we may write for each $i = 1, \ldots, d-1$

$$W(t,x)x^i = (1 + B_1)\pi \hat{f}_t(t,x)x^i + S_i(t,x) + T_i(t,x), \tag{24}$$

where

$$S_i := \pi \hat{f}_t \left(B_2 x^2 + \cdots + B_{d-1}x^{d-1}\right) x^{i-1}$$
$$T_i := K_2 x^i + D(t)(Q_1^{-1}\pi \hat{f}_t h x^{i-1}) - x\frac{\partial}{\partial x}(Q_1^{-1}\pi \hat{f}_t h x^{i-1}) - K_1 Q_1^{-1}\pi \hat{f}_t h x^{i-1}.$$

From the estimates above one may show $T_i \in \mathcal{K}(\frac{p}{p-1}, \frac{p}{p-1}; -\left(\frac{p}{p-1}\right) w_0(i))$, and so $T_i \in \mathcal{K}(b', b; -bw_0(i))$. Furthermore, $t^p x$ divides each S_i since t divides f_t and t^{p-1} divides each B_j. Thus, $S_i \in \mathcal{K}(b', b; -e' + \varrho - bw_0(i))$ where

$$\varrho := \left(\frac{p}{p-1} - b'\right)(w_1(p) - 1) + \left(\frac{p}{p-1} - b\right) w_0(1). \tag{25}$$

By Theorem 6 we may write

$$\begin{aligned} S_i &= C_{i,1}x + \cdots + C_{i,d-1}x^{d-1} + D(t)z_i \\ T_i &= A_{i,1}x + \cdots + A_{i,d-1}x^{d-1} + D(t)w_i \end{aligned} \tag{26}$$

where

$$C_{i,j} \in L(b'; b(w_0(j) - w_0(i)) - e' + \varrho) \quad \text{and} \quad A_{i,j} \in L(b'; b(w_0(j) - w_0(i))). \tag{27}$$

These define $(d-1) \times (d-1)$ matrices $C := (C_{i,j})$ and $A := (A_{i,j})$. It follows from (27) that C and A define morphisms

$$C : \mathcal{H}_{1,t}(b', b; e') \to \mathcal{H}_{1,t}(b', b; \varrho)$$
$$A : \mathcal{H}_{1,t}(b', b; 0) \to \mathcal{H}_{1,t}(b', b; 0).$$

By Lemma 9 below this means

$$\mathcal{L}_C : \mathcal{H}^{(k)}_{1,t}(b', b; e') \to \mathcal{H}^{(k)}_{1,t}(b', b; \varrho)$$
$$\mathcal{L}_A : \mathcal{H}^{(k)}_{1,t}(b', b; 0) \to \mathcal{H}^{(k)}_{1,t}(b', b; 0).$$

Finally, with $\mu := 1 + B_1$, from (24) and (26) we see that

$$\mathcal{L}_W = \mu \mathcal{L}_{\pi \hat{f}_t} + \mathcal{L}_C + \mathcal{L}_A.$$

We will first suppose $\frac{1}{p-1} < b' < \frac{p}{p-1}$. Let $\xi \in \mathcal{N}(b', b; 0)$. By the hypothesis on $\mathcal{L}_{\pi \hat{f}_t}$, there exists $\eta_1 \in V(b', b; 0)$ and $\zeta_1 \in \mathcal{M}(b', b; e')$ so that $\xi = \eta_1 + \mathcal{L}_{\pi \hat{f}_t} \zeta_1$. Hence,

$$\begin{aligned}\xi &= \eta_1 + (\mathcal{L}_C + \mathcal{L}_A)(-\mu^{-1}\zeta_1) + \mathcal{L}_W(\mu^{-1}\zeta_1) \\ &= \eta_1 + \underbrace{(\mathcal{L}_C + \mathcal{L}_A + t\frac{d}{dt})(-\mu^{-1}\zeta_1)}_{=:\nu_1} + \partial(\underbrace{\mu^{-1}\zeta_1}_{=:\zeta_1'}),\end{aligned}$$

and so $\xi = \eta_1 + \nu_1 + \partial\zeta_1'$. By our assumptions on b', and since $p > d\delta_i$, we have $\varrho > 0$. Let $\epsilon := \min\{\varrho, e'\} > 0$. Notice that $\nu_1 \in \mathcal{H}^{(k)}_{1,t}(b', b; \epsilon)$ and $\zeta_1' \in \mathcal{H}^{(k)}_{1,t}(b', b; e')$. We have just taken ξ and written it as the sum $\eta_1 + \nu_1 + \partial\zeta_1'$. We may now do the same procedure but now with ν_1, which means

$$\xi = (\eta_1 + \eta_2) + \nu_2 + \partial(\zeta_1' + \zeta_2').$$

Continuing this process, we are led to an equation of the form

$$\xi = (\eta_1 + \cdots + \eta_N) + \nu_N + \partial(\zeta_1' + \cdots + \zeta_N')$$

where $\eta_i \in V(b', b; (i-1)\epsilon)$, $\zeta_i' \in \mathcal{H}^{(k)}_{1,t}(b', b; (i-1)\epsilon + e')$, and $\nu_N \in \mathcal{N}(b', b; N\epsilon)$. Letting N tend to infinity, we see that

$$\xi = \eta + \partial\zeta'$$

where $\eta := \sum_{i=1}^{\infty} \eta_i \in V(b', b; 0)$ and $\zeta' := \sum_{i=1}^{\infty} \zeta_i' \in \mathcal{M}(b', b; e')$ as desired.

Suppose now that $b = b' = \frac{p}{p-1}$. Let $\xi \in \mathcal{N}(b', b; 0)$. Let $0 < \epsilon < 1$. Then $\xi \in \mathcal{N}(b' - \epsilon, b; 0)$. By the above argument, there exists $\eta^{(\epsilon)} \in V(b' - \epsilon, b; 0)$ and $\zeta^{(\epsilon)} \in \mathcal{M}(b' - \epsilon, b; 1 - \epsilon)$ such that $\xi = \eta^{(\epsilon)} + \partial\zeta^{(\epsilon)}$. Restricting ourselves to a strictly decreasing sequence of ϵ which tend to zero constructs sequences $\{\eta^{(\epsilon)}\}_\epsilon \subset V(\frac{1}{p-1}, b; 0)$ and $\{\zeta^{(\epsilon)}\}_\epsilon \subset \mathcal{M}(\frac{1}{p-1}, b; 0)$. These two spaces are compact in the topology of coefficientwise convergence, and so the sequences have accumulation points η and ζ, respectively. By construction, η belongs to each $V(b' - \epsilon, b; 0)$ for every $\epsilon > 0$, which forces $\eta \in V(b', b; 0)$. A similar argument holds for ζ, showing ζ is an element of $\mathcal{M}(b', b; 1)$. This proves the result for the case $b' = b = \frac{p}{p-1}$.

Lastly, suppose $b' = \frac{1}{p-1}$. For each $N \in \mathbb{Z}_{\geq 0}$, we may write

$$\xi = \sum_{\mathbf{i}} \left(\sum_{n=1}^{N} B_{n,\mathbf{i}} t^n \right) e^{\mathbf{i}} + \sum_{\mathbf{i}} \left(\sum_{n=N+1}^{\infty} B_{n,\mathbf{i}} t^n \right) e^{\mathbf{i}}.$$

Let $\epsilon > 0$. For each $1 \leq n \leq N$, since $b'w_1(n) + bw_0(\mathbf{i}) \geq (b' + \frac{\epsilon}{w_1(N)})w_1(n) + bw_0(\mathbf{i}) - \epsilon$, we see that

$$\sum_{\mathbf{i}} \left(\sum_{n=1}^{N} B_{n,\mathbf{i}} t^n \right) e^{\mathbf{i}} \in \mathcal{N}(b' + \frac{\epsilon}{w_1(N)}, b; -\epsilon).$$

Suppose $b \geq b' + \frac{\epsilon}{w_1(N)} > \frac{1}{p-1}$. It follows from the argument above that there exists $\eta^{(\epsilon,N)} \in V(b' + \frac{\epsilon}{w_1(N)}, b; -\epsilon)$ and $\zeta^{(\epsilon,N)} \in \mathcal{M}(b' + \frac{\epsilon}{w_1(N)}, b; -\epsilon + \frac{\epsilon}{w_1(N)})$ such that

$$\sum_{\mathbf{i}} \left(\sum_{n=1}^{N} B_{n,\mathbf{i}} t^n \right) e^{\mathbf{i}} = \eta^{(\epsilon,N)} + \partial\zeta^{(\epsilon,N)}.$$

We have just constructed sequences $\{\eta^{(\epsilon,N)}\}_{N=1}^{\infty} \subset V(b', b; -\epsilon)$ and $\{\zeta^{(\epsilon,N)}\}_{N=1}^{\infty} \subset \mathcal{M}(b', b; -\epsilon)$. Since both of these spaces are compact in the topology of coefficientwise convergence we may restrict ourselves to convergent subsequences with limits $\eta^{(\epsilon)} \in V(b', b; -\epsilon)$ and $\zeta^{(\epsilon)} \in \mathcal{M}(b', b; -\epsilon)$ which satisfy

$$\xi = \eta^{(\epsilon)} + \partial\zeta^{(\epsilon)}.$$

In the coefficientwise convergence topology, letting $\epsilon \to 0^+$, there exists $\eta \in V(b', b; 0)$ and $\zeta \in \mathcal{M}(b', b; 0)$ such that

$$\xi = \lim_{\epsilon \to 0^+} \eta^{(\epsilon)} + \partial \zeta^{(\epsilon)} = \eta + \partial \zeta.$$

This finishes the case when $b > b' = \frac{1}{p-1}$. The case when $b = b' = \frac{1}{p-1}$ is similar. □

Lemma 11. *Suppose $p > \delta_i$, $(p, d) = 1$, $\frac{1}{p-1} < b' \leq \frac{p}{p-1}$, and $1/(p-1) \leq b \leq p/(p-1)$. Suppose*

$$V(b', b) \cap \mathcal{L}_{\pi \hat{f}_t} \mathcal{M}(b', b) = \{0\}.$$

Then

$$V(b', b) \cap \partial \mathcal{M}(b', b) = \{0\}.$$

Proof. We will first assume $\frac{p}{p-1} > b' > \frac{1}{p-1}$. Let $\eta \in V(b', b) \cap \partial \mathcal{M}(b', b)$ be non-zero, and let $\xi \in \mathcal{M}(b', b)$ be such that $\partial \xi = \eta$. Let c be a real number such that $\xi \in \mathcal{M}(b', b; c)$ but $\xi \notin \mathcal{M}(b', b; c + \epsilon)$ for any $\epsilon > 0$. We will prove that no such c exists, contradicting the existence of a non-zero η.

Using notation from the proof of Lemma 10, we have $\partial = t\frac{d}{dt} + \mu \mathcal{L}_{\pi \hat{f}_t} + \mathcal{L}_C + \mathcal{L}_A$, and so

$$\eta = \partial \xi = \mu \mathcal{L}_{\pi \hat{f}_t} \xi + \underbrace{(t\frac{d}{dt} + \mathcal{L}_C + \mathcal{L}_A)\xi}_{=:\zeta_1'}.$$

Letting $\epsilon := \min\{\varrho, e'\} > 0$, where ϱ comes from (25), we see that $\zeta \in \mathcal{M}(b', b; c - e' + \epsilon)$. From Lemma 10, there exists $\eta_1 \in V(b', b; c - e' + \epsilon)$ and $\zeta_1 \in \mathcal{M}(b', b; c + \epsilon)$ such that

$$\begin{aligned} \zeta_1' &= \eta_1 + \partial \zeta_1 \\ &= \eta_1 + \mathcal{L}_{\pi \hat{f}_t}(\mu \zeta_1) + \underbrace{(t\frac{d}{dt} + \mathcal{L}_C + \mathcal{L}_A)\zeta_1}_{=:\zeta_2'}. \end{aligned}$$

Observe that $\zeta_2' \in \mathcal{M}(b', b; c + 2\epsilon - e')$, and

$$\eta = \mathcal{L}_{\pi \hat{f}_t}(\mu \xi + \mu \zeta_1) + \zeta_2' + \eta_1.$$

Iterating this process, but now starting with ζ_2', we are led to the equation

$$\eta = \mathcal{L}_{\pi \hat{f}_t}(\mu(\xi + \zeta_1 + \cdots + \zeta_N)) + \zeta_{N+1}' + (\eta_1 + \cdots + \eta_N)$$

with $\zeta_i \in \mathcal{M}(b', b; c + i\epsilon)$, $\zeta_{N+1}' \in \mathcal{M}(b', b; c + (N+1)\epsilon - e')$, and $\eta_i \in V(b', b; c - e' + i\epsilon)$. Hence,

$$\eta - \sum_{i=1}^{\infty} \eta_i = \mathcal{L}_{\pi \hat{f}_t}(\mu(\xi + \sum_{i=1}^{\infty} \zeta_i)).$$

It follows from the hypothesis on $\mathcal{L}_{\pi \hat{f}_t}$ that $\xi = -\sum_{i=1}^{\infty} \zeta_i \in \mathcal{M}(b', b; c + \epsilon)$, contradicting out choice of c.

The result follows for $b' = \frac{p}{p-1}$ since

$$V(b', b) \cap \partial \mathcal{M}(b', b) \subset V(b' - \epsilon, b) \cap \partial \mathcal{M}(b' - \epsilon, b) = \{0\},$$

where the last equality follows for any $\epsilon > 0$ by the preceding argument. □

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