OSCILLATORY AND FOURIER INTEGRAL OPERATORS WITH DEGENERATE CANONICAL RELATIONS

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We shall mostly survey results concerning the L^2 boundedness of oscillatory and Fourier integral operators. Many mathematicians have contributed important results to this subject. This article does not intend to give a broad overview; it mainly focusses on a few topics directly related to the work of the authors.

1. The nondegenerate situation

1.1. Oscillatory integral operators. The main subject of the article concerns oscillatory integral operators given by

$$T_\lambda f(x) = \int e^{\imath\lambda\Phi(x,y)}\sigma(x,y)f(y)dy. \tag{1.1}$$

In (1.1) it is assumed that the real-valued phase function Φ is smooth in $\Omega_L \times \Omega_R$ where Ω_L, Ω_R are open subsets of $\mathbb{R}^d$ and amplitude $\sigma \in C_0^\infty(\Omega_L \times \Omega_R)$. (The assumption that $\dim(\Omega_L) = \dim(\Omega_R)$ is only for convenience; many of the definitions, techniques and results described below have some analogues in the non-equidimensional setting.)

The L^2 boundedness properties of T_λ are determined by the geometry of the canonical relation

$$C = \{(x, \Phi_x, y, -\Phi_y) : (x,y) \in \operatorname{supp}\sigma\} \subset T^*\Omega_L \times T^*\Omega_R.$$

The best possible situation occurs when C is locally the graph of a canonical transformation; i.e., the projections π_L, π_R to $T^*\Omega_L$, $T^*\Omega_R$, resp.,

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ T^*\Omega_L & & T^*\Omega_R \end{array}$$

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are locally diffeomorphisms. In this case Hörmander [37],[38] proved that the norm of T_λ as a bounded operator on $L^2(\mathbb{R}^d)$ satisfies

$$\|T_\lambda\|_{L^2\to L^2} = O(\lambda^{-d/2}). \tag{1.2}$$

The proof consists in applying Schur's test to the kernel of $T_\lambda^* T_\lambda$; see the argument following (1.6) below.

It is also useful to study a more general class of oscillatory integrals which naturally arises when composing two different operators T_λ, $\widetilde{T}_\lambda$ and which is also closely related to the concept of Fourier integral operator. We consider the oscillatory integral kernel with frequency variable $\vartheta \in \Theta$ (an open subset of $\mathbb{R}^N$), defined by

$$K_\lambda(x,y) = \int e^{\imath\lambda\Psi(x,y,\vartheta)} a(x,y,\vartheta)d\vartheta \tag{1.3}$$

where $\Psi \in C^\infty(\Omega_L \times \Omega_R \times \Theta)$ is real-valued and $a \in C_0^\infty(\Omega_L \times \Omega_R \times \Theta)$. Let $\mathfrak{T}_\lambda$ be the associated integral operator,

$$\mathfrak{T}_\lambda f(x) = \int K_\lambda(x,y) f(y) dy. \tag{1.4}$$

Again the L^2 mapping properties of $\mathfrak{T}_\lambda$ are determined by the geometric properties of the canonical relation

$$C = \{(x, \Psi_x, y, -\Psi_y) : \Psi_\vartheta = 0\} \subset T^*\Omega_L \times T^*\Omega_R.$$

It is always assumed that C is an immersed manifold, which is a consequence of the linear independence of the vectors $\nabla_{(x,y,\vartheta)}\Psi_{\vartheta_i}$, $i = 1, \dots, N$ at $\{\Psi_\vartheta = 0\}$. In other words, Ψ is a nondegenerate phase in the sense of Hörmander [37], although Ψ is not assumed to be homogeneous.

As before, the best possible situation for L^2 estimates arises when C is locally the graph of a canonical transformation. Analytically this means that

$$\det \begin{pmatrix} \Psi_{xy} & \Psi_{x\vartheta} \\ \Psi_{\vartheta y} & \Psi_{\vartheta\vartheta} \end{pmatrix} \neq 0 \tag{1.5}$$

Under this assumption the L^2 result becomes

$$\|\mathfrak{T}_\lambda\|_{L^2\to L^2} \lesssim \lambda^{-(d+N)/2} \tag{1.6}$$

so that we discover (1.2) when $N = 0$. The proof of (1.6) could be given by using methods in [37] or alternatively by a straightforward modification of the argument in [38]. Indeed consider the Schwartz kernel H_λ of the operator $\mathfrak{T}_\lambda^* \mathfrak{T}_\lambda$ which is given by

$$H_\lambda(u,y) = \iiint e^{-\imath\lambda[\Psi(x,u,w)-\Psi(x,y,\vartheta)]} \gamma(x,u,w,y,\vartheta) dw d\vartheta dx$$

where γ is smooth and compactly supported. By using partitions of unity we may assume that σ in (1.1) has small support; thus γ has small support. Change variables $w = \vartheta + h$, and, after interchanging the order of integration, integrate parts with respect to the variables (ϑ, x). Since

$$\nabla_{x,\vartheta}[\Psi(x,u,\vartheta+h) - \Psi(x,y,\vartheta)] = \begin{pmatrix} \Psi_{xy} & \Psi_{x\vartheta} \\ \Psi_{\vartheta y} & \Psi_{yy} \end{pmatrix} \begin{pmatrix} u-y \\ h \end{pmatrix} + O(|u-y|^2 + |h|^2)$$

this yields, in view of the small support of γ,

$$\begin{aligned} |K_\lambda(u,y)| &\lesssim \int (1+\lambda|u-y|+\lambda|h|)^{-2M} dh \\ &\lesssim \lambda^{-N-d} \frac{\lambda^d}{(1+\lambda|u-y|)^M} \end{aligned}$$

if $M > d$. It follows that $\|\mathfrak{T}_\lambda^* \mathfrak{T}_\lambda\|_{L^2\to L^2} \lesssim \lambda^{-N-d}$ and hence (1.6).

1.2 Reduction of frequency variables.

Alternatively, as in the theory of Fourier integral operators, one may compose T_λ with unitary operators associated to canonical transformations, and together with stationary phase calculations, deduce estimates for operators of the form (1.3-4) from operators of the form (1.1), which involve no frequency variables; in fact this procedure turns out to be very useful when estimating operators with degenerate canonical relations.

We briefly describe the idea based on [37], for details see [25].

Consider the operator $\mathfrak{T}_\lambda$ with kernel $\int_{\mathbb{R}^N} e^{\imath\lambda\phi(x,y,z)} a(x,y,z)dz$. Let A_i, $i = 1,2$, be symmetric $d \times d$ matrices and define

$$S_{\lambda,i} g(x) = \left(\frac{\lambda}{2\pi}\right)^{d/2} \int e^{-\imath\lambda[\langle x,w\rangle + \frac{1}{2}A_i w\cdot w]} g(w)dw;$$

clearly $S_{\lambda,i}$ are unitary operators on $L^2(\mathbb{R}^d)$. A computation yields that the operator $\lambda^{-d} S_{\lambda,1}\mathfrak{T}_\lambda S_{\lambda,2}^*$ can be written as the sum of an oscillatory integral operator with kernel $O_\lambda(x,y)$ plus an operator with L^2 norm $O(\lambda^{-M})$ for any M. The oscillatory kernel $O_\lambda(x,y)$ is again of the form (1.3) where the phase function is given by

$$\Psi(x,y,\vartheta) = \langle y,\tilde{w}\rangle - \langle x,w\rangle + \frac{1}{2}(A_1\tilde{w}\cdot\tilde{w} - A_2 w\cdot w) + \phi(w,\tilde{w},z)$$

with frequency variables $\vartheta = (w,z,\tilde{w}) \in \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d$, and the amplitude is compactly supported.

One can choose A_1, A_2 so that for tangent vectors $\delta x, \delta y \in \mathbb{R}^d$ at a reference point the vector $(\delta x, A_1\delta x, \delta y, A_2\delta y)$ is tangent to the canonical relation $\widetilde{C}$ associated with $S_{\lambda,1}\mathfrak{T}_\lambda S_{\lambda,2}^*$. Let π_{space} be the projection $\widetilde{C} \to \Omega_L \times \Omega_R$ which with our choice of A_1, A_2 has invertible differential. Since the number of frequency variables $(N+2d)$ minus the rank of $\phi_{\vartheta\vartheta}$ is equal to $2d - \text{rank}\, d\pi_{\text{space}}$, we deduce that $\det\phi_{\vartheta\vartheta} \neq 0$.

In the integral defining the kernel of $S_{\lambda,1}\mathfrak{T}_\lambda S_{\lambda,2}^*$ we can now apply the method of stationary phase to reduce the number of frequency variables to zero, and gain a factor of $\lambda^{-(2d+N)/2}$. Thus we may write

$$S_{\lambda,1}\mathfrak{T}_\lambda S_{\lambda,2}^* = \lambda^{-N/2} T_\lambda + R_\lambda$$

where T_λ is an oscillatory integral operator (without frequency variables) and R_λ is an operator with L^2 norm $O(\lambda^{-M})$ for any large M. Since $S_{\lambda,i}$ are unitary the L^2 bounds for $\lambda^{N/2}\mathfrak{T}_\lambda$ and T_λ are equivalent.

1.3 Fourier integral operators. The kernel of a Fourier integral operators $\mathcal{F} : C_0^\infty(\Omega_R) \to \mathcal{D}'(\Omega_L)$ of order μ, $\mathcal{F} \in I^\mu(\Omega_L, \Omega_R; \mathcal{C})$ is locally given as a finite sum of oscillatory integrals

$$\int e^{\imath\Psi(x,y,\theta)} a(x,y,\theta) d\theta, \tag{1.7}$$

where now Ψ is nondegenerate in the sense of Hörmander [37], satisfies the homogeneity condition $\Psi(x,y,t\theta) = t\Psi(x,y,\theta)$ for $|\theta| = 1$ and $t \gg 1$, and a is a symbol of order $\mu + (d-N)/2$. We assume in what follows that $a(x,y,\theta)$ vanishes for (x,y) outside a fixed compact set. The canonical relation is locally given by $\mathcal{C} = \{(x, \Psi_x, y, -\Psi_y), \Psi_\theta = 0\}$ and we assume that

$$\mathcal{C} \subset \Big(T^*\Omega_L \setminus 0_L\Big) \times \Big(T^*\Omega_R \setminus 0_R\Big),$$

where 0_L, 0_R denote the zero-sections in $T^*\Omega_L$ and $T^*\Omega_R$. Staying away from the zero sections implies

$$|\Psi_x(x,y,\theta)| \approx |\theta| \approx |\Psi_y(x,y,\theta)| \tag{1.8}$$

for large θ (when Ψ_θ is small). Let $\beta \in C_0^\infty(1/2, 2)$ and

$$a_k(x,y,\theta) = \beta(2^{-k}|\theta|) a(x,y,\theta)$$

and let $\mathcal{F}_k$ be the dyadic localization of $\mathcal{F}$; i.e. (1.7) but with a replaced by a_k. The assumptions $\Psi_x \neq 0$ and $\Psi_y \neq 0$ can be used to show that for $k, l \geq 1$ the operators $\mathcal{F}_k$ are almost orthogonal, in the sense that $\mathcal{F}_k^* \mathcal{F}_l$ and $\mathcal{F}_k \mathcal{F}_l^*$ have operator norms $O(\min\{2^{-kM}, 2^{-lM}\})$ for any M, provided that $|k - l| \geq C$ for some large but fixed constant C. This follows from a straightforward integration by parts argument based on (1.8) and the assumption of compact (x,y) support. Using a change of variable $\theta = \lambda\vartheta$ the study of the L^2 boundedness (and L^2-Sobolev boundedness) properties is reduced to the study of oscillatory integral operators (1.3-4) and, in the nondegenerate case, an application of estimate (1.2) above. The result is that if $\mathcal{F}$ is of order μ and if the associated homogeneous canonical transformation is a local canonical graph, then $\mathcal{F}$ maps the Sobolev space L^2_α to $L^2_{\alpha-\mu}$

An important subclass is the class of conormal operators associated to phase functions linear in the frequency variables (see [37,§2.4]). The generalized Radon transforms

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} f(y)\chi(x,y) d\sigma_x(y) \tag{1.9}$$

arise as model cases. Here $\mathcal{M}_x$ are codimension ℓ submanifolds in $\mathbb{R}^d$, and $d\sigma_x$ is a smooth density on $\mathcal{M}_x$, varying smoothly in x, and $\chi \in C_0^\infty(\Omega_L \times \Omega_R)$. One assumes that the $\mathcal{M}_x$ are sections of a manifold $\mathcal{M} \subset \Omega_L \times \Omega_R$, so that the projections to Ω_L and to Ω_R have surjective differential; this assumption insures the L^1 and L^∞ boundedness of the operator $\mathcal{R}$. We refer to $\mathcal{M}$ as the associated *incidence relation.*

Assuming that $\mathcal{M}$ is given by an $\mathbb{R}^\ell$ valued defining function Φ,

$$\mathcal{M} = \{(x,y) : \Phi(x,y) = 0\}, \tag{1.10}$$

then the distribution kernel of $\mathcal{R}$ is $\chi_0(x,y)\delta(\Phi(x,y))$ where $\chi_0 \in C_0^\infty(\Omega_L \times \Omega_R)$ and δ is the Dirac measure in $\mathbb{R}^\ell$ at the origin. The assumptions on the projections to Ω_L, Ω_R imply that rank $\Phi_x =$ rank $\Phi_y = \ell$ in a neighborhood of $\mathcal{M} = \{\Phi = 0\}$. The Fourier integral description is then obtained by writing out δ by means of the Fourier inversion formula in $\mathbb{R}^\ell$,

$$\chi_0(x,y)\delta(\Phi(x,y)) = \chi_0(x,y)(2\pi)^{-\ell}\int_{\mathbb{R}^\ell} e^{\imath\tau\cdot\Phi(x,y)}d\tau; \tag{1.11}$$

this has been used in [35] where $\mathcal{R}$ is identified as a Fourier integral operator of order $-(d-\ell)/2$, see also [55]. More general conormal operators are obtained by composing Radon transforms with pseudo-differential operators (see [37]).

The canonical relation associated to the generalized Radon transform is the twisted conormal bundle of the incidence relation,

$$\mathcal{C} = N^*\mathcal{M}' = \{(x, \tau\cdot\Phi_x, y, -\tau\cdot\Phi_y) : \Phi(x,y) = 0\}. \tag{1.12}$$

We can locally (after possibly a change of coordinates) parametrize $\mathcal{M}$ as a graph so that

$$\Phi(x,y) = S(x,y') - y'' \tag{1.13}$$

with $y' = (y_1,\dots,y_{d-\ell}) \in \mathbb{R}^{d-\ell}$, $y'' = (y_{d-\ell+1},\dots,y_d) \in \mathbb{R}^\ell$, $S = (S^1,\dots,S^\ell)$. Using (1.5) with $\Psi(x,y,\tau) = \tau\cdot\Phi(x,y)$ one verifies that the condition for $N^*\mathcal{M}'$ being a local canonical graph is equivalent to the nonvanishing of the determinant

$$\det\begin{pmatrix} \tau\cdot\Phi_{xy} & \Phi_x \\ {}^t\Phi_y & 0 \end{pmatrix} = (-1)^\ell \det\begin{pmatrix} \tau\cdot S_{x'y'} & S_{x'} \\ \tau\cdot S_{x''y'} & S_{x''} \end{pmatrix} \tag{1.14}$$

for all $\tau \in S^{\ell-1}$. Under this condition $\mathcal{R}$ maps L^2 to $L^2_{(d-\ell)/2}$.

We note that the determinant in (1.14) vanishes for some τ if $\ell < d/2$. In particular if $\ell = d-1$ then the expression (1.14) is a linear functional of τ and thus, if (x,y) is fixed, it vanishes for all τ in a hyperplane. Therefore degeneracies always occur for averaging over manifolds with high codimension, in particular for curves in three or more dimensions.

2. Finite type conditions

2.1. Finite type. Different notions of finite type are useful in different situations. Here we shall restrict ourselves to maps (or pairs of maps) which have corank ≤ 1.

Let M, N be n-dimensional manifolds, $P \in M$ and $Q \in N$, and let $f : M \to N$ be a C^∞ map with $f(P_0) = Q_0$. A vector field V is a *kernel field* for the map f on a neighborhood $\mathcal{U}$ of P_0 if V is smooth on $\mathcal{U}$ and if there exists a smooth vector field W on $f(U)$ so that $Df_P V = \det(Df_P) W_{f(P)}$ for all $P \in \mathcal{U}$. If rank $Df_{P_0} \ge n-1$ then it is easy to see that there is a neighborhood of P and a nonvanishing kernel vector field V for f on $\mathcal{U}$. Moreover if $\widetilde{V}$ is another kernel field on U then $\widetilde{V} = \alpha V - \det(Df)W$ in some neighborhood of P_0, for some vector field W and smooth function α. If $Df = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$ with A an invertible $(n-1)\times(n-1)$ matrix, then $\det Df = \det A(d - c^t A^{-1} b)$ and a choice for the kernel vector field is

$$V = \frac{\partial}{\partial x_d} - A^{-1}b\cdot\nabla_{x'}. \tag{2.1}$$

Definition. Suppose that M and N are smooth n-dimensional manifolds and that $f : M \to N$ is a smooth map with $\dim \ker(Df) \le 1$ on M. We say that f is *of type k at P* if there is a nonvanishing kernel field V near P so that $V^j(\det Df)_P = 0$ for $j < k$ but $V^k(\det Df)_P \neq 0$.

This definition was proposed by Comech [13], [15] who assumes in addition that Df drops rank simply on the singular variety $\{\det Df = 0\}$.

The finite type condition is satisfied for the class of Morin singularities (folds, cusps, swallowtails, ...) which we shall now discuss.

2.2 Morin singularities. We consider as above maps $f : M \to N$ of corank ≤ 1. We say that f drops rank simply at P_0 if rank $Df_{P_0} = n-1$ and if $d(\det Df)_P \neq 0$. Then near P_0 the variety $S_1(f) = \{x : \text{rank } Df = n-1\}$ is a hypersurface and we say that f has an S_1 singularity at P with singularity manifold $S_1(f)$.

Next let $\mathfrak{S}$ be a hypersurface in a manifold $\mathcal{U}$ and let V be a vector field defined on $\mathfrak{S}$ with values in $T\mathcal{U}$ (meaning that $v_P \in T_P\mathcal{U}$ for $P \in \mathfrak{S}$). We say that v is *transversal* to $\mathfrak{S}$ at $P \in \mathfrak{S}$ if $v_P \notin T_P\mathfrak{S}$. We say that v is *simply tangent* to $\mathfrak{S}$ at P_0 if there is a one-form ω annihilating vectors tangent to $\mathfrak{S}$ so that $\langle \omega, v\rangle\big|_{\mathfrak{S}}$ vanishes of exactly first order at P_0. This condition does not depend on the particular choice of ω. Next let $P \to \ell(P) \subset T_P(V)$ be a smooth field of lines defined on $\mathfrak{S}$. Let v be a nonvanishing vector field so that $\ell(P) = \mathbb{R}v_P$. The definitions of transversality and simple tangency carry over to field of lines (and the notions do not depend on the particular choice of the vector field).

Next consider $F : \mathcal{U} \to N$ where $\dim \mathcal{U} = k \ge 2$ and $\dim N = n \ge k$ and assume that rank $DF \ge k-1$. Suppose that $\mathfrak{S}$ is a hypersurface in $\mathcal{U}$ such that rank $DF = k-1$ on $\mathfrak{S}$. Suppose that Ker DF is simply tangent to $\mathfrak{S}$ at $P \in \mathfrak{S}$. Then there is a neighborhood U of P in $\mathfrak{S}$ such that the variety $\{Q \in U : \text{rank } DF\Big|_{T_Q\mathfrak{S}} = k-2\}$ is a smooth hypersurface in $\mathfrak{S}$.

With these notions we can now recall the definition of Morin singularities ([78], [47]).

Definition. Let $1 \le r \le n$. Let $\mathfrak{S}_1, \dots, \mathfrak{S}_r$ be submanifolds of an open set $\mathcal{U} \subset M$ so that $\mathfrak{S}_k$ is of dimension $n-k$ in V and $\mathfrak{S}_1 \supset \mathfrak{S}_2 \supset \cdots \supset \mathfrak{S}_r$; we also set $\mathfrak{S}_0 := \mathcal{U}$.

We say that f has an S_{1_r} singularity in $\mathcal{U}$, with a descending flag of singularity manifolds $(\mathfrak{S}_1, \dots, \mathfrak{S}_r)$ if the following conditions hold in $\mathcal{U}$.

(i) For $P \in \mathcal{U}$, either Df_P is bijective or f drops rank simply at P.
(ii) For $1 \le i \le r$, rank $D(f\big|_{\mathfrak{S}_{i-1}})_P = n - i + 1$ for all $P \in \mathfrak{S}_{i-1} \setminus \mathfrak{S}_i$.
(iii) For $2 \le i \le r-1$, Ker $D(f\big|_{\mathfrak{S}_{i-1}})$ is simply tangent to $\mathfrak{S}_i$ at points in $\mathfrak{S}_{i+1}$.

Definition. We say that f has an $S_{1_r,0}$ singularity at P, if the following conditions hold.

(i) There exists a neighborhood $\mathcal{U}$ of P submanifolds $\mathfrak{S}_k$ of dimension $n-k$ in U so that $P \in \mathfrak{S}_r \subset \mathfrak{S}_{r-1} \subset \cdots \subset \mathfrak{S}_1$ and so that $f : \mathcal{U} \to N$ has an S_{1_r} singularity in $\mathcal{U}$, with singularity manifolds $(\mathfrak{S}_1, \dots, \mathfrak{S}_r)$.
(ii) Ker $Df_P \cap T_P(\mathfrak{S}_r) = \{0\}$.

The singularity manifolds $\mathfrak{S}_k$ are denoted by $S_{1_k}(f)$ in singularity theory (if the neighborhood is understood). An $S_{1,0}$ (or $S_{1_1,0}$) singularity is a Whitney fold; an $S_{1,1,0}$ (or $S_{1_2,0}$) singularity is referred to as a Whitney or simple cusp.

If f is given in *adapted coordinates* vanishing at P, i.e.

$$f : t \mapsto (t', h(t)) \tag{2.2}$$

then f has an S_{1_r} singularity in a neighborhood of $P = 0$ if and only if

$$(\partial/\partial t_n)^k h(0) = 0, \qquad 1 \le k \le r, \tag{2.3}$$

and the gradients

$$\nabla_t\big(\frac{\partial^k h}{\partial t_n^k}\big), k = 1, \dots r-1, \tag{2.4}$$

are linearly independent at 0. Moreover f has an $S_{1_r,0}$ singularity at P if in addition

$$(\partial/\partial t_n)^{r+1} h(0) \neq 0. \tag{2.5}$$

The singularity manifolds are then given by

$$S_{1_k}(f) = \{t : (\partial/\partial t_n)^j f(t) = 0,\, 1 \le j \le k\}.$$

In these coordinates the kernel field for f is $\partial/\partial t_n$ and the map f is of type r at P.

Normal forms of S_{1_r} singularities are due to Morin [47], who showed that there exists adapted coordinate systems so that (2.2) holds with

$$h(t) = t_1 t_n + t_2 t_n^2 + \dots + t_{r-1} t_n^{r-1} + t_n^{r+1}. \tag{2.6}$$

Finally we mention the situation of maximal degeneracy for S_1 singularities which occurs when the kernel of Df is everywhere tangential to the singularity surface $S_1(f)$. In this case we say that f is a *blowdown*; see example 2.3.3 below.

2.3. Examples. We now discuss some model examples. The first set of examples concern translation invariant averages over curves, the second set restricted X-ray transforms for rigid line complexes. The map f above will always be one of the projections $\pi_L : \mathcal{C} \to T^*\Omega_L$ or $\pi_R : \mathcal{C} \to T^*\Omega_R$. Note that $S_1(\pi_L) = S_1(\pi_R)$.

2.3.1. Consider the operator on functions in $\mathbb{R}^d$

$$\mathcal{A}f(x) = \int f(x + \Gamma(\alpha))\chi(\alpha) d\alpha \tag{2.7}$$

where $\alpha \to \Gamma(\alpha)$ is a curve in $\mathbb{R}^d$ so that $\Gamma'(\alpha), \Gamma''(\alpha), \dots, \Gamma^{(d)}(\alpha)$ are linearly independent. Then the canonical relation is given by

$$C = \{(x, \xi; x + \Gamma(\alpha), \xi) : \langle \xi, \Gamma'(\alpha)\rangle = 0\}.$$

Consider the projection π_L then it is not hard to see that $S_{1_k}(\pi_L)$ is the submanifold of $\mathcal{C}$ where in addition $\langle \xi, \Gamma^{(j)}(\alpha)\rangle = 0\}$ for $2 \le j \le k+1$. Clearly then $S_{1_{d-1}}(\pi_L) = \emptyset$ so that we have an $S_{1_{d-2,0}}$ singularity. The behavior of π_R is of course exactly the same; moreover for small perturbations the projections π_L and π_R still have at most $S_{1_{d-2,0}}$ singularities. Note that in the translation invariant setting we have

$S_{1_k}(\pi_L) = S_{1_k}(\pi_R)$, but for for small variable perturbations the manifolds $S_{1_k}(\pi_L)$, $S_{1_k}(\pi_R)$ are typically different if $k \geq 2$.

By Fourier transform arguments and van der Corput's lemma it is easy to see that $\mathcal{A}$ maps $L^2(\mathbb{R}^d)$ to the Sobolev-space $L^2_{1/d}(\mathbb{R}^d)$ and it is conjectured that this estimate remains true for variable coefficient perturbations. This is known in dimensions $d \leq 4$ (cf. §5 below).

2.3.2. Consider the example (2.7) with $d = 3$ and

$$\Gamma(\alpha) = (\alpha, \frac{\alpha^m}{m}, \frac{\alpha^n}{n})$$

where m, n are integers with $1 < m < n$.

The canonical relation C is given as the set of (x, ξ, y, ξ) where $x_2 - y_2 - (x_1 - y_1)^m/m = 0$, $x_3 - y_3 - (x_1 - y_1)^n/n = 0$, and $\xi = (\xi_1(\lambda, \mu), \lambda, \mu)$ so that

$$\xi_1 = -(x_1 - y_1)^{m-1}\lambda - (x_1 - y_1)^{n-1}\mu$$

with $(\lambda, \mu) \neq (0, 0)$.

C is thus parametrized by $(x_1, x_2, x_3, \lambda, \mu, y_1)$ and the singular variety $S_1(\pi_L)$ is given by the equation

$$(m-1)(x_1 - y_1)^{m-2}\lambda + (n-1)(x_1 - y_1)^{n-2}\mu = 0.$$

Note that $\partial/\partial y_1$ is a kernel vector field and hence π_L is of type at most $n-2$ everywhere. Note that $S_1(\pi_L)$ is a smooth submanifold only if $m = 2$. The case $m = 2$, $n = 3$ corresponds to the situation considered above (now π_L is a fold). If $m = 2$, $n = 4$ we have a simple cusp $(S_{1,1,0})$ singularity and $S_{1,1}(\pi_L)$ is the submanifold of $S_1(\pi_L)$ on which $x_1 = y_1$. If $m \geq 3$, $n > m$ then the singular variety is not a smooth manifold but the union of the two transverse hypersurfaces $\{(m-1)\lambda + (n-1)(x_1 - y_1)^{n-m}\mu = 0\}$ and $\{x_1 = y_1\}$.

2.3.3. For an example for a one-sided behavior we consider the restricted X-ray transform

$$\mathcal{R}f(x', x_d) = \chi_0(x_d) \int f(x' + t\gamma(x_d), t)\chi(t)dt \tag{2.8}$$

where γ is now the regular parametrization of a curve in $\mathbb{R}^{d-1}$ and χ_0, χ are smooth and compactly supported. We say that $\mathcal{R}$ is associated to a d dimensional line complex which is referred to as *rigid* because of the translation invariance in the x' variables.

The canonical relation is now given by

$$C = \{(x', x_d, \tau, y_d\tau \cdot \gamma'(x_d); x' + y_d\gamma(x_d), y_d, \tau, \tau \cdot \gamma(x_d))\}$$

and the singular set $S_1(\pi_L) = S_1(\pi_R)$ is the submanifold on which $\tau \cdot \gamma'(x_d) = 0$. One computes that $V_L = \partial/\partial y_d$ is a kernel vector field for π_L and $V_R = \partial/\partial x_d$ is a kernel vector field for π_R . Clearly V_L is tangential to $S_1(\pi_L)$ everywhere so that π_L is a blowdown. The behavior of the projection π_R depends on assumptions on γ. The best case occurs when $\gamma'(x_d), \dots, \gamma^{(d-1)}(x_d)$ are linearly independent everywhere. The singularity manifolds $\mathfrak{S}_k = S_{1_k}(\pi_R)$ are then given by the equations

$$\tau \cdot \gamma^{(j)}(x_d) = 0, \qquad j = 1, \dots, k,$$

and thus $S_{1_{d-1}}(\pi_R) = \emptyset$ and π_R has (at most) $S_{1_{d-2},0}$ singularities.

For the model case given here it is easy to derive the sharp L^2-Sobolev estimates. Observe that

$$R^*Rf(w) = \overline{\chi(w_d)} \iint f(w' + s\gamma(\alpha), w_d + s)|\chi_0(\alpha)|^2\chi(s)ds\, d\alpha$$

defines (modulo the cutoff function) a translation invariant operator. By van der Corput's Lemma it is easy to see that

$$\Big|\iint e^{-\imath s(\xi'\cdot\gamma(\alpha)+\xi_d)}|\chi_0(\alpha)|^2\chi(s)ds\, d\alpha\Big| \lesssim (1+|\xi|)^{-\frac{1}{d-1}}$$

and one deduces that $\mathcal{R}$ maps L^2 to $L^2_{1/(2d-2)}$.

It is conjectured that the X-ray transform for general well-curved line complexes

$$\mathcal{R}f(x', \alpha) = \chi(x', \alpha) \int f(x' + s\gamma(x', \alpha), s)\chi(s)ds \tag{2.9}$$

satisfies locally the same estimate; here the support of χ is supported in $(-\varepsilon, \varepsilon)$ for small ε and it is assumed that for each fixed x' the vectors $(\partial/\partial\alpha)^j\gamma$, $j = 1, \dots, d-1$ are linearly independent. The sharp $L^2 \to L^2_{1/(2d-2)}$ estimate is currently known in dimension $d \le 5$ (*cf.* §4-5 below).

2.4 Strong Morin singularities.

We now discuss the notion of a *strong* Morin singularities, or $S^+_{1_r}$ singularities for maps into a fiber bundle W over a base manifold B, with projection π_B. Here it is assumed that $\dim W = n$ and $\dim(B) = q \le n - r$, so that the fibers $W_b = \pi_B^{-1}b$ are $n - q$ dimensional manifolds (see [26]). The relevant W is $T^*\Omega_R$, the cotangent bundle of the base $B = \Omega_R$.

Definition. Let $b = \pi_B(f(P))$ and let $W_b = \pi_B^{-1}b$ be the fiber through $f(P)$. The map f has an $S^+_{1_r,0}$ singularity at P if

(i) f intersects W_b transversally, so that there is a neighborhood U of P such that the preimages $f^{-1}W_b \cap U$ are smooth manifolds of dimension $n - q$,

and if

(ii) $f\big|_{f^{-1}(W_b)\cap U}$ has an $S_{1_r,0}$ singularity at P.

Now let $\mathcal{C} \subset T^*\Omega_L \times T^*\Omega_R$ be a canonical relation, consider $\pi_L : \mathcal{C} \to T^*\Omega_L$ and use the natural fibration $\pi_{\Omega_L} : T^*\Omega_L \to \Omega_L$. If $\pi_L : \mathcal{C} \to T^*\Omega_L$ has an $S^+_{1_r,0}$ singularity at $c \in C$, $c = (x_0, \xi_0, y_0, \eta_0)$ then near c we can restrict π_L to $\pi_{\Omega_L}^{-1}(\{y_0\})$ and define π_{L,y_0} as the restriction of π_L to $\pi_{\Omega_L}^{-1}(\{y_0\})$ and π_{L,y_0} has an $S_{1_r,0}$ singularity at c.

We remark that for the examples in 2.3.1 both π_L and π_R have strong Morin singularities while for the example in 2.3.3 π_R has strong Morin singularities. This remains true for small perturbations of these examples.

In order to verify the occurence of strong Morin singularities for canonical relations which come up in studying averages on curves the following simple lemma is useful.

Lemma. *Let I be an open interval, let $\psi : I \to \mathbb{R}^n$ be a smooth parametrization of a regular curve not passing through* 0 *and let*

$$M = \{(t,\eta) \in I \times \mathbb{R}^n : \eta \cdot \psi(t) = 0, \text{ some } t \in I\}.$$

Let $\pi : M \to \mathbb{R}^n$ be defined by $\pi(t,\eta) = \eta$.

Then π has singularities at most $S_{1_{n-2},0}$ if and only if $\{\psi(t), \dot\psi(t), \dots, \psi^{(n-1)}(t)\}$ is a linearly independent set for all $t \in I$.

For the proof assume first the linear independence of $\psi^{(j)}(t)$. We may work near $t = 0$ and by a linear change of variables, we may assume that $\psi^{(j)}(t_0) = e_{j+1}, 0 \le j \le n-1$, where $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. Thus

$$\begin{aligned}\eta \cdot \psi(t) =& \sum_{j=0}^{n-1} \eta_{j+1} \frac{t^j}{j!}(1 + O(|t|)) \\ =& \eta_1(1 + O(|t|)) + \sum_{j=2}^{n} \eta_j \frac{t^{j-1}}{(j-1)!}(1 + O(|t|))\end{aligned}$$

with $\eta = (\eta_1, \eta')$. We can solve $\eta \cdot \psi(t) = 0$ for $\eta_1 = \eta_1(\eta', t)$,

$$\eta_1 = -\sum_{j=2}^{n} \eta_j \frac{t^{j-1}}{(j-1)!}(1 + O(|t|)).$$

Hence, (η', t) and (ξ', ξ_1) form adapted coordinates (*cf.* (2.2)) for the map π, and in these coordinates

$$\pi(\eta', t) = (\eta', \phi(\eta', t)) = \big(\eta', -\sum_{j=2}^{n} \eta_j \frac{t^{j-1}}{(j-1)!}\big(1 + O(|t|)\big)$$

where ϕ satisfies

$$\frac{\partial^j \phi}{\partial t^j}(0,0) = 0, \quad 1 \le j \le n-2, \quad \frac{\partial^{n-1}\phi}{\partial t^{n-1}}(0,0) \ne 0$$

and the differentials

$$\Big\{d(\frac{\partial^j \phi}{\partial t^j})(0,0)\Big\}_{j=1}^{n-1} = \{e_j\}_{j=2}^{n}$$

are linearly independent. Thus π has at most $S_{1_{n-2},0}$ singularities.

Conversely, assume that π has at most $S_{1_{n-2},0}$ singularities. Since ψ does not pass through the origin, we may assume that $\psi_n(t) \ne 0$ locally. Then the map π is given in adapted coordinates by

$$(\eta', t) \mapsto (\eta', -\sum_{j=1}^{n-1} \eta_j \frac{\psi_j(t)}{\psi_n(t)})$$

and the linear independence follows easily from (2.3-5).

2.5 Mixed finite type conditions. We briefly discuss mixed conditions for pairs of maps (f_L, f_R) where $f_L : M \to N_L$, $f_R : M \to N_R$ where M, N_L, N_R are all d dimensional and f_L, f_R are *volume equivalent*, i.e., there is a nonvanishing function α so that $\det Df_L = \alpha \det Df_R$ in the domain under consideration.

Let V_L, V_R be nonvanishing kernel fields on M for the maps f_L, f_R. Let U be a neighborhood of P in M. We define $\mathcal{D}_{j,k}(U)$ to be the linear space of differential operators generated spanned by operators of the form

$$a_1 V_1 \dots a_{j+k} V_{j+k}$$

where V_i are kernel fields for the maps f_L or f_R in U, and k of them are kernel fields for f_L and j of them are kernel fields for f_R. Let h be a real valued function defined in a neighborhood of $P \in M$; we say that h *vanishes of order* (j,k) at P if $Lh_P = 0$ for all $L \in \mathcal{D}^{j-1,k} \cup \mathcal{D}^{j,k-1}$. We say that (f_L, f_R) is of type (j,k) if $h \equiv \det Df_L$ vanishes of order (j,k) at $P \in M$ and if there is an operator $L \in \mathcal{D}^{j,k}$ so that $Lh_P \neq 0$. Because of the assumption of volume equivalence $\det Df_L$ in this definition can be replaced by $\det Df_R$. In the canonical example of interest here we have $M = \mathcal{C} \subset T^*\Omega_L \times T^*\Omega_R$, a canonical relation, and $f_L \equiv \pi_L$, $f_R \equiv \pi_R$ are the projections to $T^*\Omega_L$ and $T^*\Omega_R$, respectively.

3. Fourier integral operators in two dimensions

In this section we examine the regularity of Fourier integral operators in two dimensions, in which case one can get the sharp L^2 regularity properties with the possible exception of endpoint estimates. We shall assume that Ω_L, Ω_R are open subsets of $\mathbb{R}^2$, $\mathcal{C} \subset (T^*\Omega_L \setminus 0_L) \times T^*\Omega_R \setminus 0_R)$ is a *homogeneous* canonical relation and $\mathcal{F} \in I^{-1/2}(\Omega_L, \Omega_R, \mathcal{C})$, with compactly supported distribution kernels; we assume that the rank of the projection $\pi_{\text{space}} : \mathcal{C} \to \Omega_L \times \Omega_R$ is ≥ 2 everywhere. The generalized Radon transform (1.11) (with $\ell = 1$, $d = 2$) is a model case in which rank $(d\pi_{\text{space}}) = 3$.

In order to formulate the L^2 results we shall work with the Newton polygon, as in [58] where oscillatory integral operators in one dimension are considered. We recall that for a set E of pairs (a,b) of nonnegative numbers the Newton polygon associated to E is the closed convex hull of all quadrants $Q_{a,b} = \{(x,y) : x \geq a, y \geq b\}$ where (a,b) is taken from E.

Definition. For $c \in \mathcal{C}$ let $\mathcal{N}(c)$ be the Newton polygon associated to the set

$$E(c) = \{(j+1, k+1) : \mathcal{C} \text{ is of type } (j,k) \text{ at } c\}. \tag{3.1}$$

Let (t_c, t_c) the point of intersection of the boundary $\partial\mathcal{N}(c)$ with the diagonal $\{(a,a)\}$.

Using the notion of type (j,k) in §2.5 we can now formulate

3.1. Theorem. *Let $\Omega_L, \Omega_R \subset \mathbb{R}^2$ and $\mathcal{C}$ as above and let $\mathcal{F} \in I^{-1/2}(\Omega_L, \Omega_R; \mathcal{C})$, with compactly supported distribution kernel. Let $\alpha = \min_c (2t_c)^{-1}$.*

Then the operator $\mathcal{F}$ maps L^2 boundedly to $L^2_{\alpha-\varepsilon}$ for all $\varepsilon > 0$.

In the present two-dimensional situation one can reduce matters to operators with phase functions that are linear in the frequency variables (i.e., the conormal situation). We briefly describe this reduction.

First, our operator can be written modulo smoothing operators as a finite sum of operators of the form

$$\mathcal{F}f(x) = \int e^{\imath\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)d\xi \tag{3.2}$$

where a is of order $-1/2$, and has compact x support. We may also assume that $a(x,\xi)$ has ξ-support in an annulus $\{\xi : |\xi| \approx \lambda\}$ for large λ. By scaling we can reduce matters to show that the L^2 operator norm for the oscillatory integral operator T_λ defined by

$$T_\lambda g(x) = \int e^{\imath\lambda\varphi(x,\xi)}\chi(x,\xi)g(\xi)d\xi$$

is $O(\lambda^{-1/2-\alpha})$; here χ has compact support and vanishes for ξ near 0. We introduce polar coordinates in the last integral, $\xi = \sigma(\cos y_1, \sin y_1)$ and put

$$S(x, y_1) = \phi(x_1, x_2, \cos y_1, \sin y_1).$$

Then the asserted bound for $\|T_\lambda\|$ is equivalent to the same bound for the L^2 norm of $\widetilde{T}_\lambda$ defined by

$$\widetilde{T}_\lambda h(x) = \int e^{\imath\lambda\sigma S(x,y_1)}\widetilde{\chi}(y_1,\sigma)dy_1 d\sigma$$

for suitable $\widetilde{\chi}$; here we have used the homogeneity of φ. Now we rescale again and apply a Fourier transform in σ and see that the bound $\|\widetilde{T}_\lambda\| = O(\lambda^{-1/2-\alpha})$ follows from the $L^2 \to L^2_\alpha$ bound for the conormal Fourier integral operator with distribution kernel

$$\int e^{\imath\tau\Phi(x,y)} b(x,\tau)d\tau \tag{3.3}$$

where $\Phi(x,y) = S(x,y_1) - y_2$, and b is a symbol of order 0, supported in $\{|\tau| \approx \lambda\}$ and compactly supported in x.

Thus it suffices to discuss conormal operators of this form; in fact for them one can prove almost sharp $L^p \to L^p_\alpha$ estimates. Before stating these results we shall first reformulate the mixed finite type assumption from §2.5 in the present situation.

3.2. Mixed finite type conditions in the conormal situation. We now look at operators with distribution kernels of the form (3.3). The singular support of such operators is given by

$$\mathcal{M} = \{(x,y) : \Phi(x,y) = 0\}$$

and it is assumed that $\Phi_x \neq 0$, $\Phi_y \neq 0$. The canonical relation is the twisted conormal bundle $N^*\mathcal{M}'$ as in (1.12). In view of the homogeneity the type condition at $c_0 = (x_0, y_0, \xi_0, \eta_0) \in N^*\mathcal{M}'$ is equivalent with the type condition at $(x_0, y_0, r\xi_0, r\eta_0)$ for any $r > 0$ and since the fibers in $N^*\mathcal{M}'$ are one-dimensional it seems natural to formulate finite type conditions in terms of vector fields tangent to $\mathcal{M}$, and their commutators. We now describe these conditions but refer for a more detailed discussion to [67]. Related ideas have been used in the study of subelliptic operators ([36], [63]), in complex analysis ([41], [2]) and, more recently, in the study of singular Radon transforms ([11]).

Two types of vector fields play a special role: We say that a vector field V on $\mathcal{M}$ is of type $(1,0)$ if V is tangent to $\mathcal{M} \cap (\Omega_L \times \{0\})$; likewise we define V to be of type $(0,1)$ if V is tangent to $\mathcal{M} \cap (\{0\} \times \Omega_R)$. The notation is suggested by an analogous situation in several complex variables ([41], [55]).

Note that at every point $P \in \mathcal{M}$ the vector fields of type $(1,0)$ and $(0,1)$ span a two-dimensional subspace of the three-dimensional tangent space $T_P\mathcal{M}$. Thus we can pick a nonvanishing $1-$form ω which annihilates vector fields of type $(1,0)$ and $(0,1)$; we may choose $\omega = d_x\Phi - d_y\Phi$ and $X = \Phi_{x_2}\partial_{x_1} - \Phi_{x_1}\partial_{x_2}$, $Y = \Phi_{y_2}\partial_{y_1} - \Phi_{y_1}\partial_{y_2}$ are $(1,0)$ and $(0,1)$ vector fields, respectively. With this choice

$$\langle \omega, [X,Y] \rangle = -2 \det \begin{pmatrix} \Phi_{xy} & \Phi_x \\ {}^t\Phi_y & 0 \end{pmatrix} \tag{3.4}$$

which is (1.14) in the situation $\ell = 1$, $\theta = 1$ and relates $\langle \omega, [X,Y] \rangle$ to $\det d\pi_{L/R}$. Thus $N^*\mathcal{M}'$ is a local canonical graph iff $\langle \omega, [X,Y] \rangle$ does not vanish. The quantity (3.4) is often referred to as "rotational curvature" (*cf.* [55]).

Now let μ and ν be two positive integers. For a neighborhood U of P let $\mathcal{W}^{\mu,\nu}(U)$ be the module generated by vector fields $\mathrm{ad}W_1 \mathrm{ad}W_2 \dots \mathrm{ad}W_{\mu+\nu-1}(W_{\mu+\nu})$ where μ of these vector fields are of type $(1,0)$ and ν are of type $(0,1)$. The finite type condition in (2.4) can be reformulated as follows. Let $P \in \mathcal{M}$ and let $c \in N^*\mathcal{M}'$ with base point P. Then $\mathcal{C}$ is of type (j,k) at c if there is a neighborhood U of P so that for all vector fields $W \in \mathcal{W}^{j+1,k}(U) \cup \mathcal{W}^{j,k+1}(U)$ we have $\langle \omega, W \rangle_P = 0$ but there is a vector field $\widetilde{W}$ in $\mathcal{W}^{j+1,k+1}$ for which $\langle \omega, \widetilde{W} \rangle_P \neq 0$.[1] Now coordinates can be chosen so that $\Phi(x,y) = -y_2 + S(x,y_1)$ and the generalized Radon transform is given by

$$\mathcal{R}f(x) = \int \chi(x, y_1, S(x,y_1)) f(y_1, S(x,y_1)) dy_1 \tag{3.5}$$

where $S_{x_2} \neq 0$ and $\chi \in C_0^\infty(\Omega_L \times \Omega_R)$. If

$$\Delta(x,y_1) = \det \begin{pmatrix} S_{x_1 y_1} & S_{x_1} \\ S_{x_2 y_1} & S_{x_2} \end{pmatrix}$$

then at $P = (x, y_1, S(x,y_1))$ the mixed finite type condition amounts to

$$X^{j'} Y^{k'} \Delta(x,y_1) = 0 \text{ whenever } j' \leq j \text{ and } k' < k \text{ or } j' < j \text{ and } k' \leq k \tag{3.6}$$

but

$$X^j Y^k \Delta(x,y_1) \neq 0 \tag{3.7}$$

for $X = S_{x_2}\partial_{x_1} - S_{x_1}\partial_{x_2}$ and $Y = \partial_{y_1} + S_{y_1}\partial_{y_2}$. For the equivalence of these conditions see [67].

[1] Here we deviate from the terminology in [67], where the incidence relation $\mathcal{M}$ is said to be of type $(j+1, k+1)$ at P.

We now relate the last condition to the finite type condition above. Notice that

$$\mathcal{C} = \{(x_1, x_2, \tau S_{x_1}, \tau S_{x_2}; y_1, S(x, y_1), -\tau S_{y_1}, \tau)\}$$

and using coordinates (x_1, x_2, y_1, τ) a kernel vector field for the projection π_R is given by $V_R = S_{x_2}\partial_{x_1} - S_{x_1}\partial_{x_2}$; this can be identified with the vector field X on $\mathcal{M}$. Moreover a kernel vector field for the projection π_L is given by $V_L = \partial/\partial_{y_1} - S_{x_2}^{-1}S_{x_2y_1}\partial/\partial\tau$ and for any function of the form $F(x, y_1)$ we see that $(V_L - Y)(\tau F)$ equals F multiplied by a C^∞ function. Thus it is immediate that C is of type (j, k) at the point c (with coordinates (x, y_1, τ)) if conditions (3.6), (3.7) are satisfied, and this is just a condition at the base-point P.

We shall now return to the proof of Theorem 3.1 and formulate an L^p version for the conormal situation.

3.3. Theorem [67]. *Let $\Omega_L, \Omega_R \subset \mathbb{R}^2$, let $\mathcal{M} \subset \Omega_L \times \Omega_R$ so that the projections to Ω_L and Ω_R have surjective differential. Suppose that $\mathcal{F} \in I^{-1/2}(\Omega_L, \Omega_R; N^*\mathcal{M}')$ with compactly supported distribution kernel.*

For $c \in N^*\mathcal{M}'$ denote by $\widetilde{\mathcal{N}}(c)$ the closure of the image of $\mathcal{N}(c)$ under the map $(x, y) \mapsto (\frac{x}{x+y}, \frac{1}{x+y})$; i.e., the convex hull of the points $(1, 1)$, $(0, 0)$ and $(\frac{j+1}{j+k+2}, \frac{1}{j+k+2})$ where $N^*\mathcal{M}'$ is of type (j, k) at c.

Suppose that $(1/p, \alpha)$ belongs to the interior of $\widetilde{\mathcal{N}}(c)$, for every c. Then $\mathcal{F}$ is bounded from L^p to L^p_α.

The L^2 estimate of Theorem 3.1 for conormal operators follows as a special case, and for the general situation we use the above reduction. Theorem 3.3 is sharp up to the open endpoint cases (*cf.* also §3.5.1-3 below).

We now sketch the main ingredients of the proof of Theorem 3.3. We may assume that S_{x_2} is near 1 and $|S_{x_1}| \ll 1$. Suppose that $Q = (x^0, y^0) \in \mathcal{M}$ and suppose that the type (j', k') condition holds for some choice of (j', k') with $j' \le j$ and $k' \le k$ at Q, and suppose that this type assumption is still valid in a neighborhood on the support of the cutoff function χ in (3.5) (otherwise we work with partitions of unity).

Our goal is then to prove that $\mathcal{F}$ maps L^p to L^p_α for $p = (j + k + 2)/(j + 1)$ and $\alpha < 1/(j + k + 2)$.

Since we do not attempt to obtain an endpoint result, it is sufficient to prove the required estimate for operators with the frequency variable localized to $|\tau| \approx \lambda$ for large λ. We then make an additional dyadic decomposition in terms of the size of $|\Delta|$ (i.e., the rotational curvature). Define a Fourier integral operator $\mathcal{F}_{\lambda,l_0}$ by

$$\mathcal{F}_{\lambda,l_0} f(x) = \int f(y) \int e^{\imath\tau(S(x,y_1)-y_2)}\beta(x, y, \frac{|\tau|}{\lambda})\chi(2^{l_0}|\Delta(x, y_1)|)d\tau\, dy;$$

then by interpolation arguments our goal will be achieved by proving the following crucial estimates:

$$\|\mathcal{F}_{\lambda,l_0}\|_{L^p\to L^p} \le C_\gamma 2^{-l_0\gamma}, \qquad p = \frac{j+k}{j}, \quad \gamma < \frac{1}{j+k}, \tag{3.8}$$

and

$$\|\mathcal{F}_{\lambda,l_0}\|_{L^2\to L^2} \le C_\varepsilon 2^{l_0(\frac{1}{2}+\varepsilon)}\lambda^{-1/2}. \tag{3.9}$$

A variant of this interpolation argument goes back to investigations on maximal operators in [18] and [71], [72], and (3.9) can be thought of a version of an estimate for damped oscillatory integrals.

The type assumption is only used for the estimate (3.8). We note that by integration by parts with respect to the frequency variable the kernel of $\mathcal{F}_{\lambda,l_0}$ is bounded by

$$\lambda(1+\lambda|y_2-S(x,y_1)|)^{-N}\widetilde{\chi}(2_0^l\Delta(x,y_1)). \tag{3.10}$$

We can use a well known sublevel set estimate related to van der Corput's lemma (see [8]) to see that for each fixed x the set of all y_1 such that $|\Delta(x,y_1)|\le 2^{-l}$ and $|\partial_{y_1}^{k'}\Delta(x,y_1)|\approx 2^{-m}$ has Lebesgue measure bounded by $C_\varepsilon 2^{\varepsilon l_0}2^{(m-l_0)/k'}\le C_\varepsilon 2^{\varepsilon l_0}2^{(m-l_0)/k}$ if $m\le l_0$. Moreover, if $\mathfrak{S}(y,x_1)$ is implicitly defined by $y_2=S(x_1,\mathfrak{S}(y,x_1),y_1)$ then the assumption $X^{j'}Y^{k'}\Delta\neq 0$ for some (j',k'), $j'\le j$, $k'\le k$ implies that $\partial_{x_1}^{j'}[\partial_{y_1}^{k'}\Delta(x_1,\mathfrak{S}(y,x_1),y_1)]\neq 0$, for some (j',k'), $j'\le j$, $k'\le k$. Thus for fixed y the set of all x_1 for which $|\partial_{x_1}^{j'}[\partial_{y_1}^{k'}\Delta(x_1,\mathfrak{S}(y,x_1),y_1)]|\le 2^{-m}$ has Lebesgue measure $\lesssim 2^{-m/j'}\lesssim 2^{-m/j}$. The two sublevel set estimates together with (3.10) and straightforward applications of Hölder's inequality yield (3.8), see [67].

We now turn to the harder L^2 estimate (3.9). We sketch the ideas of the proof (see [66] and also [67] for some corrections).

Firstly, if $2^{l_0}\le\lambda$ we consider as above the oscillatory integral operator T_{λ,l_0} given by

$$T_{\lambda,l_0}g(x)=\int e^{\imath\lambda\tau(S(x,y_1)-y_2)}\eta(y_1,\tau)\chi(2^{l_0}|\Delta(x,y_1)|)g(y_1,\tau)d\tau dy_1 \tag{3.11}$$

with compactly supported η; it suffices to show that

$$\|T_{\lambda,l_0}\|_{L^2\to L^2}\le C_\varepsilon 2^{l_0(1+\varepsilon)/2}\lambda^{-1}. \tag{3.12}$$

If $|\Delta(x,y_1)|\le\lambda^{-1}$ we modify our definition by localizing to this set. We note that it suffices to estimate the operator $\chi_{Q'}\mathcal{F}[\chi_Q f]$ where Q and Q' are squares of sidelength $2^{-l_0\varepsilon/10}$, since summing over all relevant pairs of squares will only introduce an error $O(2^{4\varepsilon l_0/10})$ in the final estimate.

If we tried to use the standard TT^* argument we would have to have good lower bounds for $S_{y_1}(w,y_1)-S_{y_1}(x,y_1)$ in the situation where $S(w,y_1)-S(x,y_1)$ is small, but the appropriate lower bounds fail to hold if the rotational curvature is too small. Thus it is necessary to work with finer decompositions. Solve the equation $S(w,y_1)-S(x,y_1)=0$ by $w_2=u(w_1,x,y_1)$ and expand

$$\begin{aligned} S_{y_1}(w,y_1)-S_{y_1}(x,y_1)&=\\ &S_{y_1}(w_1,u(w_1,x,y_1),y_1)-S_{y_1}(x,y_1)+O(S(w,y_1)-S(x,y_1))\end{aligned} \tag{3.13}$$

and

$$S_{y_1}(w_1,u(w_1,x,y_1),y_1)-S_{y_1}(x,y_1)=\sum_{j=0}^{M}\gamma_j(x,y_1)(w_1-x_1)^{j+1}+O(2^{-l_0\varepsilon/M}) \tag{3.14}$$

where $M \gg 100/\varepsilon$. In particular

$$\gamma_0(x,y_1) = S_{y_1x_1}(x,y_1) + u_{w_1}(x_1,x,y_1)S_{y_1,x_2}(x,y_1) = \frac{\Delta(x,y_1)}{S_{x_2}(x,y_1)};$$

thus $|\gamma_0(x,y_1)| \approx 2^{-l_0}$. For the coefficient $\gamma_j(x,y_1)$ we have

$$\gamma_j(x,y_1) = S_{x_2}^{-1}V^j\Delta(x,y_1) + \sum_{k<j}\alpha_k(x,y_1)V^k(x,y_1)\Delta(x,y_1)$$

where the α_k are smooth and V is the $(1,0)$ vector field $\partial_{x_1} - S_{x_1}/S_{x_2}\partial_{x_2}$. We introduce an additional localization in terms the size of $\gamma_j(x,y_1)$. For $\vec{l} = (l_0,\dots,l_M)$, with $l_j < l_0$ for $j = 1,\dots,M$ define

$$T_{\lambda,\vec{l}}\,g(x) = \iint e^{\imath\lambda\tau S(x,y_1)}\eta(x,y_1)\prod_{j=0}^{M}\chi(2^{l_j}|\gamma_j(x,y_1)|)g(y_1,\tau)dy_1d\tau$$

which describes a localization to the sets where $|\gamma_j(x,y_1)| \approx 2^{-l_j}$. A modification of the definition is required if $|\gamma_j| \le 2^{-l_0}$ for some $j \in \{1,...,M\}$.

Since we consider at most $O((1+l_0)^M) = O(2^{\varepsilon l_0})$ such operators it suffices to bound any individual $T_{\lambda,\vec{l}}$, and the main estimate is

3.4. Proposition.

$$\|T_{\lambda,\vec{l}}\|_{L^2\to L^2} \le C_\varepsilon 2^{l_0(1+\varepsilon)/2}\lambda^{-1/2}.$$

In what follows we fix λ and $\vec{l}$ and set

$$\mathcal{T} = T_{\lambda,\vec{l}}.$$

The proof of the asserted L^2 bound for $\mathcal{T}$ relies on an orthogonality argument based on the following result (a rudimentary version of the orthogonality argument in the case of two-sided fold singularities is already in [56]).

Lemma. *For $\vec{l} = (l_0,\dots,l_m)$, with $0 \le l_j \le l_0$ for $1 \le j \le M$, let $\mathcal{P}_M(l)$ be the class of polynomials $\sum_{i=0}^M a_ih^i$ with $2^{-l_i-2} \le |a_i| \le 2^{-l_i+2}$ if $l_i < l_0$ and $|a_i| \le 2^{-l_0+2}$ if $l_i = l_0$. Then there is a constant $C = C(M)$ and numbers ν_s,μ_s, $s = 1,\dots,10^M$ so that*

(i)

$$0 \le \nu_1 \le \mu_1 \le \nu_2 \le \mu_2 \le \cdots \le \nu_M \le \mu_M \le 1 := \nu_{M+1},$$

(ii)

$$\nu_i \le \mu_i \le C\nu_i.$$

(iii)

$$\Big|\sum_{i=1}^N a_ih^i\Big| \ge C^{-1}\max\{|a_j||h|^j; j = 1,\dots M\} \text{ if } h \in [0,1]\setminus\bigcup_s[\nu_s,\mu_s].$$

Note that while μ_i and ν_i are close there may be 'large' gaps between μ_i and ν_{i+1} for which the favorable lower bound (iii) holds. The elementary but somewhat lengthy proof of the Lemma based on induction is in [66]. A shorter and more elegant proof (of a closely related inequality) based on a compactness argument is due to Rychkov [64].

In order to descibe the orthogonality argument we need some terminology. Let I be a subinterval of $[0,1]$. We say that β is a *normalized cutoff function associated to I* if β is supported in I and $|\beta^{(j)}(t)| \le |I|^{-j}$, for $j = 1, \dots, 5$ and denote by $\mathfrak{A}(I)$ the set of all normalized cutoff functions associated to I.

Fix I and β in $\mathfrak{A}(I)$; then we define another localization of $\mathcal{T} = T_{\lambda,\vec{l}}$ by

$$\mathcal{T}[\beta]g(x) = \beta(x_1)\mathcal{T}g(x). \tag{3.15}$$

It follows quickly from the definition and the property $\nu_s \le \mu_s \le C\nu_s$ that

$$\sup_{\substack{\widetilde{I} \\ |\widetilde{I}|=\mu_s}} \sup_{\widetilde{\beta}\in\mathfrak{A}(\widetilde{I})} \big\|\mathcal{T}[\widetilde{\beta}]\big\| \lesssim \sup_{\substack{I \\ |I|=\nu_s}} \sup_{\beta\in\mathfrak{A}(I)} \big\|\mathcal{T}[\beta]\big\|. \tag{3.16}$$

This is because for any interval $\widetilde{I}$ of length μ_s a function $\beta \in \mathfrak{A}(\widetilde{I})$ can be written as a sum of a bounded number of functions associated to subintervals of length ν_s.

We have to prove that also

$$\sup_{\substack{\widetilde{I} \\ |\widetilde{I}|=\nu_s}} \sup_{\beta\in\mathfrak{A}(\widetilde{I})} \big\|\mathcal{T}[\widetilde{\beta}]\big\| \lesssim \sup_{\substack{I \\ |I|=\mu_{s-1}}} \sup_{\beta\in\mathfrak{A}(I)} \big\|\mathcal{T}[\beta]\big\| + 2^{l_0(\frac12+\varepsilon)}\lambda^{-1}. \tag{3.17}$$

and

$$\sup_{\substack{I \\ |I|=\nu_1/8}} \sup_{\beta\in\mathfrak{A}(I)} \big\|\mathcal{T}[\beta]\big\| \lesssim 2^{l_0(\frac12+\varepsilon)}\lambda^{-1}. \tag{3.18}$$

By te above remark (3.17) is only obvious if This is obvious if $\mu_{s-1} \approx \nu_s$. Thus let us assume that $\mu_{s-1} \le 2^{-100M}\nu_s$ and fix $\widetilde{\beta} \in \mathfrak{A}(\widetilde{I})$, $|\widetilde{I}| = \mu_{s-1}$.

One uses the Cotlar-Stein Lemma in the form

$$\Big\|\sum A_j\Big\| \lesssim \Big[\sum_{n=-\infty}^{\infty} \sup_j \|A^*_{j+n}A_j\|^{\theta}\Big]^{1/2}\Big[\sum_{n=-\infty}^{\infty} \sup_j \|A_{j+n}A^*_j\|^{1-\theta}\Big]^{1/2}, \tag{3.19}$$

for a (finite) sum of operators $\sum_j A_j$ on a Hilbert space. (See [73, ch. VII.2]; as pointed out in [7] and elsewhere, the version (3.19) follows by a slight modification of the standard proof).

Now if J is an interval of length $\nu_s/8$ and $\widetilde{\beta} \in \mathfrak{A}(J)$ then we split $\widetilde{\beta} = \sum_n \beta_n$ where for a fixed absolute constant C the function $C^{-1}\beta_n$ belongs to $\mathfrak{A}(I_n)$ and the I_n are intervals of length μ_{s-1}; I_n and $I_{n'}$ are disjoint if $|n-n'| > 3$ and the sum extends over no more than $O(\nu_s/\mu_{s-1})$ terms and thus over no more than $O(2^{l_0})$ terms.

Now let $|I_n| = |I_{n'}| \approx \mu_{s-1}$ and $\mathrm{dist}(I_n, I_{n'}) \approx |n-n'||I|$ and assume $|n-n'||I| \le \nu_s/8$. Let β_n, $\beta_{n'}$ be normalized cutoff functions associated to I_n, $I_{n'}$. Then

$$\|\mathcal{T}[\beta_n]^*\mathcal{T}[\beta_{n'}]\| = 0 \qquad \text{if } |n-n'| > 3 \tag{3.20}$$

by the disjointness of the intervals I_n, I'_n. The crucial estimate is
(3.21)

$$\|\mathcal{T}[\beta_n]\mathcal{T}[\beta_{n'}]^*\| \lesssim |n-n'|^{-1}2^{l_0(1+\varepsilon)}\lambda^{-2} \qquad \text{if } |n-n'|>3,\ |n-n'|\le \frac{\nu_s}{8\mu_{s-1}}.$$

(3.20/21) allows us to apply (3.19) with $\theta=0$ (the standard version does not apply, as is erroneously quoted in [66]). This yields the bound

$$\|\mathcal{T}[\widetilde{\beta}]\| \le C_\varepsilon\big[\sup_n \|\mathcal{T}[\beta_n]\| + 2^{l(1+\varepsilon)/2}\lambda^{-1}\sum_{n=3}^{2^{l_0}} n^{-1}\big]$$

and thus (3.17).

To see (3.21) one examines the kernel K of $\mathcal{T}[\beta_n]\mathcal{T}[\beta_{n'}]^*$ which is given by

$$K(x,w) = \beta_n(x_1)\beta_{n'}(w_1)\int e^{-\imath\lambda\tau(S(w,y_1)-S(x,y_1))}b(x,w,y_1,\tau)dy_1d\tau \tag{3.22}$$

and by definition of μ_{s-1}, ν_s, I_n, $I_{n'}$ and the above Lemma we have

$$|S_{y_1}(x,y_1)-S_{y_1}(w,y_1)| \gtrsim 2^{-l_0}|x_1-y_1| - O(S(x,y_1)-S(w,y_1)) \tag{3.23}$$

To analyze the kernel K and prove (3.21) by Schur's test one integrates by parts once in y_1 and then many times in τ, for the somewhat lengthy details see [66], [67]. Analogous arguments also apply to the estimation of $\mathcal{T}[\beta]\mathcal{T}[\beta]^*$ when β is associated to an interval of length $\ll \nu_1$, this gives (3.18).

Remarks.

3.5.1. Phong and Stein, in the remarkable paper [58], proved sharp L^2 decay estimates for oscillatory integral operators with kernel $e^{i\lambda s(x,y)}\chi(x,y)$ in one dimensions, where s is *real analytic*. ¿From their result and standard arguments one gets an improved result for the generalized Radon transform in the special semi-translation invariant case where the curves in $\mathbb{R}^2$ are given by

$$y_2 = x_2 + s(x_1,y_1). \tag{3.24}$$

Namely, if $\mathcal{R}f(x) = \int f(y_1, x_2+s(x_1,y_1))\chi(x,y_1)dy_1$ then the endpoint $L^2\to L^2_\alpha$ estimate in Theorem 3.1 holds true. An only slightly weaker result for the case $s\in C^\infty$ has been obtained by Rychkov [64]. For related work see also some recent papers by Greenblatt [23], [24].

3.5.2. It is not known exactly which endpoint bounds hold in the general case of Theorem 3.3. As an easy case the $L^p\to L^p_{1/p}$ estimate holds if $p>n$ and a type $(0,n-2)$ condition is satisfied (in the terminology of Theorem 3.3). A similar statement for $1<p<n/(n-1)$ is obtained for type $(n-2,0)$ conditions by passing to the adjoint operator.

The interpolation idea (3.8-9) is not limited to conormal operators. Using variants of this method, sharp L^p estimates for Fourier integral operators in the nondegenerate case ([68]) were extended to certain classes with one- or two-sided fold singularities ([70], [16]). For other L^p Sobolev endpoint bounds in special cases see [74], [66], [57], [80].

3.5.3. Some endpoint inequalities in Theorem 3.3 fail: M. Christ [9] showed that the convolution with a compactly supported density on (t, t^n) fails to map $L^n \to L^n_{1/n}$. The best possible substitute is an $L^{n,2} \to L^n_{1/n}$ estimate in [69]; here $L^{n,2}$ is the Lorentz space.

3.5.4. Interpolation of the bounds in Theorem 3.3 with trivial $L^1 \to L^\infty$ bounds (with loss of one derivative) yields almost sharp $L^p \to L^q$ bounds ([66], [67]). Endpoint estimates for the case of two-sided finite type conditions are in [1]. For endpoint $L^p \to L^q$ estimates in the case (3.24), with real-analytic s, see [57], [79], [42].

3.5.5. It would be desirable to obtain almost sharp L^2 versions such as Theorem 3.1 for more general oscillatory integral operators with a corank one assumption. Sharp endpoint L^2 results where one projection is a Whitney fold (type 1) and the other projection satisfies a finite type condition are due to Comech [15].

3.5.6. Interesting bounds for the semi-translation invariant case (3.24) where only lower bounds on s_{xy} (or higher derivatives) are assumed were obtained by Carbery, Christ and Wright [6]. Related is the work by Phong, Stein and Sturm ([60], [62], [61]), with important contributions concerning the stability of estimates.

4. Operators with one-sided finite type conditions

We now discuss operators of the form (1.1) and assume that one of the projections, π_L, is of type $\leq r$ but make no assumption on the other projection, π_R. The role of the projections can be interchanged by passing to the adjoint operator.

4.1. Theorem [25],[26],[28]. *Suppose π_L is of corank ≤ 1 and type $\leq r$, and suppose that* $\det d\pi_L$ *vanishes simply. If $r \in \{1, 2, 3\}$ then*

$$\|T_\lambda\|_{L^2 \to L^2} \lesssim \lambda^{-(d-1)/2-1/(2r+2)} \tag{4.1}$$

It is conjectured that this bound also holds for $r > 3$. The estimate (4.1) is sharp in cases where the other projection exhibits maximal degeneracy. In fact if π_L is a fold and π_R is a blowdown then more information is available such as a rather precise description of the kernel of $T_\lambda T_\lambda^*$, *cf.* Greenleaf and Uhlmann [32], [33]. Applications include the restricted X-ray transform in three dimensions for the case where the line complexes are admissible in the sense of Gelfand ([21], [30], [34]); for an early construction and application of a Fourier integral operator with this structure see also [43].

In the discussion that follows we shall replace the assumption that $\det d\pi_L$ vanishes simply (i.e., $\nabla_{x,z} \det \pi_L \neq 0$) by the more restrictive assumption

$$\nabla_z \det \pi_L \neq 0. \tag{4.2}$$

In the case $r = 1$ this is automatically satisfied, and it is shown in [26], [28] that in the cases $r = 2$ and $r = 3$ one can apply canonical transformations to reduce matters to this situation. For the oscillatory integral operators coming from the restricted X-ray transform for well-curved line complexes, the condition (4.2) is certainly satisfied. We shall show that for general r the estimate (4.1) is a consequence of sharp estimates for oscillatory integral operators satisfying two-sided finite type conditions of order $r - 1$, in $d - 1$ dimensions. The argument is closely related to Strichartz estimates and can also be used to derive $L^2 \to L^q$ estimates (an early version can be found in Oberlin [48]).

We shall now outline this argument. After initial changes of variables in x and z separately we may assume that

$$\Phi_{x'z'}(0,z) = I_{d-1}, \qquad \Phi_{x'z_d}(0,z) = 0,$$
$$\Phi_{x'z'}(x,0) = I_{d-1}, \qquad \Phi_{x_dz'}(x,0) = 0; \tag{4.3.1}$$

moreover by our assumption on the type we may assume that

$$\Phi_{x_dz_d^{r+1}}(0,0) \neq 0 \tag{4.3.2}$$

and that $\Phi_{x_dz_d^j}(x,z)$ is small for $j \leq r$. We may assume that the amplitudes are supported where $|x| + |z| \leq \varepsilon_0 \ll 1$.

We form the operator $T_\lambda T_\lambda^*$ and write

$$T_\lambda T_\lambda^* f(x'x_d) = \int \mathcal{K}^{x_d,y_d}[f(\cdot,y_d)](x')dy_d \tag{4.4}$$

where the kernel of $\mathcal{K}^{x_dy_d}$ is given by

$$K^{x_dy_d}(x',y') = \int e^{\imath\lambda[\Phi(x',x_d,z)-\Phi(y',y_d,z)]}\sigma(x,z)\overline{\sigma(y,z)}dz.$$

We split $K^{x_dy_d} = H^{x_dy_d} + R^{x_dy_d}$ where $H^{x_dy_d}(x',y')$ vanishes when $|x_d - y_d| \leq \lambda^{-1}$ and when $|x'-y'| \gtrsim \varepsilon|x_d - y_d|$, for some ε with $\varepsilon_0 \ll \varepsilon \ll 1$.

Notice that by (4.3.1)

$$\Phi_{x'}(x',x_d,z) - \Phi_{x'}(y',y_d,z) = x' - y' + O(\varepsilon_0|x_d - y_d|)$$

and by an integration by parts argument we get

$$|R^{x_dy_d}(x',y')| \leq C_N(1+\lambda|x'-y'|)^{-N}$$

for any N, in the relevant range $|x'-y'| \gtrsim \varepsilon|x_d - y_d|$. Thus the corresponding operator $\mathcal{R}^{x_dy_d}$ is bounded on $L^2(\mathbb{R}^{d-1})$ and satisfies

$$\|\mathcal{R}^{x_dy_d}\|_{L^2\to L^2} \leq C_N'\lambda^{-d+1}(1+\lambda|x_d-y_d|)^{-N+d-1}, \tag{4.5}$$

for any N. For the main contribution $\mathcal{H}^{x_dy_d}$ we are aiming for the estimate

$$\|\mathcal{H}^{x_dy_d}\|_{L^2\to L^2} \lesssim \lambda^{1-d-\frac{1}{r+1}}|x_d-y_d|^{-\frac{1}{r+1}}. \tag{4.6}$$

¿From (4.4), (4.5), (4.6) and the $L^2(\mathbb{R})$ boundedness of the operator with kernel $|x_d - y_d|^{-1/(r+1)}\chi_{[-1,1]}(x_d - y_d)$ the bound (4.1) follows in a straightforward way.

Now observe that the operator $\mathcal{H}^{x_dy_d}$ is local on cubes of diameter $\approx |x_d - y_d|$ and we can use a trivial orthogonality argument to put the localizations to cubes together. For a single cube we may then apply a rescaling argument. Specifically, let $c \in \mathbb{R}^d$ and define

$$\widetilde{H}_c^{x_dy_d}(u,v) = H^{x_dy_d}(c+u|x_d-y_d|, c+v|x_d-y_d|).$$

Then for the corresponding operators we have

$$\|\mathcal{H}^{x_d y_d}\|_{L^2\to L^2} \lesssim |x_d - y_d|^{d-1} \sup_c \|\widetilde{\mathcal{H}}_c^{x_d y_d}\|_{L^2\to L^2}. \tag{4.7}$$

Note that $\widetilde{\mathcal{H}}_c^{x_d y_d}$ does not vanish only for small c. A calculation shows that the kernel of $\widetilde{\mathcal{H}}_c^{x_d y_d}$ is given by an oscillatory integral

$$\int e^{\imath\mu\Psi^{\pm}(u,v,z;c,\alpha,y_d)} b(u,v,z;c,x_d,y_d)dz; \qquad \alpha = |x_d - y_d|,\ \mu = \lambda|x_d - y_d|, \tag{4.8}$$

with small parameters $c, \alpha = |x_d - y_d|, y_d$, and the phase function is given by

$$\Psi^{\pm}(u,v,z;\alpha,y_d,c) = \langle u - v, \Phi_{x'}(0,z)\rangle \pm \Phi_{x_d}(0,z) + \rho^{\pm}(u,v,z;\alpha,y_d,c).$$

Here the choice of Ψ^+ is taken if $x_d > y_d$ and Ψ^- is taken if $x_d < y_d$; for the error we have $\rho^{\pm} = O(\alpha(|y_d| + c))$ in the C^∞ topology. Observe in particular that for $\alpha = 0$ we get essentially the localization of a translation invariant operator.

We now examine the canonical relation associated to the oscillatory integral, when $\alpha = 0$. In view of (4.3.1) the critical set $\{\nabla_z \Psi^{\pm} = 0\}$ for the phase function at $\alpha = 0$ is given by $\{(u,v,z) : v = u + g(z), \Phi_{x_d z_d}(0,z) = 0\}$ for suitable $g(z)$; in view of (4.2) this defines a smooth manifold. Consequently the canonical relation

$$\mathcal{C}_{\Psi^{\pm}}\Big|_{\alpha=0} = \{u, \Psi_u^{\pm}, v, \Psi_v^{\pm}) : \Psi_z^{\pm} = 0\}$$

is a smooth manifold. By (4.2) we may assume (after performing a rotation) that $\Phi_{x_d z_d z_1} \neq 0$ and then solve the equation $\Phi_{x_d z_d}(0,z) = 0$ near the origin in terms of a function $z_1 = \tilde{z}_1(z'', z_d)$. The projection π_L is given by

$$(u, z'', z_d) \to (u, \Phi_{x'}(0, z_1^{\pm}(z'', z_d), z'', z_d)$$

and $\partial/\partial z_d$ is a kernel field for π_L. Implicit differentiation reveals that $\partial_{z_d}^k z_1^{\pm} - \Phi_{z_d x_d z_1}^{-1} \Phi_{x_d z_d^{k+1}}$ belongs to the ideal generated by $\Phi_{x_d z_d^j}$, $j \le k$ and thus, by our assumption (4.3.2) we see that π_L is of type $\le r - 1$. The same holds true for π_R, by symmetry considerations. Although we have verified these conditions for $\alpha = 0$ they remain true for small α since Morin singularities are stable under small perturbations.

We now discuss estimates for the oscillatory integral operator $S_\mu^{\pm}$ whose kernel is given by (4.8) (we suppress the dependence on c, α, y_d.) The number of frequency variables is $N = d$ and thus we can expect the uniform bound

$$\|S_\mu^{\pm}\|_{L^2\to L^2} \lesssim \mu^{-\frac{d-2}{2} - \frac{1}{r+1} - \frac{d}{2}} \tag{4.9}$$

for small α. Indeed, the case $\alpha = 0$ of (4.9) is easy to verify; because of the translation invariance we may apply Fourier transform arguments together with the method of stationary phase and van der Corput's lemma. Given (4.9) we obtain from (4.7) and from (4.9) with $\mu = \lambda|x_d - y_d|$ that

$$\|\mathcal{H}^{x_d y_d}\|_{L^2\to L^2} \lesssim |x_d - y_d|^{d-1} \mu^{-(d-1)-\frac{1}{r+1}} \lesssim \lambda^{-(d-1)-\frac{1}{r+1}} |x_d - y_d|^{-\frac{1}{r+1}}.$$

Of course the Fourier transform argument does not extend to the case where α is merely small. However if $r = 1$ the estimate follows from (1.6) (with d replaced by $d-1$ and $N = d$) since then $\mathcal{C}_{\Psi^\pm}$ is a local canonical graph. Similarly, if $r = 2$ then the canonical relation $\mathcal{C}_{\Psi^\pm}$ projects with two-sided fold singularities so that the desired estimate follows from known estimates for this situation (see the pioneering paper by Melrose and Taylor [44], and also [53], [19], [27]). For the case $r = 3$, inequality (4.6) follows from a recent result by the authors [28] discussed in the next section, plus the reduction outlined in §1.2. The case $r \geq 4$ is currently open.

Remarks.

4.2.1. The argument above can also be used to prove $L^2 \to L^q$ estimates (see [48], [25], [26]). Assume $r = 1$ and thus assume that $\pi_L : \mathcal{C} \to T^*\Omega_L$ projects with Whitney folds. Then a stationary phase argument gives that

$$\|\mathcal{K}^{x_d y_d}\|_{L^1 \to L^\infty} \lesssim (1 + \lambda|x_d - y_d|)^{-1/2} \tag{4.10}$$

and interpolation with (4.5-6) yields $L^{q'} \to L^q$ estimates for $\mathcal{K}^{x_d y_d}$ and then $L^2 \to L^q$ bounds for T_λ. The result [25] is

$$\|T_\lambda\|_{L^2 \to L^q} \lesssim \lambda^{-d/q}, \qquad 4 \leq q \leq \infty. \tag{4.11}$$

The estimate (4.10) may be improved under the presence of some curvature assumption. Assume that the projection of the fold surface $S_1(\pi_L)$ to Ω_L is a submersion, then for each $x \in \Omega_L$ the projection of $S_1(\pi_L)$ to the fibers is a hypersurface Σ_x in $T^*_x\Omega_L$. Suppose that for every x this hypersurface has l nonvanishing principal curvatures (this assumption is reminiscent of the so-called cinematic curvature hypothesis in [46]). Then (4.10) can be replaced by

$$\|\mathcal{K}^{x_d y_d}\|_{L^1 \to L^\infty} \lesssim (1 + \lambda|x_d - y_d|)^{-(l+1)/2}$$

and (4.11) holds true for a larger range of exponents, namely

$$\|T_\lambda\|_{L^2 \to L^q} \lesssim \lambda^{-d/q}, \qquad \frac{2l+4}{l+1} \leq q \leq \infty. \tag{4.12}$$

The version of this estimate for Fourier integral operators [25], with $l = 1$, yields Oberlin's sharp $L^p \to L^q$ estimates [48] for the averaging operator (2.7) in three dimension (assuming that Γ is nondegenerate), as well as variable coefficient perturbations. It also yields sharp results for certain convolution operators associated to curves on the Heisenberg group ([65], see §7.3 below) and for estimates for restricted X-ray transforms associated to well curved line complexes in $\mathbb{R}^3$ ([25]).

In dimensions $d > 3$ the method yields $L^2 \to L^q$ bounds ([26]) which should be considered as partial results, since in most interesting cases the endpoint $L^p \to L^q$ estimates do not involve the exponent 2.

4.2.2. The analogy with the cinematic curvature hypothesis has been exploited by Oberlin, Smith and Sogge [52] to prove nontrivial $L^4 \to L^4_\alpha$ estimates for translation invariant operators associated to nondegenerate curves in $\mathbb{R}^3$. Here it is crucial to apply a square function estimate due to Bourgain [3] that he used in proving bounds for cone multipliers. The article [51] contains an interesting counterexample for the failure of $L^p \to L^p_{1/p-\varepsilon}$ estimates when $p < 4$.

4.2.3. Techniques of oscillatory integrals have been used by Oberlin [49] to obtain essentially sharp $L^p \to L^q$ estimates for the operator (2.7) in four dimension, see also [29] for a related argument for the restricted X-ray transform in four dimensions, in the rigid case (2.8).

4.2.4. More recently, a powerful combinatorial method was developed by Christ [10] who proved essentially sharp $L^p \to L^q$ estimates for the translation invariant model operator (2.7) in all dimensions (for nondegenerate Γ). $L^p \to L^q$ bounds for the X-ray transform in higher dimensions, in the model case (2.8), have been obtained by Burak-Erdoğan and Christ ([4], [5]); these papers contain even stronger mixed norm estimates. Christ's combinatorial method has been further developed by Tao and Wright [75] who obtained almost sharp $L^p \to L^q$ estimates for variable coefficient analogues.

5. Two-sided type two singularities

We consider again the operator (1.1) and discuss the proof of the following result mentioned in the last section.

5.1. Theorem [28]. *Suppose that both π_L and π_R are of type ≤ 2. Then for $\lambda \ge 1$*

$$\|T_\lambda\|_{L^2 \to L^2} = O(\lambda^{-(d-1)/2-1/4}).$$

A slightly weaker version of this result is due to Comech and Cuccagna [17] who obtained the bound $\|T^\lambda\| \le C_\varepsilon \lambda^{-(d-1)/2-1/4+\varepsilon}$ for $\varepsilon > 0$.

The proof of the endpoint estimate is based on various localizations and almost orthogonality arguments. As in §2 we start with localizing the determinant of $d\pi_{L/R}$ and its derivatives with respect to a kernal vector field. The form (5.2) below of this first decomposition can already be found in [15], [17].

We assume that the amplitude is supported near the origin and assume that (4.3.1) holds. Let $\Phi^{z'x'} = \Phi^{-1}_{x'z'}$, $\Phi^{x'z'} = \Phi^{-1}_{z'x'}$; then kernel vector fields for the projections π_L are given by

$$\begin{aligned} V_R &= \partial_{x_d} - \Phi_{x_d z'} \Phi^{z'x'} \partial_{x'}, \\ V_L &= \partial_{z_d} - \Phi_{z_d x'} \Phi^{x'z'} \partial_{z'}, \end{aligned} \tag{5.1}$$

respectively. Also let $h(x,z) = \det \Phi_{xz}$ and by the type two assumption we can assume that $|V_L^2 h|$, $|V_R^2 h|$ are bounded below. Emphasizing the amplitude in (1.1) we write $T_\lambda[\sigma]$ for the operator T_λ and will introduce various decompositions of the amplitude.

Let $\beta_0 \in C^\infty(\mathbb{R})$ be an even function supported in $(-1,1)$, and equal to one in $(-1/2, 1/2)$ and for $j \ge 1$ let $\beta_j(s) = \beta_0(2^{-j}s) - \beta_0(2^{-j+1}s)$. Denote by ℓ_0 that is the largest integer ℓ so that $2^\ell \le \lambda^{1/2}$ (we assume that λ is large). Define

$$\begin{aligned} \sigma_{j,k,l}(x,z) &= \sigma(x,z)\beta_1(2^l h(x,z))\beta_j(2^{l/2} V_R h(x,z))\beta_k(2^{l/2} V_L h(x,z)) \\ \sigma^0_{j,k,\ell_0}(x,z) &= \sigma(x,z)\beta_0(2^{\ell_0} h(x,z))\beta_j(2^{\ell_0/2} V_R h(x,z))\beta_k(2^{\ell_0/2} V_L h(x,z)); \end{aligned} \tag{5.2}$$

thus if $j,k > 0$ then $|h| \approx 2^{-l}$, $|V_L h| \approx 2^{k-l/2}$, $|V_R h| \approx 2^{j-l/2}$ on the support of $\sigma_{j,k,l}$.

It is not hard to see that the estimate of Theorem 5.1 follows from

5.2. Proposition. *We have the following bounds:*
(i) For $0 < l < \ell_0 = [\log_2(\sqrt{\lambda})]$

$$\|T_\lambda[\sigma_{j,k,l}]\|_{L^2\to L^2} \lesssim \lambda^{-(d-1)/2} \min\{2^{l/2}\lambda^{-1/2}; 2^{-(l+j+k)/2}\}. \tag{5.3}$$

(ii)

$$\|T_\lambda[\sigma^0_{j,k,\ell_0}]\|_{L^2\to L^2} \lesssim \lambda^{-(d-1)/2-1/4} 2^{-(j+k)/2}. \tag{5.4}$$

We shall only discuss (5.3) as (5.4) is proved similarly. In what follows j, k, l will be fixed and we shall discuss the main case where $0 < k \le j \le l/2$, $2^l \le \lambda^{1/2}$. As in the argument in §2 standard T^*T arguments do not work and further localizations and almost orthogonality arguments are needed. These are less straightforward in the higher dimensional situation considered here, and the amplitudes will be localized to nonisotropic boxes of various sides depending on the geometry of the kernel vector fields.

For $P = (x^0, z^0) \in \Omega_L \times \Omega_R$ let $a_P = \big(-\Phi^{x'z'}(P)\Phi_{z'x_d}(P), 1\big)$ and $b_P = (-\Phi^{z'x'}(P)\Phi_{x'z_d}(P), 1)$ so that $V_L = \langle a_P, \partial_x\rangle$, $V_R = \langle b_P, \partial_z\rangle$. Let $\pi^\perp_{a_P}$, $\pi^\perp_{b_P}$ be the orthogonal projections to the orthogonal complement of $\mathbb{R}a_P$ in $T_{x^0}\Omega_L$ and $\mathbb{R}b_P$ in $T_{z^0}\mathbb{R}^d$, respectively. Suppose $0 < \gamma_1 \le \gamma_2 \ll 1$ and $0 < \delta_1 \le \delta_2 \ll 1$ and let

$$B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$$

denote the box of all (x, z) for which $|\pi^\perp_{a_P}(x - x^0)| \le \gamma_1$, $|\langle x - x^0, a_P\rangle| \le \gamma_2$, $|\pi^\perp_{b_P}(z - z^0)| \le \delta_1$, $|\langle z - z^0, b_P\rangle| \le \delta_2$. We always assume

$$\gamma_1 \le \gamma_2,\ \ \delta_1 \le \delta_2 \tag{5.5}$$

We say that $\chi \in C_0^\infty$ is a *normalized cutoff function associated to* $B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ if it is supported in $B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ and satisfies the (natural) estimates

$$|(\pi_a^\perp \nabla_x)^{m_L}\langle a, \nabla_x\rangle^{n_L}(\pi_b^\perp\nabla_z)^{m_R}\langle b, \nabla_z\rangle^{n_R}\chi(x, z)| \le \gamma_1^{-m_L}\gamma_2^{-n_L}\delta_1^{-m_R}\delta_2^{-n_R}$$

whenever $m_L + n_L \le 10d$, $m_R + n_L \le 10d$.

We denote by $\mathfrak{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ the class of all normalized cutoff functions associated to $B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$.

Suppose that $(\gamma_1, \gamma_2, \delta_1, \delta_2) = (\varepsilon 2^{-l}, \varepsilon 2^{-j-l/2}, \varepsilon 2^{-l}, \varepsilon 2^{-k-l/2})$. It turns out that $h = \det \Phi_{xz}$ changes only by $O(\varepsilon 2^{-l})$ in the box $B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ but is in size comparable to 2^{-l}. This enables one to apply a TT^* argument and one obtains the correct bound $O(2^{l/2}\lambda^{-d/2})$ for the operator norm of $T_\lambda[\chi\sigma]$ assuming that $\chi \in \mathfrak{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ for some fixed P. This step had already been carried out by Comech and Cuccagna [17]. Let

$$\mathcal{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2) := \sup\{\|T_\lambda[\chi\sigma_{j,k,l}]\| :\ \chi \in \mathfrak{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2)\}$$

then, for $2^l \le \lambda^{1/2}$,

$$\sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) \lesssim 2^{l/2}\lambda^{-d/2}. \tag{5.6}$$

If one uses that $|V_R h| \approx 2^{j-l/2}$, $|V_L h| \approx 2^{k-l/2}$ one also gets

$$\sup_P \mathcal{A}_P(2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}) \lesssim 2^{-(l+j+k)/2}\lambda^{-(d-1)/2}. \tag{5.7}$$

Initially one obtains these estimates for boxes of size $(\varepsilon 2^{-l}, \varepsilon 2^{-j-l/2}, \varepsilon 2^{-l}, \varepsilon 2^{-k-l/2})$ but the ε may be removed since we can decompose any $B_P(\gamma_1, \gamma_2, \delta_1, \delta_2)$ into no more than $O(\varepsilon^{-2d})$ boxes of dimensions $(\varepsilon\gamma_1, \varepsilon\gamma_2, \varepsilon\delta_1, \varepsilon\delta_2)$. ¿From this one deduces

$$\mathcal{A}_P(\gamma_1, \gamma_2, \delta_1, \delta_2) \le C_\varepsilon \sup_Q \mathcal{A}_Q(\varepsilon\gamma_1, \varepsilon\gamma_2, \varepsilon\delta_1, \varepsilon\delta_2). \tag{5.8}$$

In order to put the localized pieces together we need some orthogonality arguments. For the sharp result we need to prove various inequalities of the form

$$\begin{aligned} &\sup_P \mathcal{A}_P(\gamma_{1,\text{large}}, \gamma_{2,\text{large}}, \delta_{1,\text{large}}, \delta_{2,\text{large}}) \\ &\qquad\qquad \lesssim \sup_Q \mathcal{A}_Q(\gamma_{1,\text{small}}, \gamma_{2,\text{small}}, \delta_{1,\text{small}}, \delta_{2,\text{small}}) + E(j,k,l) \end{aligned} \tag{5.9}$$

where the error term satisfies

$$E(j,k,l) \lesssim \lambda^{-(d-1)/2} \min\{2^{l/2}\lambda^{-1}, 2^{-(l+j+k)/2}\} \tag{5.10}$$

or a better estimate.

In the argument it is crucial that we assume

$$\min\{\frac{\gamma_{1,\text{small}}}{\gamma_{2,\text{small}}}, \frac{\delta_{1,\text{small}}}{\delta_{2,\text{small}}}\} \gtrsim \max\{\gamma_{2,\text{large}}, \delta_{2,\text{large}}\} \tag{5.11}$$

since from (5.11) one can see that the orientation of small boxes $B_Q(\gamma_{\text{small}}, \delta_{\text{small}})$ does not significantly change if Q varies in the large box $B_P(\gamma_{\text{large}}, \delta_{\text{large}})$.

5.3 Proposition. *Let $k \le j \le l/2, 2^l \le \lambda^{1/2}$. There is $\varepsilon > 0$ (chosen independently of k, j, l, λ) so that the inequality (5.9) holds with the choices of*

(i)

$$\begin{aligned} (\gamma_{\text{large}}, \delta_{\text{large}}) &= (\varepsilon 2^{j+k-l}, \varepsilon 2^{k-l/2}, \varepsilon 2^{j+k-l}, \varepsilon 2^{k-l/2}) \\ (\gamma_{\text{small}}, \delta_{\text{small}}) &= (2^{-l}, 2^{-j-l/2}, 2^{-l}, 2^{-k-l/2}), \end{aligned} \tag{5.12}$$

(ii)

$$\begin{aligned} (\gamma_{\text{large}}, \delta_{\text{large}}) &= (\varepsilon 2^{j-l/2}, \varepsilon 2^{j-l/2}, \varepsilon 2^{k-l/2}, \varepsilon 2^{k-l/2}) \\ (\gamma_{\text{small}}, \delta_{\text{small}}) &= (2^{j+k-l}, 2^{k-l/2}, 2^{j+k-l}, 2^{k-l/2}), \end{aligned} \tag{5.13}$$

(iii)

$$\begin{aligned} (\gamma_{\text{large}}, \delta_{\text{large}}) &= (\varepsilon, \varepsilon, \varepsilon, \varepsilon) \\ (\gamma_{\text{small}}, \delta_{\text{small}}) &= (2^{j-l/2}, 2^{j-l/2}, 2^{k-l/2}, 2^{k-l/2}). \end{aligned} \tag{5.14}$$

A combination of these estimates (with 5.8) yields the desired bound (5.3); here the outline of the argument is similar to the one given in §3. For each instance we are given a cutoff function $\zeta \in \mathfrak{A}_P(\gamma_{\text{large}}, \delta_{\text{large}})$ and we decompose

$$\zeta = \sum_{(X,Z)\in\mathbb{Z}^d\times\mathbb{Z}^d} \zeta_{XZ}$$

where the ζ_{XZ} is, up to a constant, a normalized cutoff function associated to a box of dimensions $(\gamma_{\text{small}}, \delta_{\text{small}})$; the various boxes have bounded overlap, and comparable orientation. More precisely if (P,Q) is a reference point in the big box $B_P(\gamma_{\text{large}}, \delta_{\text{large}})$ then each of the small boxes is comparable to a box defined by the conditions $|\pi^{\perp}_{a_P}(x-x_X)| \le \gamma_1$, $|\langle x-x_X, a_P\rangle| \le \gamma_2$, $|\pi^{\perp}_{b_P}(z-z_Z)| \le \delta_1$, $|\langle z-z_Z, b_P\rangle| \le \delta_2$.

If T_{XZ} denotes the operator $T_\lambda[\zeta_{XZ}\sigma_{j,k,l}]$ then in each case we have to show that for large N

$$\begin{aligned}\|T_\lambda[\zeta_{XZ}](T_\lambda[\zeta_{X'Z'}])^*\|_{L^2\to L^2} + \|(T_\lambda[\zeta_{XZ}])^* T_\lambda[\zeta_{X'Z'}]\|_{L^2\to L^2}\\ \lesssim \lambda^{1-d}\min\{2^l\lambda^{-2}, 2^{-l-j-k}\}(|X-X'|+|Z-Z'|)^{-N}\end{aligned}$$

if $|X-X'|+|Z-Z'| \gg 1$.

For the estimation in the case (5.12) it is crucial that in any fixed large box $V_L h$ does not change by more than $O(\varepsilon 2^{k-l/2})$ and thus is comparable to $2^{k-l/2}$ in the entire box; similarly $V_R h$ is comparable to $2^{j-l/2}$ in the entire box. For the orthogonality we use that $\Phi_{x'z'}$ is close to the identity. In the other extreme case (5.14) $V_R h$ and $V_L h$ change significantly in the direction of kernel fields and this can be exploited in the orthogonality argument. (5.13) is an intermediate case. This description is an oversimplification and we refer the reader to [28] for the detailed discussion of each case.

6. Geometrical conditions on families of curves

We illustrate some of the results mentioned before by relating conditions involving strong Morin singularities to various conditions on vector fields and their commutators.

6.1. Left and right commutator conditions and strong Morin singularities. projections We first look at an incidence relation $\mathcal{M}$ with canonical relation $\mathcal{C} = N^*\mathcal{M}$ as in (1.12) and assume $\ell = d-1$ so that $\dim \mathcal{M} = d+1$. As in §3 [67], we have two distinguished classes of vector fields on $\mathcal{M}$, namely vector fields of type $(1,0)$ which are also tangent to $\mathcal{M}\cap(\Omega_L\times 0)$ and vector fields of type $(0,1)$ which are tangent to $\mathcal{M}\cap(\{0\}\times\Omega_R)$. Note that for each point P the corresponding distinguished tangent spaces $T^{1,0}_P\mathcal{M}$ and $T^{0,1}_P\mathcal{M}$ are one-dimensional. If Φ is the $\mathbb{R}^{d-1}$-valued defining function for $\mathcal{M} = \{\Phi(x,y) = 0\}$ then a nonvanishing $(1,0)$ vector field X and a nonvanishing $(0,1)$ vector field Y are given by

$$X = \sum_{j=1}^d a_j(x,y)\frac{\partial}{\partial x_j}, \quad Y = \sum_{k=1}^d b_k(x,y)\frac{\partial}{\partial y_k} \tag{6.1}$$

where $(-1)^{j-1}a_j(x,y)$ is the determinant of the $(d-1)\times(d-1)$ matrix obtained from the $(d-1)\times d$ matrix Φ'_x by omitting the j^{th} column, and $(-1)^{k-1}b_j(x,y)$ is the determinant of the $(d-1)\times(d-1)$ matrix obtained from Φ'_y by omitting the j^{th} column.

The canonical relation $N^*\mathcal{M}'$ in (1.12) can be identified with a subbundle $T^{*,\perp}\mathcal{M}$ of $T^*\mathcal{M}$ whose fiber at $P\in\mathcal{M}$ is the ℓ-dimensional space of all linear functionals in $T^*_P\mathcal{M}$ which annihilate vectors in $T^{1,0}_P\mathcal{M}$ and vectors in $T^{0,1}_P\mathcal{M}$,

$$T^{*,\perp}_P\mathcal{M} = (T^{1,0}_P\mathcal{M}\oplus T^{0,1}_P\mathcal{M})^{\perp}.$$

Concretely, if $\imath:\mathcal{M}\to\Omega_L\times\Omega_R$ denotes the inclusion map and $\imath^*$ the pullback of $\imath$ (or restriction operator) acting on forms in $T^*(\mathfrak{X}\times\mathfrak{Y})$, then

$$T^{*,\perp}\mathcal{M} = \{(P,\imath^*_P\lambda) : (P,\lambda)\in\mathcal{C}\}$$

Finite type conditions can be formulated in terms of iterated commutators of $(1,0)$ and $(0,1)$ vector fields ([67]). Here they are used to characterize the situation of strong Morin singularities (*cf.* §2.4). Let $x^0\in\Omega_L$, let $\mathcal{M}_{x^0}=\{y\in\Omega_R:(x^0,y)\in\mathcal{M}\}$ and let

$$\mathfrak{N}_{L,x^0} := \pi_L^{-1}(\{x^0\}\times T^*_{x_0}\Omega_L) = \{(y,\lambda): y\in\mathcal{M}_{x^0},\ \lambda\in T^{*,\perp}_{(x^0,y)}\mathcal{M}\}.$$

Let π_{L,x_0} the restriction of π_L to $\mathfrak{N}_{L,x_0}$ as a map to $T^*_{x_0}\Omega_L$, then π_L has strong Morin singularities if for fixed x_0 the map π_{L,x_0} has Morin singularities.

Similarly, if $y^0\in\Omega_L$, let $\mathcal{M}^{y^0}=\{x\in\Omega_L:(x,y^0)\in\mathcal{M}\}$ then the adjoint operator $\mathcal{R}^*$ is an integral operator along the curves $\mathcal{M}^{y_0}$; now we define $\mathfrak{N}_{R,y^0}$ as the set of all (x,λ) where $x\in\mathcal{M}_{y^0}$, $\lambda\in T^{*,\perp}_{(x,y^0)}\mathcal{M}$, and $\pi_{R,y^0}:\mathfrak{N}_{R,y^0}\to T^*_{y^0}\Omega_R$ is the restriction of the map π_R.

Proposition.

(a) Let $x^0\in\Omega_L$ and $y^0\in\mathcal{M}_{x^0}$ and let $P=(x^0,y^0)$. The following statements are equivalent.

(i) *Near P, the only singularities of π_{L,x^0} are $S_{1_k,0}$ singularities, for $k\le d-2$.*
(ii) *The vectors $(\mathrm{ad}Y)^m X$, $m=1,\dots,d-1$ are linearly independent at P.*

(b) Let $y^0\in\Omega_R$ and $x^0\in\mathcal{M}^{y^0}$ and let $P=(x^0,y^0)$. The following statements are equivalent.

(i) *Near P, the only singularities of π_{R,y^0} are $S_{1_k,0}$ singularities, for $k\le d-2$.*
(ii) *The vectors $(\mathrm{ad}X)^m Y$, $m=1,\dots,d-1$ are linearly independent at P.*

It suffices to verify statement (a). There are coordinate systems $x=(x',x_d)$ near x^0, vanishing at x^0 and $y=(y',y_d)$ near y^0, vanishing at y^0 so that near P the manifold $\mathcal{M}$ is given by $y'=S(x,y_d)$ with

$$S(x,y_d)=x'+x_d g(y_d)+O(|x|^2)$$

where $g(0)=0$.

In these coordinates we compute the vector fields X and Y in (6.1) and find

$$\begin{aligned}(-1)^{d-1}a_j &= g_j(y_d)+O(|x|), \qquad j=1,\dots,d-1,\\ (-1)^{d-1}a_d &= 1+O(|x|),\end{aligned}$$

and

$$b_j = x_d \frac{\partial g_j}{\partial y_d} + O(|x'|^2 + |x'||x_d|), \qquad j = 1, \dots, d-1,$$
$$b_d = 1.$$

By induction one verifies that for $m = 1, 2, \dots$

$$(-1)^{d-1}(\mathrm{ad}Y)^m X = \sum_{j=1}^{d} v_j^m \frac{\partial}{\partial x_j} + \sum_{j=1}^{d} w_j^m \frac{\partial}{\partial y_j}$$

where

$$v_j^m = \frac{\partial^m g_j}{\partial y_d^m} + O(|x|), \qquad j = 1, \dots, d-1,$$
$$v_d^m = O(|x|)$$

and

$$w_j^m = -\frac{\partial^m g_j}{\partial y_d^m} + O(|x|), \qquad j = 1, \dots, d-1,$$
$$w_d^m = O(|x|)$$

Consequently we see that the linear independence of the vector fields $(\mathrm{ad}Y)^m X$ at P is equivalent with the linear independence of $\partial^m g_j/(\partial y_d^m)$ at $y_d = 0$.

Next, the map $\pi_{L,x^0} : \mathfrak{N}_{L,x^0} \to T^*_{x^0}\Omega_L$ is in the above coordinates given by

$$(y_d, \tau) \mapsto \tau \cdot S_x(0, y_d) = (\tau_1, \dots, \tau_{d-1}, \sum_{i=1}^{d-1} \tau_i g_i(y_d))$$

and from (2.3-5) we see that the statement (i) is also equivalent with the linear independence of the vectors $\partial^m g_j/(\partial y_d^m)$ at $y_d = 0$.

This proves the proposition.

6.2. Families of curves defined by exponentials of vector fields. Let now $\{\gamma_t(\cdot)\}_{t\in I}$ be a one-parameter family of diffeomorphisms of $\mathbb{R}^n$ which we can also consider as a family of parametrized curves,

$$t \mapsto \gamma_t(x) := \gamma(x, t).$$

We shall assume that x varies in an open set Ω, the open parameter interval I is a small neighborhood of 0 and that $\gamma_0 = Id$ and $\dot\gamma \neq 0$, where $\dot\gamma$ denotes $\frac{d}{dt}(\gamma_t)$. Thus for each x, $t \mapsto \gamma(x,t)$ defines a regular curve passing through x. As in the article by Christ, Nagel, Stein and Wainger [11], we may write such a family as

$$\gamma_t(x) := \gamma(x,t) = \exp(\sum_{i=1}^{N} t^i X_i)(x) \mod O(t^{N+1}) \tag{6.2}$$

for some vector fields $X_1, X_2, \dots$, and $N \in \mathbb{N}$. The generalized Radon transform is now defined by

$$\mathcal{R}f(x) = \int f(\gamma(x,t))\chi(t)dt$$

and incidence relation $\mathcal{M}$ is given by

$$\mathcal{M} = \{(x, \gamma(x,t)) : x \in \mathbb{R}^n, t \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R}^n. \tag{6.3}$$

Besides using the projections π_L and π_R, there are other ways of describing what it means for the family $\{\gamma_t(\cdot)\}$ to be maximally nondegenerate, in either a one- or two-sided fashion. One is given in terms the structure of the pullback map with respect to the diffeomorphisms $\gamma_t(\cdot)$, and another is given by the linear independence of certain linear combinations of the vector fields X_j and their iterated commutators. We formulate the conditions on the right, with the analogous conditions on the left being easily obtained by symmetry.

6.3. Strong Morin singularities and pull-back conditions. We are working with (6.2) and formulate the *pullback condition* $(P)_R$. Form the curve

$$\Gamma_R(x,t) = \frac{d}{ds}\big(\gamma_{s+t} \circ \gamma_t^{-1}(x)\big)\big|_{s=0}, \tag{6.4}$$

so that $\Gamma_R(x,\cdot) : \mathbb{R} \to T_x\mathbb{R}^n$. Let $\Gamma_R^{(\nu)}(x,t) = (\partial/\partial t)^\nu \Gamma_R(x,t)$ for $\nu = 0, 1, \dots$

Definition. The family of curves $\{\gamma(x,\cdot)\}_{x\in\Omega}$ satisfies condition $(P)_R$ at x if the vectors $\Gamma_R^{(\nu)}(x,0)$, $\nu = 0, \dots, n-1$ are linearly independent.

Let $\mathcal{M}$ be the incidence relation for our averaging operator.

Proposition. *Let $c_0 = (x_0, \xi_0, x_0, \eta_0) \in N^*\mathcal{M}'$. Then condition $(P)_R$ is satisfied at x_0 if and only if π_R has only $S^+_{1_k,0}$ singularities at c, with $k \le d-2$.*

To see this, note that $\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^n$ is the image of the immersion $(x,t) \mapsto (x, \gamma(x,t))$. Thus $(x, \xi; y, \eta)$ belongs to $N^*\mathcal{M}'$ if and only if $y = \gamma(x,t)$ for some $t \in \mathbb{R}$ and $(D\Phi_{(x,t)})^*(\xi, -\eta) = (0,0) \in T^*_{(x,t)}\mathbb{R}^{n+1}$. This yields

$$N^*\mathcal{M}' = \big\{(x, (D_x\gamma)^*(\eta); \gamma(x,t), \eta) : x \in \mathbb{R}^n, t \in \mathbb{R}, \eta \cdot \dot\gamma(x,t) = 0\big\}.$$

For each fixed t, let $y = \gamma_t(x)$, so that $x = \gamma_t^{-1}(y)$ and $\dot\gamma_t(x) = \dot\gamma_t(\gamma_t^{-1}(y)) = \frac{d}{ds}(\gamma_{t+s} \circ \gamma_t^{-1}(y)) = \Gamma_R(y,t)$. We thus have a parametrization of the canonical relation,

$$N^*\mathcal{M}' = \Big\{(\gamma_t^{-1}(y), (D_x\gamma)^*(\eta); y, \eta) : y \in \mathbb{R}^n, t \in \mathbb{R}, \eta \perp \Gamma_R(y,t))\Big\}, \tag{6.5}$$

which is favorable for analyzing the projection π_R. Indeed the equivalence of $(P)_R$ with the strong cusp condition follows immediately from the Lemma in §2.4.

6.4. Pullback and commutator conditions. The *bracket condition* $(B)_R$ for families of curves (6.2) states the linear independence of vector fields $\widehat{X}_i$, $i = 1, \dots, n$ where $\widehat{X}_1 = X_1$, $\widehat{X}_2 = X_2$ and for $k = 2, \dots, n$

$$\widehat{X}_k := X_k + \sum_{m=2}^{k-1} \sum_{I=(i_1,\dots,i_m)} a_{I,k}[X_{i_1}, [X_{i_2}, ..., [X_{i_{m-1}}, X_{i_m}] \dots] \tag{6.6}$$

with universal coefficients $a_{I,k}$ which can be computed from the coefficients of the Campbell-Hausdorff formula ([40,Ch.V.5], see also the exposition in [11]). In particular

$$\begin{aligned} \widehat{X}_1 &= X_1, \quad \widehat{X}_2 = X_2, \\ \widehat{X}_3 &= X_3 - \frac{1}{6}[X_1, X_2] \\ \widehat{X}_4 &= X_4 - \frac{1}{4}[X_1, X_3] + \frac{1}{24}[X_1, [X_1, X_2]] \\ \widehat{X}_5 &= X_5 - \frac{3}{10}[X_1, X_4] - \frac{1}{10}[X_2, X_3] + \frac{1}{15}[X_1, [X_1, X_3]] \\ &\quad + \frac{1}{30}[X_2, [X_1, X_2]] - \frac{1}{120}[X_1, [X_1, [X_1, X_2]]]. \end{aligned} \tag{6.7}$$

See [56], [26] for the computation of the vector fields $\widehat{X}_3$, $\widehat{X}_4$ and their relevance for folds and cusps.

Assuming $(P)_R$ we shall now show that $(B)_R$ holds and how one can determine the coefficients in (6.6). By Taylor's theorem in the s variable

$$\gamma_{s+t} \circ \gamma_t^{-1} = \exp\Big(\phi(t, X_1, \dots, X_n, \dots) + s\psi(t, X_1, \dots, X_n, \dots) + O(s^2)\Big), \tag{6.8}$$

and then, by an application of the Campbell-Hausdorff formula (essentially [26, Eq. (6.4)]), we can rewrite this as

$$\exp\Big(O(s^2)\Big) \circ exp\Big(\phi + s\psi\Big).$$

¿From this it follows that

$$\Gamma_R(x, t) = \psi(t, X_1, \dots, X_n, \dots)$$

and thus condition $(P)_R$ becomes the linear independence of $\psi, \psi', ..., \psi^{(n-1)}$. We will work modulo $O(s^2) + O(st^{n+1})$ and so can assume that there are only n vector fields, $X_1, \dots, X_n$. Compute

$$\begin{aligned} \gamma_{s+t} \circ \gamma_t^{-1} =& \exp(\sum_{i=1}^{n}(s+t)^i X_i) \circ \exp(-\sum_{i=1}^{n} t^i X_i) \\ =& \exp((\sum t^i X_i + s \sum i t^{i-1} X_i) + O(s^2)) \circ \exp(-\sum t^i X_i) \\ =& \exp(\sum_{i=1}^{n}(t+is)t^{i-1} X_i) \circ \exp(-\sum_{i=1}^{n} t^i X_i) \mod O(s^2) \\ =& \exp(B) \circ exp(A) \end{aligned}$$

with $A = -\sum_{i=1}^{n} t^i X_i$ and $B = \sum_{i=1}^{n}(t+is)t^{i-1} X_i$. Now, the explicit Campbell-

Hausdorff formula (see [40]) can be written as

$$\exp(B)\circ\exp(A)$$
$$= \exp\Big(A+B+\frac{1}{2}[A,B]+\sum_{m=3}^{\infty}\sum_{\substack{I=(i_1,\dots,i_m)\\ \in\{1,2\}^m}} c_I \mathrm{ad}(C_{i_1})...\mathrm{ad}(C_{i_{m-1}})(C_{i_m})\Big)$$

(6.9)
$$= \exp\Big(A+B+\frac{1}{2}[A,B]+\sum_{m=3}^{\infty}\sum_{\substack{J=(j_1,\dots,j_{m-2})\\ \in\{1,2\}^{m-2}}} \tilde{c}_J \mathrm{ad}(C_{j_1})...\mathrm{ad}(C_{j_{m-2}})([A,B])\Big)$$

where $C_1 = A, C_2 = B$ and

$$\tilde{c}_J = c_{(J,1,2)} - c_{(J,2,1)}.$$

The first few terms are given by

$$\begin{aligned} A+B+&\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]] \\ &-\frac{1}{48}[A,[B,[A,B]]]-\frac{1}{48}[B,[A,[A,B]]]\dots . \end{aligned} \tag{6.10}$$

For notational convenience, we let the sum start at $m=2$ instead of $m=3$ and set $\tilde{c}_\emptyset = 1/2$, and for the higher coefficients we get $\tilde{c}_{(1)} = -\tilde{c}_{(2)} = 1/12$ and $\tilde{c}_{(1,2)} = \tilde{c}_{(2,1)} = -1/48$. These are enough to calculate the coefficients in $(B)_R$ in dimensions less than or equal to five which is the situation corresponding to at most $S^+_{1,1,1,0}$ (strong swallowtail) singularities.

Returning to $(P)_R$, since we have $C_1 = A, C_2 = B$, we can use the Kronecker delta notation to write $C_j = (-1)^j\sum_{i=1}^n (t+\delta_{j2}is)t^{i-1}X_i$. Now

$$\gamma_{s+t}\circ\gamma_t^{-1} = \exp\Big(A+B+\sum_{m=2}^{\infty}\sum_{\substack{J=(j_1,\dots,j_{m-2})\\ \in\{1,2\}^{m-2}}} \tilde{c}_J \mathrm{ad}(C_{j_1})...\mathrm{ad}(C_{j_{m-2}})([A,B])\Big)+O(s^2)$$

which modulo $O(s^2)$ is equal to

$$\begin{aligned} \exp\Big(&-\sum_{i=1}^n t^iX_i+\sum_{i=1}^n (t^i+ist^{i-1})X_i \\ &-\sum_{m=2}^{\infty}\sum_{\substack{J=(j_1,\dots,j_{m-2})\\ \in\{1,2\}^{m-2}}} \tilde{c}_J \mathrm{ad}\big((-1)^{j_1}\sum_{i_1}(t+\delta_{j_12}i_1s)t^{i_1-1}X_{i_1}\big)... \\ &\dots\quad \mathrm{ad}\big((-1)^{j_{m-2}}\sum_{i_{m-2}}(t+\delta_{j_{m-2}2}i_{m-2}s)t^{i_{m-2}-1}X_{i_{m-2}}\big) \\ &\cdot\Big(\Big[\sum_{i_{m-1}=1}^n -t^{i_{m-1}-1}X_{i_{m-1}},\sum_{i_m=1}^n (t+i_ms)t^{i_m-1}X_{i_m}\Big]\Big)\quad\Big]\Big), \end{aligned}$$

which, again modulo $O(s^2)$, is equal to

$$\exp\Big(\phi(t,X_1,\dots,X_n)+s\Big[\sum_{i=1}^n it^{i-1}X_i-\sum_{m=2}^\infty\Big(\sum_{\substack{J\\ \in\{1,2\}^{m-2}}}(-1)^{\sum_{l=1}^{m-2}j_l}\tilde c_J\Big)\times$$
$$\sum_{i_1,\dots,i_{m-2}}\sum_{i_{m-1}<i_m}(i_m-i_{m-1})\cdot\mathrm{ad}(X_{i_1})\cdot\dots$$
$$\dots\cdot\mathrm{ad}(X_{i_{m-2}})\big([X_{i_{m-1}},X_{i_m}]\big)t^{-1+\sum_{l=1}^m i_l}\Big]\Big).$$

¿From this we obtain

$$\begin{aligned}\Gamma_R(x,t)=&\sum_{i=1}^n it^{i-1}X_i-\sum_{m=2}^\infty\Big(\sum_{\substack{J\\ \in\{1,2\}^{m-2}}}(-1)^{\sum_{l=1}^{m-2}j_l}\tilde c_J\Big)\times\\ &\sum_{i_1,\dots,i_{m-2}}\sum_{i_{m-1}<i_m}(i_m-i_{m-1})\cdot\mathrm{ad}(X_{i_1})\cdot\dots\\ &\dots\cdot\mathrm{ad}(X_{i_{m-2}})\big([X_{i_{m-1}},X_{i_m}]\big)t^{-1+\sum_{l=1}^m i_l}\\ :=&\sum_{i=1}^n it^{i-1}\widehat X_i.\end{aligned}$$

Since the $\tilde c_J$'s are known (*cf.* [40,Ch.V.5], [77]) this allows one to compute the $\widehat X_i$'s and this shows that the condition $(P)_R$ is equivalent with a bracket condition $(B)_R$ for some coefficients $a_{I,k}$.

To illustrate this, we restrict to $n\le 5$ to get a manageable expression we work mod $O(t^5)$ and use (6.10); the expression for $\Gamma_R(x,t)$ becomes then

$$\begin{aligned}&\sum_i it^{i-1}X_i-\frac12\sum_{i_1<i_2}(i_2-i_1)[X_{i_1},X_{i_2}]t^{i_1+i_2-1}\\ &+\frac16\sum_{i_1}\sum_{i_2<i_3}(i_3-i_2)\cdot[X_{i_1},[X_{i_2},X_{i_3}]]t^{i_1+i_2+i_3-1}\\ &-\frac1{24}\sum_{i_1,i_2}\sum_{i_3<i_4}(i_4-i_3)\cdot[X_{i_1},[X_{i_2},[X_{i_3},X_{i_4}]]]t^{i_1+i_2+i_3+i_4-1}\end{aligned}$$

which becomes

$$\begin{aligned}&X_1+2tX_2+3t^2X_3+4t^3X_4+5t^4X_5\\ &-\frac12[X_1,X_2]t^2-[X_1,X_3]t^3-\frac32[X_1,X_4]t^4-\frac12[X_2,X_3]t^4\\ &+\frac16[X_1,[X_1,X_2]]t^3+\frac13[X_1,[X_1,X_3]]t^4+\frac16[X_2,[X_1,X_2]]t^4\\ &-\frac1{24}[X_1,[X_1,[X_1,X_2]]]t^4\\ =&\widehat X_1+2t\widehat X_2+3t^2\widehat X_3+4t^3\widehat X_4+5t^4\widehat X_5\end{aligned}$$

where the $\widehat X_i$ are given in (6.7). Thus condition $(B)_R$ in dimension $n\le 5$ is the linear independence of the $\widehat X_i$ for $0\le i\le n-1$.

6.5. Curves on some nilpotent groups.

Let G be an n dimensional nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\gamma : \mathbb{R} \to G$ be a smooth curve and define

$$G_R(t) = (DR_{\gamma(t)})^{-1}(\gamma'(t)),$$

where DR_g denotes the differential of right-translation by $g \in G$. Note that $G_R : \mathbb{R} \to T_0G = \mathfrak{g}$ defines a curve in the Lie algebra $\mathfrak{g}$.

Lemma. *The pullback condition $(P)_R$ for the family of curves $t \mapsto x \cdot \gamma(t)^{-1}$ is satisfied if and only if the vectors $G_R(t), G_R'(t), \dots, G_R^{(n-1)}(t)$ are linearly independent everywhere.*

To prove this, compute

$$\begin{aligned}
\Gamma_R(x,t) =& \frac{d}{ds}\Big(\gamma_{s+t}(x \cdot \gamma(t))\Big)|_{s=0} \\
=& \frac{d}{ds}(x \cdot \gamma(t) \cdot \gamma(s+t)^{-1})|_{s=0} \\
=& -x \cdot \gamma(t) \cdot \gamma^{-1}(t) \cdot \gamma'(t) \cdot \gamma^{-1}(t) \\
=& -x \cdot \gamma'(t) \cdot \gamma^{-1}(t) \\
=& -x \cdot \big(DR^{-1}_{\gamma(t)}(\gamma'(t)\big) = -x \cdot G_R(t),
\end{aligned}$$

from which the equivalence is obvious.

The condition that $G_R, \dots G_R^{(n-1)}$ be linearly independent came up in work of Secco [65], who proved under this condition the sharp $L^{3/2} \to L^2$ boundedness result for the convolution operator

$$\mathcal{R}f(x) = \int f(x \cdot \gamma(t)^{-1})\chi(t)dt$$

on the Heisenberg group $\mathbb{H}$ (thus $n = 3$). For the model family of cubics $\gamma(t) = (t, t^2, \alpha t^3)$, one easily computes that $G_R(t) = (1, 2t, (3\alpha + \frac{1}{6})t^2)$, so that her condition is satisfied if and only if $\alpha \neq -\frac{1}{6}$.

We further illustrate the Lemma above by analyzing a two-parameter family of quartics on a four-dimensional, three-step nilpotent group, which we denote $\mathbb{M}$, due to its relation with the Mizohata operator. The Lie algebra $\mathfrak{m}$ of $\mathbb{M}$ is spanned by $Y_j, 1 \leq j \leq 4$, satisfying

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = Y_4,$$

with all other commutators equal zero. Thus, Y_1 and Y_2 satisfy the same commutator relations as real and imaginary parts of the operator $\frac{\partial}{\partial x} + i\frac{x^2}{2}\frac{\partial}{\partial y}$, *cf.* [45].

The group multiplication is given by

$$\begin{aligned}
(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (& x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1), \\
& x_4 + y_4 + \frac{1}{2}(x_1y_3 - x_3y_1) + \frac{1}{12}(x_1 - y_1)(x_1y_2 - x_2y_1)\big).
\end{aligned}$$

For $\alpha, \beta \in \mathbb{R}$, we define curves $\gamma(t) = (t, s^2, \alpha t^3, \beta t^4)$ and ask for which values of the parameters the vectors $G_R(t), \dots, G_R'''(t)$ are linearly independent.

We derive this in two different ways: first by the above Lemma and then using the bracket condition. To form G_R, we first calculate the derivative of $R_y(x) = x \cdot y$, acting on a tangent vector $X = (X_1, X_2, X_3, X_4) \in \mathfrak{m} = T_0\mathcal{M}$:

$$DR_y(X) = (X_1, X_2, X_3 + \frac{1}{2}y_2 X_1 - \frac{1}{2}y_1 X_2, X_4 + \frac{6y_3 - y_1 y_2}{12}X_1 + \frac{y_1^2}{12}X_2 - \frac{y_1}{2}X_3).$$

Computing the inverse of this and applying it for $y = \gamma(t) = (t, t^2, \alpha t^3, \beta t^4)$, one calculates

$$\begin{aligned} G_R(s) &= \Big(DR_{\gamma(t)}\Big)^{-1}(\dot\gamma(t)) = \Big(DR_{\gamma(t)}\Big)^{-1}(1, 2t, 3\alpha t^2, 4\beta t^3) \\ &= (1, 2t, (\frac{6\alpha + 1}{2})t^2, (\alpha + 4\beta + \frac{1}{6})t^3). \end{aligned}$$

Thus, $G_R^{(i)}$, $i = 0, \dots, 3$ are linearly independent if and only if $\alpha + \frac{1}{6} \neq 0$ and $\alpha + 4\beta + \frac{1}{6} \neq 0$.

Alternatively we may quickly rederive by using the bracket condition $(B)_R$ for $n = 4$. We have

$$\gamma(x,t) = x \cdot (t, t^2, \alpha t^3, \beta t^4)^{-1} = \exp\big(t(-Y_1) + t^2(-Y_2) + t^3(-\alpha Y_3) + t^4(-\beta Y_4)\big)(x),$$

where $Y_1, \dots, Y_4$ is the above basis for $\mathfrak{m}$, so we have the representation as in (1.1) with

$$X_1 = -Y_1, \quad X_2 = -Y_2, \quad X_3 = -\alpha Y_3, \quad X_4 = -\beta Y_4$$

and thus condition $(B)_R$ says that the vector fields

$$-Y_1,\ -Y_2,\ -\alpha Y_3 - \frac{1}{6}[-Y_1, -Y_2],\ -\beta Y_4 - \frac{1}{4}[-Y_1, -\alpha Y_3] + \frac{1}{24}[-Y_1, [-Y_1, -Y_2]]$$

are linearly independent, which is equivalent with the linear independence of the vector fields Y_1, Y_2, $(\alpha + \frac{1}{6})Y_3$ and $(\frac{\alpha}{4} + \beta + \frac{1}{24})Y_4$.

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