THE CALDERÓN PROBLEM FOR CONORMAL POTENTIALS, I: GLOBAL UNIQUENESS AND RECONSTRUCTION

Allan Greenleaf, Matti Lassas and Gunther Uhlmann

1. Introduction

The goal of this paper is to establish global uniqueness and obtain reconstruction, in dimensions $n \geq 3$, for the Calderón problem in the class of potentials conormal to a smooth submanifold H in $\mathbb{R}^n$. In the case of hypersurfaces, the potentials considered here may have any singularity weaker than that of the delta function δ_H on the hypersurface H; in general, these potentials correspond to conductivities which are in $C^{1+\epsilon}$ and thus fail to be covered by previously known results.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $H \subset \Omega$ a smooth submanifold of codimension k, and $q \in I^\mu(H)$ a real conormal distribution of order μ with $\mu < 1-k$. Thus, if $H = \{x : F_j(x) = 0, 1 \leq j \leq k\}$ is a local representation of H by means of defining functions with $\{\nabla F_j : 1 \leq j \leq k\}$ linearly independent on H, then locally $q(x)$ has the Fourier integral representation

$$q(x) = \int_{\mathbb{R}^k} e^{i \sum_j F_j(x)\cdot\theta_j} a(x,\theta) d\theta, \quad a \in S^\mu_{1,0}, \tag{1.1}$$

where $S^\mu_{1,0}$ denotes the standard class of symbols of order μ and type $(1,0)$ on $\mathbb{R}^n \times (\mathbb{R}^k \backslash 0)$. (Here, we use the order convention of [12] rather than [16].) A general element $q \in I^\mu(H)$ is a locally finite sum of such expressions. We assume throughout that $\mathrm{supp}(q)$ is compact in Ω. If $-k < \mu < 0$, then q satisfies $|q(x)| \leq C \cdot \mathrm{dist}\,(x,H)^{-k-\mu}$, so that $q \in L^{\frac{k}{k+\mu}-\epsilon}(\Omega), \forall \epsilon > 0$, and no better in general; in particular, a general element of $I^\mu(H)$ is unbounded. For comparison, surface measure $\delta_H \in I^0(H)$ and, in the hypersurface case, a Heaviside discontinuity across H belongs to $I^{-1}(H)$.

Rather than working with the Dirichlet-to-Neumann map, Λ_q, we state our main results in terms of the Cauchy data, $\mathcal{CD}_q$, of sufficiently regular solutions of the Schrödinger equation

$$(\Delta + q(x))u(x) = 0 \text{ on } \Omega. \tag{1.2}$$

This is more flexible, since $\mathcal{CD}_q$ can be defined for potentials q for which Λ_q is either not defined (for example, if $\lambda = 0$ is a Dirichlet eigenvalue) or is not known to be defined (due to the low regularity of $q(x)$); it is perhaps more natural as well.

This work was supported in part by National Science Foundation grants DMS-9877101, DMS-0070488 and Finnish Academy project 172434. The third author was also supported by a John Simon Guggenheim Fellowship. The second and third author are grateful for the hospitality of the Mathematical Sciences Research Institute in Berkeley, CA.

It will be convenient to write $\mu = \nu - k$. Assume that $\nu_0(k) < \nu < 1$, where $\nu_0(k) \stackrel{def}{=} max(\frac{2}{3}, 1 - \frac{k}{4})$. Fix p and r satisfying

$$(1.3) \qquad 2 \leq r < \frac{k}{2(1-\nu)} \stackrel{def}{=} r_0(k,\nu) < p_0(k,\nu) \stackrel{def}{=} \frac{2k}{k-\nu} < p < \infty.$$

(If $\nu \leq \nu_0(k)$, just pick p and r for some $\nu' > \nu_0(k)$.) Fixing a smooth function $\psi \in C^\infty(\mathbb{R}^n), \psi \equiv 1$ near $\partial\Omega$ and $int\{\psi = 0\} \cap \Omega \neq \phi$, define the norm

$$(1.4) \qquad ||f||_{X^{p,r}} = ||f||_{L^p(\Omega)} + ||\Delta f||_{L^{p'}(\Omega)} + ||\psi f||_{W^{2,r}(\Omega)},$$

where p' is the dual exponent to p and $W^{2,r}$ is the standard Sobolev space of $f \in \mathcal{D}'(\Omega)$ having two derivatives in $L^r(\Omega)$. Set

$$(1.5) \qquad X^{p,r}(\Omega) = \{f \in \mathcal{D}'(\Omega) : ||f||_{X^{p,r}} < \infty\}$$

and note that the Schrödinger operator $\Delta + q$ maps $X^{p,r}(\Omega) \to L^{p'}(\Omega)$ continuously. We denote throughout this paper by n the unit outer normal to Ω.

Definition. *For a potential $q \in I^\mu(H)$ with $H \cap supp(q) \subset int\{\psi = 0\}$, the Cauchy data of the Schrödinger operator $\Delta + q$ relative to $X^{p,r}(\Omega)$ is*

$$(1.6) \qquad \mathcal{CD}_q = \mathcal{CD}_q^{p,r} = \Big\{(u|_{\partial\Omega}, \frac{\partial u}{\partial n}|_{\partial\Omega}) : u \in X^{p,r}(\Omega), (\Delta + q)u = 0 \ on\ \Omega\Big\}.$$

By Sobolev embedding, $\mathcal{CD}_q$ is a subspace of $W^{2-\frac{1}{r},r}(\partial\Omega) \times W^{1-\frac{1}{r},r}(\partial\Omega)$. Observe that if the Dirichlet-to-Neumann map Λ_q *is* defined (on $W^{2-\frac{1}{r},r}(\partial\Omega)$, say), then $\mathcal{CD}_q$ is simply its graph. We will construct certain nontrivial exponentially growing solutions $u \in X^{p,r}(\Omega)$, so that, for the potentials considered, $\mathcal{CD}_q$ is in fact nontrivial. We can now state our first result.

Theorem 1. *Suppose that for $j = 1, 2$, $H_j \subset\subset \Omega$ are submanifolds of codimension k_j. Suppose further that $q_j \in I^{\mu_j}(H_j)$ are real potentials with $\nu_0(k_j) - k_j < \mu_j < 1 - k_j$ and $supp(q_j) \subset\subset \Omega$. Let p, r satisfy $2 \leq r < min(r_0(k_1,\nu_1), r_0(k_2,\nu_2))$ and $max(p_0(k_1,\nu_1), p_0(k_2,\nu_2)) < p < \infty$ and suppose that $\psi \equiv 0$ on a neighborhood of $H_1 \cup H_2$. Then $\mathcal{CD}_{q_1} = \mathcal{CD}_{q_2}$ relative to $X^{p,r}(\Omega)$ implies that $q_1 = q_2$ on Ω.*

We also show that under the same assumptions as in Theorem 1 for the potential we have a reconstruction procedure, that is we can reconstruct q from $\mathcal{CD}_q$ (see Theorem 2 in section 3 for more details).

Global uniqueness was established in [32] for $n \geq 3$ (for smooth potentials) and [26] for $n = 2$; for $n \geq 3$ this was extended to $q \in L^\infty$ in [27]. The regularity was further lowered to $q \in L^{\frac{n}{2}}$ in unpublished work of R. Lavine and A. Nachman and to potentials of small norm in the Fefferman-Phong class in [4]. Note that for $-\frac{n-2}{n}k \leq \mu < 0, k < \frac{n}{2}$, a general element of $I^\mu(H)$ fails to be in $L^{\frac{n}{2}}(\Omega)$.

The isotropic conductivity problem, where one considers the Dirichlet-to-Neumann map for $L_\gamma = \nabla(\gamma \cdot \nabla)$, can be reduced to the Schrödinger problem via the substitution $q = -\frac{\Delta(\gamma^{\frac{1}{2}})}{\gamma^{\frac{1}{2}}}$, and thus the analogue of the Theorem holds for conductivities

$\gamma_j \in I^{-k-1-\epsilon}(H_j) \hookrightarrow C^{1+\epsilon}(\overline{\Omega}), \forall 0 < \epsilon < 1$. Currently, the best general global uniqueness result known for $n \geq 3$ is for $\gamma \in C^{\frac{3}{2}}$, proved in [28], building on [2] and using the general argument of [32], while the best known result for $n = 2$ is $\gamma \in W^{1,p}(\Omega)$, $p > 2$, proved in [3] using the $\overline{\partial}$ technique of [1,25,26]. Global uniqueness for piecewise-analytic conductivities was proven [20], and special types of jump discontinuities were treated in [17].

Here, we will follow the general argument of [32], although employing a different integral identity so as to avoid difficulties when applying Green's Theorem. It is this identity that makes $X^{p,r}$ a convenient space for the problem; indeed, both sides of $\int_\Omega \Delta u \cdot v - u \cdot \Delta v dx = \int_{\partial\Omega} \partial_n u \cdot v - u \cdot \partial_n v d\sigma$ are continuous with respect to $|| \cdot ||_{X^{p,r}}$ and thus Green's Theorem holds for $u, v \in X^{p,r}(\Omega)$.

We now start the proof of Thm. 1, so as to motivate the technicalities that follow.

Given a submanifold H of codimension k and a potential $q \in I^\mu(H)$ with $\mu < 1-k$, we will construct exponentially growing solutions of (1.2) belonging to $X^{p,r}(\Omega)$, of the form $v(x) = e^{\rho\cdot x}(1+\psi(x,\rho))$, with $\rho \in \mathbb{C}^n$ satisfying $\rho\cdot\rho = 0$. Let $v_1 \in X^{p,r}(\Omega)$ be a solution of $(\Delta + q_1)v_1 = 0$. By the hypothesis of Thm. 1, there is a solution $v_2 \in X^{p,r}(\Omega)$ of $(\Delta + q_2)v_2 = 0$ with

$$v_2|_{\partial\Omega} = v_1|_{\partial\Omega} \quad \text{and} \quad \frac{\partial v_2}{\partial n}|_{\partial\Omega} = \frac{\partial v_1}{\partial n}|_{\partial\Omega}. \tag{1.7}$$

Let $w_2 \in X^{p,r}(\Omega)$ be any other solution to $(\Delta + q_2)w_2 = 0$. Then,

$$(\Delta + q_2)(v_1 - v_2) = (\Delta + q_2)v_1 = (\Delta + q_1 + (q_2 - q_1))v_1 = (q_2 - q_1)v_1,$$

so that

$$\begin{aligned}
\int_\Omega (q_2 - q_1)v_1 w_2 dx &= \int_\Omega (\Delta + q_2)(v_1 - v_2) \cdot w_2 dx \\
&= \int_\Omega (\Delta + q_2)(v_1 - v_2) \cdot w_2 - (v_1 - v_2) \cdot (\Delta + q_2) w_2 dx \\
&= \int_\Omega \Delta(v_1 - v_2) \cdot w_2 - (v_1 - v_2) \cdot \Delta w_2 dx \\
&= \int_{\partial\Omega} \frac{\partial}{\partial n}(v_1 - v_2) \cdot w_2 - (v_1 - v_2) \cdot \frac{\partial}{\partial n} w_2 d\sigma \\
&= 0,
\end{aligned} \tag{1.8}$$

where the application of Green's Theorem is valid since $v_1 - v_2$ and $w_2 \in X^{p,r}$ and the last equality holds by (1.7). If we carry this out for the solutions v_1 and w_2 constructed below for complex frequencies ρ_1 and ρ_2 satisfying $\rho_1 + \rho_2 = -i\xi$, with $\xi \in \mathbb{R}^n \backslash 0$, then we have, as in [32],

$$\begin{aligned}
0 &= \int_\Omega (q_1 - q_2) e^{(\rho_1+\rho_2)\cdot x} \Big(1 + \psi_1(x,\rho_1)\Big)\Big(1 + \psi_2(x,\rho_2)\Big) dx \\
&= \widehat{(q_1 - q_2)}(\xi) + \int_\Omega e^{-i\xi\cdot x}(q_1 - q_2)(\psi_1 + \psi_2 + \psi_1\psi_2) dx
\end{aligned} \tag{1.9}$$

If one can do this for pairs (ρ_1, ρ_2) with $|\rho_j| \to \infty$ and show that the last integral $\to 0$ as $|\rho| \to \infty$, then $\hat{q_1}(\xi) = \hat{q_2}(\xi)$; doing this for all $\xi \in \mathbb{R}^n$ will finish the proof of Thm. 1.

As is well known, $v(x) = e^{\rho \cdot x}(1 + \psi(x))$ is a solution of the Schrödinger equation iff ψ is a solution of

$$(\Delta_\rho + q)\psi = -q(x) \text{ where } \Delta_\rho = \Delta + 2\rho \cdot \nabla. \tag{1.10}$$

We will show in Prop. 2.6 that (1.10) is uniquely solvable, with some decay in $|\rho|$, in a Banach space of finite-regularity conormal distributions associated with H, yielding exponentially growing solutions $v_j \in X^{p,r}$ to (1.2) which allow the argument above to be carried out. In §3, this result is extended to a hybrid global space, and this is applied to obtain reconstruction of the potential from the Cauchy data, following the general argument of [25]. Finally, in §4 we show that uniqueness can fail in a weak formulation of the problem for potentials with very strong singularities on a hypersurface, with blow-up rates corresponding to those of distributions conormal of order greater than 1 for H.

We would like to thank Steve McDowall for valuable discussions, and the referee for pointing out an error in the original version of this paper.

2. Uniqueness for conormal potentials

As described in the Introduction, to prove Thm.1, it suffices to construct exponentially growing solutions to (1.2) of the form $v(x) = e^{\rho \cdot x}(1 + \psi(x, \rho))$ for $\rho \cdot \rho = 0, |\rho| \to \infty$ so that the second integral in (1.9) tends to 0 as $|\rho| \to \infty$. To do this for potentials $q \in I^\mu(H)$, the standard space of (infinite-regularity) conormal distributions of order μ associated with the codimension k submanifold H, we will also need to formulate Banach spaces of finite-regularity conormal distributions in $\mathbb{R}^n$ associated with H. Rather than working in unnecessary generality, we will restrict ourselves to the spaces needed here; unlike [23],[22], where several other types of finite-regularity conormal spaces are defined using iterated regularity with respect to Lie algebras of tangent vector fields, we impose the finite-regularity assumption directly on the symbols in the oscillatory representations of the distributions, using symbol classes modelled on those of [33].

For $l \in \mathbb{R}$, let C^l_* denote the Zygmund space of order l on $\mathbb{R}^n$ [33]. Thus, if $\psi_0(D) + \sum_{i=1}^\infty \psi_i(D) = I$ is a Littlewood-Paley decomposition, with $\psi_i(\xi) = \psi_1(\frac{\xi}{2^{i-1}}), i \geq 1$, then

$$||u||_{C^l_*} = \sup_i 2^{li} ||\psi_i(D)u||_{L^\infty(\mathbb{R}^n)}.$$

Recall that if $l \geq 0, l \notin \mathbb{Z}$, then $C^l_* = C^{[l], l-[l]}(\mathbb{R}^n)$.

Now fix an order $m \in \mathbb{R}$, an $N \in \mathbb{N}$, and a sequence $\vec{\delta} = (\delta_1, \delta_2, ..., \delta_N)$ of numbers $0 \leq \delta_j \leq 1, 1 \leq j \leq N$. For any multi-index $\alpha \in \mathbb{Z}^k_+$, let $\delta(\alpha) = \sum_{j=1}^{|\alpha|} \delta_j$, setting $\delta(0) = 0$ for convenience, and $|\vec{\delta}| = \sum_{j=1}^N \delta_j$.

Definition 2.1. *(i)*

$$C^l_* S^{m,\vec{\delta}}(\mathbb{R}^n \times \mathbb{R}^k) = \left\{ a(x,\theta) : ||\partial^\alpha_\theta a(\cdot, \theta)||_{C^l_*} \leq C_\alpha (1 + |\theta|)^{m - \delta(\alpha)}, \forall |\alpha| \leq N \right\}$$

and

$$||a||_{C^l_* S^{m,\vec{\delta}}} = \max_{0\le\alpha\le N} (1+|\theta|)^{-m+\delta(\alpha)} ||\partial^\alpha_\theta a(\cdot,\theta)||_{C^l_*}.$$

(ii) If H is a smooth codimension k submanifold with compact closure, then $C^l_* I^{m,\vec{\delta}}(H)$ is the space of locally finite sums of distributions of the form $u(x) = \int_{\mathbb{R}^k} e^{iF(x)\cdot\theta} a(x,\theta)d\theta$ with $a \in C^l_* S^{m,\vec{\delta}}$, where $F(x) = (F_1(x),\dots,F_k(x))$ are local defining functions for H.

Remarks. (1) $C^l_* S^{m,\vec{\delta}}$ is a Banach space with respect to the norm defined in (i), and $C^l_* I^{m,\vec{\delta}}(H)$ inherits this structure.
(2) $\partial_\theta : C^l_* S^{m,\vec{\delta}} \to C^l_* S^{m-\delta_1,\vec{\delta}'}$, with $\vec{\delta}' = (\delta_2,\delta_3,...,\delta_N)$ and $\partial_x : C^l_* S^{m,\vec{\delta}} \to C^{l-1}_* S^{m,\vec{\delta}}$ continuously.
(3) $C^l_* I^{m,\vec{\delta}}(H)$ is well-defined, since changing the defining functions of H locally corresponds to a change of variable in x, which leaves the symbol class invariant.
(4) The usual (infinite-regularity) conormal space $I^m(H)$ has a continuous inclusion with respect to its Fréchet structure: $I^m(H) \hookrightarrow C^l_* I^{m,\vec{\delta}}(H)$ for any l, N and $\vec{\delta}$.

Proposition 2.2. *Let $H \subset \Omega \subset\subset \mathbb{R}^n$, with codim$(H) = k$, $l \notin \mathbb{Z}$ and $\vec{\delta} = (\delta_1,\dots,\delta_N)$.*

(i) If $l > 1, -k \le m < -\frac{k}{2}, N \ge 1, \delta_1 > 0$ and $m - |\vec{\delta}| < -k$, then

$$C^l_* I^{m,\vec{\delta}}(H) \hookrightarrow L^p(\Omega) \text{ continuously for all } 1 \le p < \frac{k}{m+k}. \tag{2.1}$$

(ii) If $l > 2$, $N \ge 2$, $\delta_1 > 1 - \frac{k}{2}$ and $|\vec{\delta}| > 1$, then for each smooth function $\tilde{F}$ vanishing on H,

$$v \in C^l_* I^{-k,\vec{\delta}}(H) \implies \tilde{F}\cdot v \in W^{1,p}(\Omega) \text{ for } 1 \le p < \frac{k}{1-\delta_1} \tag{2.2}$$

and thus, for $G \subset\subset \overline{\Omega}\backslash H$, $C^l_* I^{-k,\vec{\delta}}(H) \hookrightarrow W^{1,p}(G)$ continuously.
(iii) Suppose $l > 3$. For $\max(\frac{2}{3}, 1-\frac{k}{4}) \overset{def}{=} \nu_0(k) < \nu < 1$, set

$$\vec{\delta}_0 = (\nu, 1-\nu, 2\nu-1, 1-\nu, 1-\nu, 3\nu-2).$$

Then, for any $\tilde{F}$ vanishing on H,

$$v \in C^l_* I^{-k,\vec{\delta}_0}(H) \implies \tilde{F}^3\cdot v \in W^{2,r}(\Omega) \text{ for } 1 \le r < \frac{k}{2(1-\nu)} \tag{2.3}$$

and thus, for $G \subset\subset \overline{\Omega}\backslash H$, $C^l_* I^{-k,\vec{\delta}_0}(H) \hookrightarrow W^{2,r}(G)$ continuously.

Proof. (i) Since L^p and $C^l_* I^{m,\vec{\delta}}(H)$ are diffeomorphism-invariant, it suffices to assume that, with respect to coordinates $x = (x',x'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$,

$$H = \{x'' = 0\} \quad \text{and} \quad u(x) = \int_{\mathbb{R}^k} e^{ix''\cdot\theta} a(x,\theta)d\theta, \quad a \in C^l_* S^{m,\vec{\delta}}.$$

We then have

$$u(x',x'') = \int e^{ix''\cdot\theta}[a(x',0,\theta)+\sum_{0<|\alpha''|<[l]} \frac{1}{(\alpha'')!}\partial_{x''}^{\alpha''} a(x',0,\theta)(x'')^{\alpha''}+R_{[l]}(x',x'',\theta)]d\theta,$$

where

$$R_{[l]}(x,\theta) = \sum_{|\alpha''|=[l]} b_{\alpha''}(x,\theta)(x'')^{\alpha''} \text{ with } |b_{\alpha''}(x,\theta)| \le C(1+|\theta|)^m, \forall\alpha''.$$

Since $a(x',0,\cdot) \in L^q(\mathbb{R}^k_\theta), \forall q > -\frac{k}{m}$ and $-\frac{k}{m} < 2$, the Hausdorff-Young inequality implies that $\int e^{ix''\cdot\theta}a(x',0,\theta)d\theta \in L^{q'}(\mathbb{R}^k_{x''}), \forall 2 \le q' < \frac{k}{m+k}$, uniformly in x', and thus belongs to $L^p(\Omega)$ for all $2 \le p < \frac{k}{m+k}$; since it is compactly supported, the range is in fact $1 \le p < \frac{k}{m+k}$. For the second term, note that for each function $a_{\alpha''}(x',\theta) \overset{def}{=} \frac{1}{(\alpha'')!}\partial^{\alpha''}a(x',0,\theta)$,

$$\int e^{ix''\cdot\theta}a_{\alpha''}(x',\theta)(x'')^{\alpha''}d\theta = \int (\frac{1}{i}\partial_\theta)^{\alpha''}(e^{ix''\cdot\theta})a_{\alpha''}(x,\theta)d\theta =$$

$$= \int e^{ix''\cdot\theta}(\frac{-1}{i}\partial_\theta)^{\alpha''}(a_{\alpha''}(x,\theta))d\theta,$$

whose amplitude is $\le C(1+|\theta|)^{m-\delta(\alpha'')}$, and hence is treated by the same argument as for $a_0 = a(x',0,\theta)$. In the final term, we integrate by parts:

$$\int e^{ix''\cdot\theta}b_{\alpha''}(x,\theta)(x'')^{\alpha''}d\theta = \int e^{ix''\cdot\theta}(i\partial_\theta)^{\alpha''}b_{\alpha''}(x,\theta)d\theta,$$

and since $|\partial_\theta^{\alpha''}b(x,\theta)| \le C(1+|\theta|)^{m-|\vec{\delta}|} \in L^1(\mathbb{R}^k_\theta)$, uniformly in x', this yields a bounded function of $x \in \Omega$.

(ii) If $v \in C^l_* I^{-k,\vec{\delta}}(H)$, then $v \in L^p, \forall p < \infty$, by part (i). From

$$v(x) = \int e^{ix''\cdot\theta}a(x,\theta)d\theta, \quad a \in C^l_* S^{-k,\vec{\delta}},$$

one finds

$$\begin{aligned}\partial_{x_j}v(x) =& \int e^{ix''\cdot\theta}(i\theta_j a + \partial_{x_j}a)d\theta \\ &\in C^l_* I^{1-k,\vec{\delta}}(H) + C^{l-1}_* I^{-k,\vec{\delta}}(H)\end{aligned}$$

for $n-k+1 \le j \le n$; if $1 \le j \le n-k$, only the second term is present. The second term is covered by part (i) and hence is in $L^p, \forall p < \infty$. If the first term is multiplied by some $x_{j_0}, n-k+1 \le j_0 \le n$ and integrated by parts, it becomes an element of $C^l_* I^{1-k-\delta_1,\vec{\delta}'}(H)$, with $\vec{\delta}' = (\delta_2,\dots,\delta_N)$, which by (2.1) is in $L^p, 1 \le p < \frac{k}{1-\delta_1}$. Since any $\tilde{F}$ vanishing on H can be represented as a linear combination of x_{j_0}'s with smooth coefficients, $\tilde{F}(x)\nabla v \in L^p$ and so $v \in W^{1,p}(G)$ for any set G on which $|F|$ is bounded below.

(iii) By (i) and (ii) above, both v and $\tilde{F}\cdot\nabla v\in L^r$. Now, arguing as in (ii), for $n-k+1\le j,j'\le n$,

$$\partial^2_{x_jx_{j'}}v=\int e^{ix''\cdot\theta}\Big(-\theta_j\theta_{j'}a+i(\theta_ja_{x_{j'}}+\theta_{j'}a_{x_j})+a_{\theta_j\theta_{j'}}\Big)d\theta$$
$$\in\quad C^l_*I^{2-k,\vec{\delta}_0}(H)+C^{l-1}_*I^{1-k,\vec{\delta}_0}(H)+C^{l-2}_*I^{-k,\vec{\delta}_0}(H),$$

with simpler expressions if one or both of j or j' is $\le n-k$. By (i), the last term is in $L^r,\forall r<\infty$, if $l>3,N\ge 1$. Integrating by parts and using (ii), x_{j_0} times the second term is in $L^r, 1\le r<\frac{1}{1-\delta_1}=\frac{1}{1-\nu}$ if $l>2,N\ge 2$. As for the first term, $x_{j_0}x_{j_1}x_{j_2}$ times it is seen, after integrating by parts three times, to be in $C^l_*I^{2-k-(\delta_1+\delta_2+\delta_3),\vec{\delta}_0''}(H)=C^l_*I^{2-k-2\nu,\vec{\delta}_0''}(H)$ for $\vec{\delta}_0''=(1-\nu,1-\nu,3\nu-2)$, which, by (i), $\hookrightarrow L^r(\Omega),\forall 1\le r<\frac{k}{2(1-\nu)}$, if $l>2$, since $-k\le 2-k-2\nu<-\frac{k}{2}$ if $1-\frac{k}{4}<\nu\le 1$ and $2-k-|\vec{\delta}|=2-3\nu-k<k$ if $\nu>\frac{2}{3}$. Thus, for any $\tilde{F}$ vanishing on H, $\tilde{F}^3\cdot C^l_*I^{-k,\vec{\delta}_0}(H)\hookrightarrow W^{2,r}(\Omega)$ if $1\le r<\frac{k}{2(1-\nu)}, l>3$. $\square$

We also have

Proposition 2.3. *If $A\in\Psi^r(\mathbb{R}^n)$ is properly supported with $r<0$, then for any $l,m\in\mathbb{R}$ and any $\vec{\delta}$, and any $\epsilon>0$,*

$$A:C^l_*I^{m,\vec{\delta}}(H)\to C^l_*I^{m+r+\epsilon,\vec{\delta}}(H)$$

and $||A||$ is bounded by a finite number of semi-norms of $\sigma(A)$ in $S^r_{1,0}(\mathbb{R}^n\times(\mathbb{R}^n\setminus 0))$.

Proof. Write

$$Af(x)=\int e^{i(x-y)\cdot\xi}a(x,y,\xi)f(y)d\xi dy,\quad a\in S^r_{1,0}$$

and

$$u(y)=\int e^{iF(y)\cdot\theta}b(y,\theta)d\theta,\quad b\in C^l_*S^{m,\vec{\delta}}.$$

Then

$$Au(x)=\int e^{i[(x-y)\cdot\xi+F(y)\cdot\theta]}a(x,y,\xi)b(y,\theta)d\theta d\xi dy$$
$$=\int e^{iF(x)\cdot\theta}c(x,\theta)d\theta,$$

where

$$c(x,\theta)=\int e^{i[(x-y)\cdot\xi+(F(y)-F(x))\cdot\theta]}a(x,y,\xi)b(y,\theta)d\xi dy \tag{2.4}$$

and we need to show that $c=c[b]\in C^l_*S^{m+r+\epsilon,\vec{\delta}},\forall\epsilon>0$. Furthermore, since H is compact, we may assume that the symbol $a(x,y,\xi)$ vanishes for x,y outside some compact set.

Let $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$. Setting $b_i(y,\theta) = \psi_i(D_y)(b(y,\theta))$ and $c_j(x,\theta) = \psi_j(D_x)(c(x,\theta))$, we have

$$||\partial_\theta^\alpha b_i(\cdot,\theta)||_{L^\infty} \leq C2^{-li}\langle\theta\rangle^{m-\delta(\alpha)}, \forall|\alpha| \leq N,$$

and want to prove that

$$||\partial_\theta^\alpha c_j(\cdot,\theta)||_{L^\infty} \leq C_\epsilon 2^{-lj}\langle\theta\rangle^{m+r+\epsilon-\delta(\alpha)}, \forall|\alpha| \leq N.$$

We first consider the case of $\alpha = 0$. Let $\{\tilde{\psi}_i\}_{i=0}^\infty, \{\tilde{\tilde{\psi}}_i\}_{i=0}^\infty$ be bounded families in $S^0_{1,0}$ with $\tilde{\psi}_i = 1$ on $\mathrm{supp}(\psi_i)$ and $\tilde{\tilde{\psi}}_i = 1$ on $\mathrm{supp}(\tilde{\psi}_i)$; then, one can write $c_j = T_{ij}(b_i)$, where the operator T_{ij} has the Schwartz kernel

$$K_{ij}(x,y;\theta) = \int e^{i[(x-z)\cdot\zeta+(z-w)\cdot\xi+(F(w)-F(z))\cdot\theta+(w-y)\cdot\eta]}\psi_j(\zeta)a(z,w,\xi)\tilde{\psi}_i(\eta)d\zeta dz d\xi dw d\eta.$$

It suffices to show that $\sum_{i=0}^\infty ||T_{ij}(b_i)||_{L^\infty} \leq C_\epsilon 2^{-lj}\langle\theta\rangle^{m+r+\epsilon}$.

As in the proof of Prop. 2.2, we may assume that $F(x) = x''$, where $x = (x',x'') \in \mathbb{R}^{n-k}\times\mathbb{R}^k$; for $\theta \in \mathbb{R}^k$, let $\theta^* = (0,\theta) = DF^*(x)(\theta), \forall x$. Thus, the phase function of $K_{ij}(x,y;\theta)$ is

$$\phi = (x-z)\cdot\zeta + (z-w)\cdot\xi + (w''-z'')\cdot\theta + (w-y)\cdot\eta.$$

Let $\chi \in C_0^\infty(\mathbb{R}), \chi(t) = 1$ for $|t| \leq 1/2, \mathrm{supp}(\chi) \subset \{|t| \leq 3/4\}$. Write

$$\begin{aligned} K_{ij}(x,y;\theta) = &\int e^{i\phi}\psi_j(\zeta)\tilde{\psi}_i(\eta)[\chi(\frac{|\xi-\zeta-\theta^*|}{|\theta|}) + (1-\chi)(\frac{|\xi-\zeta-\theta^*|}{|\theta|})]\times \\ &a(z,w,\xi)d\zeta d\eta d\xi dz dw \\ =&K^0_{ij}(x,y;\theta) + K^\infty_{ij}(x,y;\theta) \end{aligned}$$

with $T_{ij} = T^0_{ij} + T^\infty_{ij}$ the corresponding operator decomposition.

K^∞_{ij} may be analyzed by noting that $|\xi-\zeta-\theta^*| \geq c\max(|\xi-\zeta|,|\theta|)$ on $\mathrm{supp}(1-\chi)$, and thus, using

$$(\frac{\xi-\zeta-\theta^*}{i|\xi-\zeta-\theta^*|^2}\cdot\nabla_z)(e^{i\phi}) = e^{i\phi},$$

we may integrate by parts $n+M$ times in z and then integrate in ξ to obtain

$$K^\infty_{ij}(x,y;\theta) = \int e^{i[(x-z)\cdot\zeta+(w''-z'')\cdot\theta+(w-y)\cdot\eta]}\psi_j(\zeta)\tilde{\psi}_i(\eta)A(z,w;\theta)d\zeta d\eta dz dw$$

with $|\partial_z^\alpha\partial_w^\beta A(z,w;\theta)| \leq C_{\alpha,\beta,M}\langle\theta\rangle^{-M}, \forall\alpha,\beta \in \mathbb{Z}^n_+, M \in \mathbb{N}$. If $\max(2^i,2^j) < \frac{1}{4}\langle\theta\rangle$, we then simply integrate in all variables to obtain $|K^\infty_{ij}| \leq c2^{n(i+j)}\langle\theta\rangle^{-M}$. If $\langle\theta\rangle < \frac{1}{4}\min(2^i,2^j)$, then $|\zeta+\theta^*| \geq c|\zeta| \geq c2^j$ and $|\eta+\theta^*| \geq c|\eta| \geq c2^i$, so we may first integrate by parts in z and w to obtain $|K^\infty_{ij}| \leq c2^{-M'(i+j)}\langle\theta\rangle^{-M}$.

In the following, we denote $a \sim b$ if either $a,b < 4$ or $\frac{1}{4}a \leq b \leq 4a$.

If $\langle\theta\rangle \sim 2^i > 2^{j+1}$ or $\langle\theta\rangle \sim 2^j > 2^{i+1}$, we may integrate by parts in z or w alone to obtain $|K^\infty_{ij}| \leq c2^{-M'i}\langle\theta\rangle^{-M}$ or $|K^\infty_{ij}| \leq c2^{-M'j}\langle\theta\rangle^{-M}$, respectively, $\forall M'$. If $2^{i+1} < \langle\theta\rangle < 2^{j-1}$, then $|\zeta+\theta^*| \geq c2^j$ and $|\eta+\theta^*| \geq c|\theta^*|$, so integration by parts in z,w gives $|K^\infty_{ij}| \leq c2^{-M'j}\langle\theta\rangle^{-M}$; similarly, for $2^{j+1} < \langle\theta\rangle < 2^{i-1}$ we obtain $|K_{ij}^\infty| \le c2^{-M'i}\langle\theta\rangle^{-M}$. Finally, if $\langle\theta\rangle \sim 2^i \sim 2^j$, we simply integrate and get $|K_{ij}^\infty| \le c2^{n(i+j)}\langle\theta\rangle^{-M} \le c\langle\theta\rangle^{-(M-2n)}$. Using $||b_i||_{L^\infty} \le c2^{-li}\langle\theta\rangle^m$ and summing in i yields $\sum_{i=0}^\infty ||T_{ij}^\infty(b_i)||_{L^\infty} \le c2^{-M'j}\langle\theta\rangle^{-M}, \forall M, M'$.

On the other hand, K_{ij}^0 may be analyzed using stationary phase in ξ and w: First rewrite

(2.5)

$$K_{ij}^0(x,y;\theta) = \int e^{i[(x-z)\cdot\zeta+(z-y)\cdot\eta]}\psi_j(\zeta)\tilde{\psi}_i(\eta)\times$$
$$\Big[\int e^{i|\theta|[w''\cdot\frac{\theta}{|\theta|}-w\cdot\xi)]}|\theta|^n a(z,z+w,|\theta|(\xi+\frac{\eta}{|\theta|}))\chi(|\xi-\frac{\theta^*}{|\theta|}-\sigma|)d\xi dw\Big]_{\sigma=\frac{\zeta-\eta}{|\theta|}}dzd\zeta d\eta.$$

The domain of integration for the inner integral contains a critical point only if $|\sigma|<2$. In this case the unique nondegenerate critical point is $\xi=\frac{\theta^*}{|\theta|}$, $w=0$. Applying the method stationary phase and analysing error terms using [16,Thm.7.7.7] for $|\sigma|<2$, or just using integration by parts for $|\sigma|\ge 2$, we see that the inner integral is $a(z,z,\eta+\theta^*)\chi(\frac{|\eta-\zeta|}{|\theta|})+e_1(z,\frac{\eta}{|\theta|},\frac{\zeta}{|\theta|},\eta+\theta^*)$, with $e_1\in S_{1,0}^{r-1}((\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)\times\mathbb{R}^n\backslash 0)$ and thus

$$\text{(2.6)}\qquad \begin{aligned} K_{ij}^0(x,y;\theta) = &\int e^{i[(x-z)\cdot\zeta+(z-y)\cdot\eta]}\psi_j(\zeta)\tilde{\psi}_i(\eta)\times\\ &[a(z,z,\eta+\theta^*)\chi(\frac{|\eta-\zeta|}{|\theta|})+e_1(z,\frac{\eta}{|\theta|},\frac{\zeta}{|\theta|},\eta+\theta^*)]d\zeta dzd\eta.\end{aligned}$$

Note that $S_{1,0}^{r-1}$ seminorms of e_1 are uniformly bounded over compact subsets of $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$. We observe for the phase that $d_z((x-z)\cdot\zeta+(z-y)\cdot\eta)=-\zeta+\eta$ and $(d_\zeta+d_\eta)((x-z)\cdot\zeta+(z-y)\cdot\eta)=x-y$ and we may use these to integrate by parts, M and M' times, respectively, to obtain, for $i\le j-2$,

$$\begin{aligned}|K_{ij}^0(x,y;\theta)| \le &C\int 2^{-Mj}(1+2^i|x-y|)^{-M'}\tilde{\psi}_j(\zeta)\langle\theta\rangle^r|\eta|^{-M'}\tilde{\tilde{\psi}}_i(\eta)d\zeta dzd\eta\\ \le &C2^{-(M-n)j}(1+2^i|x-y|)^{-M'}2^{-(M'-n)i}\langle\theta\rangle^r.\end{aligned}$$

Here, we have used $|-\zeta+\eta|\ge\frac{1}{2}|\zeta|\ge 2^{j-1}$ and

$$|(d_\zeta+d_\eta)^{M'}(\psi_j(\zeta)a(z,z,\eta+\theta^*)\tilde{\psi}_i(\eta))\le\begin{cases} C|\eta|^{r-M'}\tilde{\psi}_j(\zeta)\tilde{\tilde{\psi}}_i(\eta), |\theta|\le c2^i\\ C\langle\theta\rangle^r|\eta|^{-M'}\tilde{\psi}_j(\zeta)\tilde{\tilde{\psi}}_i(\eta), |\theta|\ge c2^i.\end{cases}$$

Thus, $\int|K_{ij}^0(x,y)|dy\le C2^{-(M-n)j}2^{-M'i}\langle\theta\rangle^r$, so that

$$\sum_{i=0}^{j-2}||T_{ij}^0(b_i)||_{L^\infty}\le C\sum_{i=0}^{j-2}2^{-(M-n)j}2^{-M'i}\langle\theta\rangle^r2^{-li}\langle\theta\rangle^m\le Cc^{-lj}\langle\theta\rangle^{m+r}$$

for $M>n+l$. Similarly, for $i\ge j+2$, we obtain the estimate

$$|K_{ij}^0(x,y)|\le C2^{-(M-n)i}(1+2^j|x-y|)^{-M'}2^{-(M'-n)j}\langle\theta\rangle^r,$$

leading to

$$\sum_{i=j+2}^{\infty} ||T_{ij}^0(b_i)||_{L^\infty} \leq C \sum_{i=j+2}^{\infty} 2^{-M'j}2^{-(M-n)i}\langle\theta\rangle^r 2^{-li}\langle\theta\rangle^m \leq C2^{-lj}\langle\theta\rangle^{m+r}$$

for $M' > l, M > n - l$.

To analyze the contributions from $|i - j| \leq 1$, we perform in (2.6) an additional stationary phase in z and ζ, with critical point $z = x$, $\zeta = \eta$. Rewriting the integral as in formula (2.5) we see that,

$$K_{ij}^0(x,y) = \int e^{i(x-y)\cdot\eta}\psi_j(\eta)[a(x,x,\eta+\theta^*) + e_2(x, \frac{\eta}{|\theta|}, \frac{\eta}{|\theta|}, \eta+\theta^*)]\tilde{\psi}_i(\eta)d\eta$$

with $e_2 \in S_{1,0}^{r-1}((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^n\backslash 0)$. Since ψ_j are uniformly bounded in $S_{1,0}^0$, the $S_{1,0}^{r-1}$ seminorms of e_2 are uniformly bounded over compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Then

$$K_{ij}^0(x,y) = e^{-i\theta^*\cdot x}\Big[\int e^{i(x-y)\cdot\eta}[\psi_j(\eta-\theta^*)h(x,\eta;\,\theta)\tilde{\tilde{\psi}}_i(\eta-\theta^*)]\tilde{\psi}_i(\eta-\theta^*)d\eta\Big]e^{i\theta^*\cdot y}$$

where $h(\cdot\,,\cdot;\,\theta) \in S_{1,0}^r(\mathbb{R}^n \times \mathbb{R}^n\backslash 0)$,

$$h(x,\eta;\,\theta) = a(x,x,\eta) + e_2(x, \frac{\eta-\theta^*}{|\theta|}, \frac{\eta-\theta^*}{|\theta|}, \eta)$$

is a symbol depending on the parameter θ, $|\theta| > 1$.

Note that multiplication by the exponentials does not affect the $L^\infty \to L^\infty$ operator norm, while

$$\psi_j(\eta-\theta^*)h(x,\eta;\,\theta)\tilde{\tilde{\psi}}_i(\eta-\theta^*) \in \begin{cases} S_{1,0}^r(\mathbb{R}^n \times \mathbb{R}^n\backslash 0), & |\theta| \leq c2^j \\ \langle\theta\rangle^r \cdot S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n\backslash 0), & |\theta| \geq c2^j \end{cases}$$

uniformly in θ. Acting on functions with Fourier transforms supported in $B(0,R)$, pseudodifferential operators of order 0 are bounded on L^∞ with norm $\leq c\log R$ [33]. Hence, for $|i-j| \leq 1$,

$$||T_{ij}(b_i)||_{L^\infty} \leq \begin{cases} Cj2^{rj}2^{-lj}\langle\theta\rangle^m \leq C_\epsilon 2^{-lj}\langle\theta\rangle^{m+r+\epsilon}, & |\theta| \leq c2^j \\ Cj2^{-lj}\langle\theta\rangle^{m+r} \leq C_\epsilon 2^{-lj}\langle\theta\rangle^{m+r+\epsilon}, & |\theta| \geq c2^j, \end{cases}$$

as desired.

To handle the derivatives $\partial_\theta^\alpha c$, note first that the application of ∂_θ to the right hand side of (2.4) yields two terms. Since $\partial_\theta b(x,\theta) \in C_*^l S^{m-\delta_1,(\delta_2,\ldots)}$, the oscillatory integral with amplitude $a(x,y,\xi)\partial_\theta b(x,\theta)$ has C_*^l norm $\leq C\langle\theta\rangle^{m+r-\delta_1}$, as desired. On the other hand, as here $F(x) = x''$, we see that if the derivative hits the phase, then the amplitude becomes $i(F(y) - F(x)) \cdot a \cdot b = i(y'' - x'') \cdot a \cdot b$. Writing $y'' - x'' = -\partial_{\xi''}[(x-y)\cdot\xi + (F(y) - F(x))\cdot\theta]$, we may then integrate by parts in ξ. The resulting amplitude, $i\partial_\xi a \cdot b$ gives a term whose C_*^l norm is $\leq C\langle\theta\rangle^{m+r-1}$. Higher derivatives $\partial_\theta^\alpha c, |\alpha| \leq N$, are handled similarly. □

To obtain the boundedness of M_q on some of the finite-regularity conormal spaces, we will need the following; similar results are in [13].

Lemma 2.4. *Let $a, b \in C(\mathbb{R}^n \times \mathbb{R}^k)$ satisfy*

$$||a(\cdot,\theta)||_{C_*^l} \le C\langle\theta\rangle^m \quad \text{and} \quad ||b(\cdot,\theta)||_{C_*^{l'}} \le C\langle\theta\rangle^{m'}$$

with $l, l' > 0, l, l' \notin \mathbb{N}$ and $m + m' < -k$. Then the partial convolution $a *_{\mathbb{R}^k} b$ satisfies

$$||a *_{\mathbb{R}^k} b(\cdot,\theta)||_{C_*^{l''}} \le C\langle\theta\rangle^{m''},$$

with $l'' = \min(l, l')$ and $m'' = \max\big((m+k)_{\tilde{+}} + m', m + (m'+k)_{\tilde{+}}, m + m' + k\big)$. Here, $t_{\tilde{+}} = t_+ = \max(t, 0)$ if $t \neq 0$ and $t_{\tilde{+}} = \epsilon$ for $t = 0$, with $\epsilon > 0$ arbitrarily small.

Sketch of proof. This follows easily by decomposing

$$\begin{aligned}(a *_{\mathbb{R}^k} b)(x,\theta) =& \int_{\mathbb{R}^k} a(x,\sigma)b(x,\theta-\sigma)d\sigma \\ =&\Big(\int_{|\sigma|\le\frac{|\theta|}{2}} + \int_{|\theta-\sigma|\le\frac{|\theta|}{2}} + \int_{|\sigma|\ge\frac{|\theta|}{2},\ |\theta-\sigma|\ge\frac{|\theta|}{2}}\Big)a(x,\sigma)b(x,\theta-\sigma)d\sigma.\end{aligned}$$

and estimating each of the three terms using the fact that $C_*^l \times C_*^{l'} \hookrightarrow C_*^{l''}$ continuously.

This is then used in the proof of

Proposition 2.5. *Let $H \subset \mathbb{R}^N$ be codimension k. For $q \in I^\mu(H)$, let M_q denote multiplication by $q(x)$. Suppose $\frac{2}{3} - k \le \mu < 1 - k$. Then, for any $l > 0, l \notin \mathbb{N}$,*

$$M_q : C_*^l I^{-k,\vec{\delta}_0}(H) \to C_*^l I^{\tilde{\mu},\vec{\delta}_0}(H) \quad \textit{continuously} \tag{2.7}$$

where, if we write $\mu = \nu - k$ with $\frac{2}{3} \le \nu < 1$, $\vec{\delta}_0$ is as in Prop. 2.2(iii):

$$\vec{\delta}_0 = (\nu, 1-\nu, 2\nu-1, 1-\nu, 1-\nu, 3\nu-2) \tag{2.8}$$

and $\tilde{\mu} = \mu + \epsilon$ for any $0 < \epsilon < -\mu$.

Proof. As before, we can assume that $H = \{x'' = 0\}$,

$$u(x) = \int_{\mathbb{R}^k} e^{ix''\cdot\theta} b(x,\theta)d\theta, \quad b \in C_*^l I^{-k,\vec{\delta}_0}(H)$$

and

$$q(x) = \int_{\mathbb{R}^k} e^{ix''\cdot\theta} a(x,\theta)d\theta, \quad a \in S_{1,0}^\mu \hookrightarrow C^{l'} S^{\mu,(1,1,\dots)}$$

for any $l' > 0$.

Hence, $M_q u(x) = \int e^{ix''\cdot\theta}(a * b)(x,\theta)d\theta$, where $*$ denotes the k-dimensional convolution in the θ variable. By Lemma 2.4,

$$||a * b(x,\theta)||_{C_*^l} \le C\langle\theta\rangle^{\max(\mu,\mu+\epsilon,\mu)} = C\langle\theta\rangle^{\mu+\epsilon}, \forall\epsilon > 0.$$

Since $\partial_\theta a \in C_*^{l'} S^{\mu-1,(1,1,\dots)}$ and $\mu - 1 < -k$,

$$||\partial_\theta(a * b)(x,\theta)||_{C_*^l} = ||(\partial_\theta a) * b(x,\theta)||_{C_*^l} \leq C\langle\theta\rangle^{\max(-k,\mu-1+\epsilon,\mu-1)} = C\langle\theta\rangle^{-k}$$

(for $\epsilon < -\mu$), which gives a gain of $\geq \mu - (-k) = \nu$, consistent with $\delta_1 = \nu$. (The additional gain of ϵ we choose to ignore.) Noting that $\partial_\theta b \in C_*^l S^{-k-\nu,(1-\nu,2\nu-1,\dots,3\nu-2)}$ by Remark 2 above, we have

$$||\partial_\theta^2(a * b)||_{C_*^l} = ||(\partial_\theta a) * (\partial_\theta b)||_{C_*^l} \leq C\langle\theta\rangle^{\max(-k-\nu,\nu-k-1,-k-1)} = C\langle\theta\rangle^{\nu-k-1}$$

since $\nu \geq \frac{1}{2}$, which is a gain of $\delta_2 = -k - (\nu - k - 1) = 1 - \nu$. Since $\partial_\theta^2 a \in C_*^{l'} S^{\mu-2,(1,1,\dots)}$ and $\partial_\theta b$ is as noted above,

$$||\partial_\theta^3(a * b)||_{C_*^l} = ||(\partial_\theta^2 a) * (\partial_\theta b)||_{C_*^l} \leq C\langle\theta\rangle^{\max(-k-\nu,\nu-k-2)} = C\langle\theta\rangle^{-k-\nu},$$

which gives a gain of $\delta_3 = \nu - k - 1 - (-k - \nu) = 2\nu - 1$. Since $\partial_\theta^2 a \in C_*^{l'} S^{\mu-2,(1,1,\dots)}$ and $\partial_\theta^2 b \in C_*^l S^{-k-1,(2\nu-1,1-\nu,\dots)}$,

$$||\partial_\theta^4(a * b)||_{C_*^l} = ||(\partial_\theta^2 a) * (\partial_\theta^2 b)||_{C_*^l} \leq C\langle\theta\rangle^{\max(-k-1,\nu-k-2,2\nu-k-3)} = C\langle\theta\rangle^{-k-1},$$

which gives a gain of $\delta_4 = -k - \nu - (-k - 1) = 1 - \nu$. Continuing in this fashion, we may estimate

$$||\partial_\theta^5(a * b)||_{C_*^l} = ||(\partial_\theta^2 a) * (\partial_\theta^3 b)||_{C_*^l} \leq C\langle\theta\rangle^{\nu-k-2},$$

which is consistent with $\delta_5 = 1 - \nu$ if $\nu \geq \frac{2}{3}$, and

$$||\partial_\theta^6(a * b)||_{C_*^l} = ||(\partial_\theta^3 a) * (\partial_\theta^3 b)||_{C_*^l} \leq C\langle\theta\rangle^{-2\nu-k},$$

which is consistent with $\delta_6 = 3\nu - 2$. The x derivatives, lowering the Zygmund space index and not involving any gain in $\langle\theta\rangle$, are handled in the obvious way. Hence, $a * b \in C_*^l S^{\mu+\epsilon,\vec{\delta}_0}$. $\square$

Now recall some facts concerning the Faddeev Green's function [10], G_ρ. As is well-known (see,e.g., [32], where this is used implicitly), the families $\{|\rho|G_\rho : \rho\cdot\rho = 0\}$ and $\{G_\rho : \rho\cdot\rho = 0\}$ are uniformly bounded in $\Psi^0(K)$ and $\Psi^{-1}(K)$, respectively, for $K \subset\subset \mathbb{R}^n$ and hence interpolation implies

$$\left\{|\rho|^{1-t}G_\rho : \rho\cdot\rho = 0\right\} \subset \Psi^{-t}(K) \text{ is bounded }, \forall t \in [0,1]. \tag{2.9}$$

We can now state a local analogue for the finite-regularity conormal spaces of the result of [32] concerning solvability of inhomogeneous equations involving $\Delta_\rho + q(x)$ in weighted L^2 spaces.

Proposition 2.6. *If $q \in I^{\mu}(H)$ with $\frac{2}{3}-k \le \mu < 1-k$, $l > 3, l \notin \mathbb{N}$ and $0 \le \sigma \le 1$, then for $\vec{\delta}_0$ as in (2.8), the inhomogeneous equation*

$$(\Delta_\rho + q)w = g \in C^l_* I^{1-k-\sigma,\vec{\delta}_0}(H) \tag{2.10}$$

has, for $|\rho|$ large, a unique solution $w \in C^l_* I^{-k,\vec{\delta}_0}(H)$, with $\|w\| \le \frac{C}{|\rho|^\sigma}\|g\|$.

Proof. Applying G_ρ to both sides of (2.10), using (2.9) for $t = 1-\sigma$ and Prop. 2.3, we are reduced to showing that

$$(I + G_\rho M_q)w = G_\rho g \in C^l_* I^{-k,\vec{\delta}_0}(H)$$

has a unique solution for $|\rho|$ sufficiently large, with $\|w\| \le C\|G_\rho g\|$. By Prop. 2.5, $M_q : C^l_* I^{-k,\vec{\delta}_0}(H) \to C^l_* I^{\mu+\epsilon,\vec{\delta}_0}(H)$ for any $0 < \epsilon < -\mu$. Note that $t = \mu + \epsilon - (-k) < 1$, so we can use Prop. 2.5 and (2.9) with this value of t to obtain

$$C^l_* I^{-k,\vec{\delta}_0}(H) \stackrel{M_q}{\to} C^l_* I^{\mu+\epsilon,\vec{\delta}_0}(H) \stackrel{G_\rho}{\to} C^l_* I^{-k,\vec{\delta}_0}(H)$$

with norm $\le C(q)|\rho|^{t-1+\epsilon'} \to 0$ as $|\rho| \to \infty$. Hence, for $|\rho|$ sufficiently large, $||G_\rho M_q|| < \frac{1}{2}$ and $I + G_\rho M_q$ is invertible on $C^l_* I^{-k,\vec{\delta}_0}(H)$. $\square$

We may now complete the proof of Thm. 1 as described in §1. Construct two solutions ψ_j, $j = 1, 2$, to (1.10) for potentials $q_j \in I^{\mu_j}(H_j)$. Note that $-q_j$, the right hand side of (1.10), belongs to $I^{\mu_j}(H) \hookrightarrow C^l_* I^{1-k_j-\sigma_j,\vec{\delta}}(H)$, with $\sigma_j > 0$ since $\mu_j < 1 - k_j$. Thus, we may apply Prop. 2.6 with $g = -q_j, j = 1, 2$, and then form as above the corresponding solutions, $v_1(x,\rho_1) = e^{\rho_1\cdot x}(1 + \psi_1(x,\rho_1))$ of $(\Delta + q_1)v_1 = 0$ and $w_2(x,\rho_2) = e^{\rho_2\cdot x}(1 + \psi_2(x,\rho_2))$ of $(\Delta + q_2)w_2 = 0$, with $\|\psi_j\|_{C^l_* I^{-k_j,\vec{\delta}}} \le C|\rho|^{-\sigma_j}$. The solutions v_1 and w_2 belong to $X^{p,r}(\Omega)$, with p, r as in (1.3). In fact, each is in $L^p(\Omega)$, and in $W^{2,r}(\Omega)$ away from H by Prop. 2.2(i) and (iii), resp., since $r < \frac{1}{2(1-\nu)}$. Furthermore, since $q_1 \in L^{\frac{k_1}{\nu_1}-\epsilon}, \forall \epsilon > 0$, we have $\Delta v_1 = -q_1 v_1 \in \left(L^{\frac{k_1}{\nu_1}-\epsilon}\right) \times (L^s), \forall \epsilon > 0, \forall s < \infty$. Since $p > \frac{k_1}{k_1-\nu_1} \implies p' < \frac{k_1}{\nu_1}$, we thus have $\Delta v_1 \in L^{p'}(\Omega)$, and similarly for w_2. These solutions are constructed for all large $|\rho|$. Since $n \ge 3$, for any $\xi \in \mathbb{R}^n\backslash 0$ and $\lambda \ge c|\xi|$, one can find $\rho_1, \rho_2 \in \{\rho\cdot\rho = 0\}$ with $|\rho_1| \simeq |\rho_2| \ge \lambda$ and $\rho_1 + \rho_2 = -i\xi$.

By the assumption that $\mathcal{CD}_{q_2} = \mathcal{CD}_{q_1}$, there exists a $v_2 \in X^{p,r}(\Omega)$ such that (1.7) holds. Applying (1.8) and (1.9), it suffices to show that

$$\int_\Omega e^{-i\xi\cdot x}(q_1 - q_2)(\psi_1 + \psi_2 + \psi_1\psi_2)dx \to 0 \text{ as } \lambda \to \infty. \tag{2.11}$$

Since each $q_j \in L^{\frac{k_j}{\nu_j}-\epsilon}(\Omega), \forall \epsilon > 0$, and, by (1.3), $p > (\frac{k_j}{\nu_j})', j = 1, 2$, we have

$$\int_\Omega |q_1 - q_2| \cdot |\psi_j| dx \le \frac{c}{\lambda^{\epsilon_j}}, \quad \epsilon_j > 0$$

as $\lambda \to \infty$ by Hölder's inequality. In fact, $p > 2(\frac{k_j}{\nu_j})'$ by (1.3), so we have

$$\int_\Omega |q_1 - q_2| \cdot |\psi_1| \cdot |\psi_2| dx \le \frac{c}{\lambda^{\epsilon_{12}}}, \quad \epsilon_{12} > 0,$$

as well. Thus, we have shown that $\mathcal{CD}_{q_1} = \mathcal{CD}_{q_2}$ implies that $\widehat{q_1 - q_2}(\xi) = 0, \forall \xi$, and hence $q_1 = q_2$, concluding the proof of Thm.1.

3. Reconstruction of the potential

In this section we prove that the potential q can be obtained constructively from the Cauchy data of $\Delta + q$. We follow here the general method of [25]; see also [26] and [19]. However, there are some additional difficulties in our case because we deal with the set of Cauchy data instead of the Dirichlet-to-Neumann map. Moreover, we work with more complicated function spaces due to the singularities of the potential. We will show:

Theorem 2. *Suppose that Ω is Lipschitz and $H \subset\subset \Omega$ is a submanifold of codimension k. Suppose further that $q \in I^{\mu}(H)$ is a real potential with $\nu_0(k) - k < \mu < 1 - k$ and $supp(q) \subset\subset \Omega$. Let p, r satisfy $2 \leq r < r_0(k, \nu)$ and $\max(p_0(k,\nu), -\frac{k}{\mu}) < p < \infty$. Then q can be reconstructed on Ω from the Cauchy data $\mathcal{CD}_q$ of $\Delta + q$ on $X^{p,r}(\Omega)$.*

To start the discussion of reconstruction, we first show how to obtain a global analogue of Prop. 2.6. For $s \in \mathbb{R}$ and $-1 < \delta < 0$, let $W^{s,2}_{\delta}(\mathbb{R}^n)$ be the weighted Sobolev space denoted by H^s_δ in [32]. In [32], it is shown that for $0 \leq t \leq 1$,

$$||G_\rho f||_{W^{s+t,2}_{\delta}} \leq \frac{C}{|\rho|^{1-t}} ||f||_{W^{s,2}_{\delta+1}}. \tag{3.1}$$

Now, let $\text{supp}(q) \subset \Omega' \subset\subset \Omega$ and let $\chi_0 + \chi_\infty \equiv 1$ be a partition of unity subordinate to the open cover $\Omega \cup (\Omega')^c = \mathbb{R}^n$. Fix $\frac{2}{3} - k < \mu < 1 - k$, $l > 3$ and let $\vec{\delta}_0$ be as in (2.8). For $m \leq 1 - k$ and $s \leq -2$, define

$$||f||_{Y^{m,s}_{\delta}(\mathbb{R}^n)} = ||\chi_0 \cdot f||_{C^l_* I^{m,\vec{\delta}_0}(H)} + ||\chi_\infty \cdot f||_{W^{s,2}_{\delta}(\mathbb{R}^n)}. \tag{3.2}$$

For $m \leq -k$, elements of $I^m(H)$ are in $L^p_{loc}(\mathbb{R}^n), \forall p < \infty$, and hence $I^m_{comp}(H) \hookrightarrow W^{s,2}_{\delta}(\mathbb{R}^n), \forall m \leq 1-k, s \leq -1$. Combining (3.1) with (2.9) and Prop. 2.3, we have, for $0 \leq t \leq 1$,

$$||G_\rho f||_{Y^{m-t,s+t}_{\delta}(\mathbb{R}^n)} \leq \frac{C}{|\rho|^{1-t}} ||f||_{Y^{m,s}_{\delta+1}(\mathbb{R}^n)}, \quad m \leq 1-k-t, s \leq -1-t, -1 < \delta < 0. \tag{3.3}$$

Finally, since $\text{supp}(q) \subset\subset \Omega$, it follows from (2.7) that

$$M_q : Y^{-k,s}_{\delta} \to Y^{\mu+\epsilon,s}_{\delta}, \quad \forall \epsilon > 0. \tag{3.4}$$

Arguing as in the proof of Prop. 2.6, but replacing the local finite-regularity spaces with the global spaces $Y^{m,s}_{\delta}(\mathbb{R}^n)$, we obtain the following result:

Proposition 3.1. *If $q \in I^\mu(H)$ with $\frac{2}{3} - k < \mu < 1 - k, s \leq -2, -1 < \delta < 0$ and $0 \leq \sigma \leq 1$, then the inhomogeneous equation*

$$(\Delta_\rho + q) w = g \in Y^{1-k-\sigma,s}_{\delta+1}(\mathbb{R}^n) \tag{3.5}$$

has, for $|\rho|$ large, a unique solution $w \in Y^{-k,s+1-\sigma}_{\delta}(\mathbb{R}^n)$ with $||w|| \leq \frac{C}{|\rho|^\sigma} ||g||$.

Next we will construct the boundary values of the exponentially growing solutions on $\partial\Omega$. For this purpose we use the Green's function $G^q_\rho(x,y)$ defined by

$$(\Delta + q) G^q_\rho(\cdot, y) = \delta_y \quad \text{in} \quad \mathbb{R}^n, \quad e_\rho(\cdot) G^q_\rho(\cdot, y) \in Y^{-k,s}_{\delta}(\mathbb{R}^n), \tag{3.6}$$

where $y \in \mathbb{R}^n \setminus \overline{\Omega}$, $e_\rho(x) = \exp(-\rho \cdot x)$ and $s < -n/2$. When $|\rho|$ is large enough, the equation (3.6) has a unique solution by Prop. 3.1. Next we consider the case when ρ is fixed and sufficiently large.

As $\mathrm{supp}(q) \subset\subset \Omega$, we see that $\partial\Omega$ has a neighborhood V such that in $V \times V$ Green's function $G_\rho^q(x,y)$ has the same singularities as the Green's function G_0^0 (for the zero potential and $\rho = 0$), that is, $G_\rho^q(x,y) - G_0^0(x,y) \in C^\infty(V \times V)$.

Using the Green's function (3.6) we define the corresponding single and double layer potentials

$$S_q\phi(y) = \int_{\partial\Omega} G_\rho^q(x,y)\phi(x)dS_x, \quad K_q\phi(y) = \int_{\partial\Omega} (\frac{\partial}{\partial n(x)} G_\rho^q(x,y))\phi(x)dS_x, \quad y \notin \partial\Omega$$

which define continuous operators $S_q : W^{1-\frac{1}{r},r}(\partial\Omega) \to X|_\Omega \oplus W^{2,r}_{loc}(\mathbb{R}^n \setminus \Omega)$ and $K_q : W^{2-\frac{1}{r},r}(\partial\Omega) \to X|_\Omega \oplus W^{2,r}_{loc}(\mathbb{R}^n \setminus \Omega)$. Here $X \subset \mathcal{D}'(\mathbb{R}^n)$ is the space with the norm $||\chi_0 \cdot f||_{C^l_* I^{m,\delta_0}(H)} + ||\chi_\infty \cdot f||_{W^{2,r}_\delta(\mathbb{R}^n)}$. These layer potentials can be considered as operators on the boundary $\partial\Omega$, defined in principal value sense. Since $\partial\Omega$ is Lipschitz, it follows from the results of [7] that these operators are continuous, $S_q^{\partial\Omega} : W^{1-\frac{1}{r},r}(\partial\Omega) \to W^{2-\frac{1}{r},r}(\partial\Omega)$ and $K_q^{\partial\Omega} : W^{2-\frac{1}{r},r}(\partial\Omega) \to W^{2-\frac{1}{r},r}(\partial\Omega)$. Similarly, on $\partial\Omega$ we define normal derivatives of the layer potentials,

$$\partial_n S_q^{\partial\Omega}\phi(y) = \text{p.v.} \int_{\partial\Omega} (\frac{\partial}{\partial n(y)} G_\rho^q(x,y)\phi(x)dS_x,$$

$$\partial_n K_q^{\partial\Omega}\phi(y) = \text{p.v.} \int_{\partial\Omega} (\frac{\partial}{\partial n(y)} \frac{\partial}{\partial n(x)} G_\rho^q(x,y))\phi(x)dS_x$$

which are continuous operators $\partial_n S_q^{\partial\Omega} : W^{1-\frac{1}{r},r}(\partial\Omega) \to W^{1-\frac{1}{r},r}(\partial\Omega)$ and $\partial_n K_q^{\partial\Omega} : W^{2-\frac{1}{r},r}(\partial\Omega) \to W^{1-\frac{1}{r},r}(\partial\Omega)$.

Next we consider the Calderón projector [5]. We start with the operator

$$A_q(\phi,\psi) = \left(-S_q^{\partial\Omega}\phi + (-\frac{1}{2} + K_q^{\partial\Omega})\psi, -(\frac{1}{2} + \partial_n S_q^{\partial\Omega})\phi + \partial_n K_q^{\partial\Omega}\psi\right).$$

Proposition 3.2. *Let $Z = W^{2-\frac{1}{r},r}(\partial\Omega) \times W^{1-\frac{1}{r},r}(\partial\Omega)$. Then the operator*

$$A_q : Z/Ker(A_q) \to Z$$

is semi-Fredholm. Moreover, $-A_q : Z \to Z$ is a projection operator with range $\mathcal{CD}_q$ and kernel independent of q. In particular, $\mathcal{CD}_q$ is a closed subspace of Z.

Proof. First we show that kernel of A_q does not depend on q. Assume that $(\phi,\psi) \in Z$. We consider the function

$$u_{\phi,\psi}(y) = -S_q(\phi) + K_q(\psi), \quad y \in \mathbb{R}^n \setminus \partial\Omega$$

and the trace operators

$$T_+ : W^{2,r}(\mathbb{R}^n \setminus \Omega) \to W^{2-\frac{1}{r},r}(\partial\Omega) \times W^{1-\frac{1}{r},r}(\partial\Omega),$$

$$T_- : W^{2,r}(\Omega) \to W^{2-\frac{1}{r},r}(\partial\Omega) \times W^{1-\frac{1}{r},r}(\partial\Omega)$$

defined by $T_\pm u = (u|_{\partial\Omega}, \partial_n u|_{\partial\Omega})$. As the Green's functions $G^q_\rho(x,y)$ have the same singularities near $\partial\Omega \times \partial\Omega$ as the standard Green's function of $\mathbb{R}^n$, we can use the standard jump relations for layer potentials (see e.g. [8]). We conclude that

$$T_- u_{\phi,\psi} = A_q(\phi,\psi), \quad T_+ u_{\phi,\psi} = (\phi,\psi) + A_q(\phi,\psi).$$

Thus we get that $u = u_{\phi,\psi} \in (e_\rho)^{-1} Y^{-k,s}_\delta$ and it is the unique solution of

$$(\Delta + q)u = g_{\phi,\psi} = \psi\delta_{\partial\Omega} + \nabla\cdot(n\phi\delta_{\partial\Omega}) \quad \text{in} \quad \mathbb{R}^n, \tag{3.7}$$

satisfying $e_\rho(\cdot)u \in Y^{-k,s}_\delta(\mathbb{R}^n)$.

Now, if $(\phi,\psi) \in \mathrm{Ker}(A_q)$ we have that

$$(\phi,\psi) = (\phi,\psi) + T_- u_{\phi,\psi} = T_+ u_{\phi,\psi}.$$

Thus, $v = u_{\phi,\psi}$ is the solution of the scattering problem

$$\Delta v = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega, \quad T_+ v = (\phi,\psi), \quad e_\rho(\cdot)v \in W^{s,2}_\delta(\mathbb{R}^n \setminus \Omega). \tag{3.8}$$

On other hand, assume that (3.8) has a solution, and let v_0 be the zero-continuation of v, that is $v_0|_{\mathbb{R}^n\setminus\Omega} = v$, $v|_\Omega = 0$. Then we conclude that v_0 is a solution of the problem (3.7), and as this solution is unique, we see that $v_0 = u_{\phi,\psi}$. This shows that $(\phi,\psi) \in \mathrm{Ker}(A_q)$ if and only if the problem (3.8) has a solution. This is obviously independent of q and thus we see that

$$T_- + (e_\rho + u_{\phi,\psi}) = (\phi_\rho + \phi, \psi_\rho + \psi) + A_q(\phi,\psi) \in \mathrm{Ran}(A_q). \tag{3.9}$$

Applying the projection $I + A_q$ to both sides of (3.9) and using $A_q(\phi,\psi) = 0$, we see that

$$0 = (I + A_q)(\phi_\rho + \phi, \psi_\rho + \psi) = (I + A_q)(\phi_\rho, \psi_\rho) + (\phi,\psi).$$

As A_q and (ϕ_ρ,ψ_ρ) are known, we can thus determine (ϕ,ψ) and the Cauchy data of $v(x)$ on $\partial\Omega$.

Next we consider range of A_q. A standard application of Green's formula (see, e.g., [8, Th. 3.1]) shows that if $v \in X^{p,r}$ satisfies

$$(\Delta + q)v = 0 \quad \text{in} \quad \Omega,$$

and $(\phi,\psi) = T_- v$, then $v = -u_{\phi,\psi}$. (Observe the negative sign which is due to the fact that we use exterior normal vector n.) Also, by for $(\phi,\psi) \in Z$ we have $\chi_0 u_{\phi,\psi} \in C^l_* I^{-k,\vec{\delta}_0}(H) \subset L^{t_1}(\Omega)$ for any $t_1 < \infty$ by Prop. 2.2. As $q \in I^\mu(H) \subset L^{t_2}(\Omega)$ for $1 < t_2 < \frac{k}{k+\mu}$ we have $\chi_0 \Delta u_{\phi,\psi} = -\chi_0 q u_{\phi,\psi} \in L^{p'}(\Omega)$ for $\frac{1}{p} < 1 - \frac{1}{t_2}$, i.e. $p > -\frac{k}{\mu}$. Hence $u_{\phi,\psi} \in X^{p,r}$. Thus the set of all solutions of the Schrödinger equation in $X^{p,r}$ equals to the set of solutions $u_{\phi,\psi}$, $(\phi,\psi) \in Z$. As $T_- u_{\phi,\psi} = A_q(\phi,\psi)$, we obtain that the range of A_q equals to $\mathcal{CD}_q$.

Now, when the potential is equal to zero, the Dirichlet-to-Neumann operator $\Lambda_0 : u|_{\partial\Omega} = \partial_n u|_{\partial\Omega}$ is well defined, $\Lambda_0 : W^{2-\frac{1}{r},r}(\partial\Omega) \to W^{1-\frac{1}{r},r}(\partial\Omega)$. The Cauchy data $\mathcal{CD}_0$ is the graph of the operator Λ_0 and is thus closed. Thus we see that the range of A_0 is a closed subspace, and therefore the operator $A_0 : Z/\mathrm{Ker}(A_0) \to Z$ has zero kernel and closed range. Thus it is a semi-Fredholm operator. Now, consider the operator $A_q - A_0$. Using (3.9) we know that the operator

$$A_q - A_0 : Z/\mathrm{Ker}(A_0) \to Z$$

is well defined and compact. As compact perturbations of semi-Fredholm operators are also semi-Fredholm, we conclude that A_q is also semi-Fredholm.

It remains to show that $-A_q$ is a projection. This can be seen similarly to the smooth case. Indeed, if $(\phi, \psi) \in \mathrm{Ran}(A_q)$, $(\phi, \psi) = A_q(\tilde{\phi}, \tilde{\psi})$ we see that the solution $u_{\tilde{\phi},\tilde{\psi}}$ has the trace $T_- u_{\tilde{\phi},\tilde{\psi}} = (\phi, \psi)$. Hence Green's formula gives

$$u_{\tilde{\phi},\tilde{\psi}} = -(-S_q\phi + K_q\psi) \quad \text{in} \quad \Omega.$$

Taking trace T_- from both sides we obtain that $(\phi, \psi) = -A_q(\phi, \psi)$, i.e. $(-A_q)^2 = -A_q$. Thus, Prop. 3.2 is proven. $\square$

Now we can construct the boundary values of the exponentially growing solutions from the Cauchy data. As we are given $\mathcal{CD}_q = \mathrm{Ran}(A_q)$, and we know $\mathrm{Ker}(A_q) = \mathrm{Ker}(A_0)$, we can construct the projection $-A_q$. Next, let $(\phi_\rho, \psi_\rho) = T_+ e_\rho$ be the boundary values of the incoming plane wave. Consider the solution $v(x) = e^{\rho\cdot x}(1 + \psi(x, \rho)) = e^{\rho\cdot x} + u_0$ and let $(\phi, \psi) = T_+ u_0$. Then $(\phi, \psi) \in \mathrm{Ker}(A_q)$ and $u_0 = u_{\phi,\psi}$ in $\mathbb{R}^n \setminus \Omega$. Moreover, as v is solution of Schrödinger equation inside Ω, we have

$$T_+(e_\rho + u_{\phi,\psi}) = (\phi_\rho + \phi, \psi_\rho + \psi) + A_q(\phi, \psi) \in \mathrm{Ran}(A_q). \tag{3.9}$$

Applying with projection $I + A_q$ to (3.9) and using $A_q(\phi, \psi) = 0$, we see that

$$0 = (1 + A_q)(\phi_0 + \phi, \psi_0 + \psi) = (1 + A_q)(\phi_0, \psi_0) + (\phi, \psi).$$

As A_q and (ϕ_0, ψ_0) are known, we find (ϕ, ψ) and the Cauchy data of $v(x)$ on $\partial\Omega$.

So far, we have constructed the Cauchy data of the solutions $v_{\rho_1}(x) = e^{\rho_1\cdot x}(1 + \psi(x, \rho_1))$ for all sufficiently large ρ_1. Thus if we consider complex frequencies ρ_1 and ρ_2 satisfying $\rho_1 + \rho_2 = -i\xi$, with $\xi \in \mathbb{R}^n \setminus 0$, an application of Green's formula yields

$$\begin{aligned}\hat{q}(\xi) &= \lim_{|\rho_1|\to\infty} \int_\Omega q(x) e^{\rho_1\cdot x}\Big(1 + \psi_1(x, \rho_1)\Big) \cdot e^{\rho_2\cdot x}\, dx \\ &= \lim_{|\rho_1|\to\infty} \int_{\partial\Omega} \Big(v_{\rho_1} \cdot \partial_n e^{\rho_2\cdot x} - \partial_n v_{\rho_1} \cdot e^{\rho_2\cdot x}\Big)\, dx\end{aligned}$$

This proves Theorem 2. $\square$

4. Non-uniqueness for highly singular potentials

We next discuss how very strong singularities of the potential can cause non-uniqueness in a closely related inverse problem. Due to the strength of the singularities, the Schrödinger equation has to be interpreted in a weak sense. Let us consider the boundary value problem

$$(\Delta + q + E)u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = f \tag{4.1}$$

with the potential q having the form

$$q(x) = -\text{dist }(x, H)^{\mu} c_0(x), \tag{4.2}$$

near H, where dist is the Euclidean distance, H is a closed hypersurface bounding a region $\Omega_0 \subset\subset \Omega$, $\mu < -2$, and $c_0(x)$ is a smooth function, satisfying $c_0(x) > C_0 > 0$ in some neighborhood V of H.

Elements of $I^{-1-\mu}(H)$ satisfy the pointwise estimate $|q(x)| \leq C\text{dist }(x, H)^{\mu}$, but a q satisfying (4.2) is not even locally integrable and thus need not define a distribution. Hence, the solutions of (4.1) cannot be formulated in the usual sense of distributions. Instead, we define the solution of (4.1) (if it exists) to be the solution of the following convex minimization problem: Find u such that

$$G(u) = \inf G(v) \tag{4.3}$$

where $G = G_{q+E} : \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = f\} \to \mathbb{R} \cup \{\infty\}$ is the convex functional

$$G_{q+E}(v) = \int_{\Omega} (|\nabla v(x)|^2 - (q(x) + E)|v(x)|^2) dx$$

Here, since the function $-q(x)$ is bounded from below, we define $G(u) = \infty$ when $q|v|^2$ is not in $L^1(\Omega)$.

Proposition 4.1. *The Cauchy data*

$$\mathcal{CD}_{q+E} = \Big\{(u|_{\partial\Omega}, \frac{\partial u}{\partial n}|_{\partial\Omega}) : u \in H^1(\Omega), \ u \text{ is a minimizer of } G_{q+E}\}.$$

does not depend on q in Ω_0. In particular, if the solution of (4.3) is unique, u vanishes identically in Ω_0.

Remark. We note that potentials having singularities similar to (4.2) as above has been used to produce counterexamples to strong unique continuation, e.g. potentials $q(x) = c/|x|^{2+\varepsilon}$ in [11]. Recently, counterexamples have been found for weak unique continuation for L^1-potentials [18], but here we need to construct potentials for which *all* solutions vanish inside H. Finally, we wish to emphasize that since the solutions of (4.1) considered here are not defined in the usual sense of distributions, but rather as solutions of a convex minimization problem, the solutions we construct do not give new counterexamples for the unique continuation problem.

Proof. Obviously we can assume that $q(x) \leq 0$ everywhere. We start first with the case where $E = 0$ and $f \in C^{\infty}(\partial\Omega)$.

As the potential q is not in the Kato class ([6,p.62]), consider instead a decreasing sequence of smooth functions $q_n \in C^{\infty}(\Omega)$, $q_{n+1}(x) \leq q_n(x)$, for which $q_n(x) = q(x)$ when $d(x, H) > \frac{1}{n}$ and in some neighborhood V of H

$$q_n(x) \leq \max(-c_1 n^{-\mu}, q(x)) \tag{4.4}$$

where $0 < c_1 < C_0$. Let G_n be the functionals defined as G with q replaced with q_n. The functionals G_n have unique minimizers u_n which satisfy in classical sense

$$(\Delta + q_n)u_n = 0 \quad \text{in} \quad \Omega, \quad u_n|_{\partial\Omega} = f. \tag{4.5}$$

Now, let $f \in C^\infty(\partial\Omega)$ be fixed. Let $F \in H^1(\Omega)$ be a function for which $F|_{\partial\Omega} = f$ and $F = 0$ in some neighborhood of H. By definition of the potentials q_n, for sufficiently large n_0 we have $G(F) = G_n(F) = G_{n_0}(F)$ for $n \geq n_0$. Thus for the minimizers u_n of G_n we have $G_n(u_n) \leq C = G_{n_0}(F)$. Next, by choosing a subsequence, we can assume that the sequences $\int |\nabla u_n(x)|^2 dx$ and $\int (-q_n(x))|u_n(x)|^2 dx$ are decreasing when $n \to \infty$. Next, let us denote by $C_1, C_2 \leq C$ the constants

$$C_1 = \lim_{n\to\infty} \int_\Omega |\nabla u_n(x)|^2 dx, \quad C_2 = \lim_{n\to\infty} \int_\Omega (-q_n(x))|u_n(x)|^2 dx = C_2.$$

Now, we see that u_n are uniformly bounded in $H^1(\Omega)$ and thus by choosing a subsequence we can assume that there is $\tilde{u} \in H^1(\Omega)$ such that $u_n \to \tilde{u}$ weakly in $H^1(\Omega)$. Moreover,

$$\int_\Omega |\nabla \tilde{u}(x)|^2 dx \leq C_1 \tag{4.6}$$

As compact operators map weakly converging sequences to strongly converging ones, we have the norm-convergence $u_n \to \tilde{u}$ in $L^2(\Omega)$. Thus for $n_0 > 0$ we have

$$-\int_\Omega q_{n_0}(x)|\tilde{u}(x)|^2 dx = \lim_{n\to\infty} \int_\Omega (-q_{n_0}(x))|u_n(x)|^2 dx \leq$$

$$\leq \lim_{n\to\infty} \int_\Omega (-q_n(x))|u_n(x)|^2 dx \leq C_2.$$

Since this is valid for any n_0 we have by monotone convergence theorem

$$-\int_\Omega q(x)|\tilde{u}(x)|^2 dx \leq C_2. \tag{4.7}$$

As $G \leq G_n$,

$$\inf G(u) \leq \lim_{n\to\infty} \min G_n(u) = C_1 + C_2.$$

Now, by (4.6) and (4.7) we have $G(\tilde{u}) \leq C_1 + C_2$ and thus $\tilde{u}$ is a minimizer of G. Now, as $G = G_n + (G - G_n)$ where G_n is a strictly convex functional and $G - G_n$ is a convex functional, G is strictly convex. Thus the minimizer is unique. Hence we see that, for every f, the solution $\tilde{u}$ of the minimization problem (4.3) exists, is unique, and is given as the L^2-limit of the functions u_n. We note that the above analysis was based to the fact that the minimization problems for the G_n epi-converge to the minimization problem for G [30].

Recalling that Ω_0 is the region bounded by H, consider functions u_n restricted to Ω_0. Let $t \mapsto B_t$ be the Brownian motion in $\mathbb{R}^n$ starting from x at time $t = 0$, i.e., $B_0 = x$. As the q_n are strictly negative smooth functions, they are in the Kato class and the pair (Ω, q_n) is gaugeable (see [6], sect. 4.3 and Th. 4.19). By [6], Th. 4.7, the solution u_n can be represented by the Feynman-Kac formula

$$u_n(x) = E(\exp\left(\int_0^\tau q_n(B_t)dt\right) f(B_\tau))$$

where $\tau = \tau_{\partial\Omega}$ is the first time when the process hits the boundary, i.e., $B_t \in \partial\Omega$. Here, we assume B_t is a version of Brownian motion for which all realizations are continuous curves (see [21] or [6], Th. 1.4). If $x \in \Omega_0$, the realizations of Brownian motion have to hit H prior to hitting $\partial\Omega$. Denote the first hitting time for H by τ_H; thus the first hitting point is B_{τ_H}, and $\tau_H < \tau_{\partial\Omega}$. (The stopping time τ_H is measurable function in the probability space, see [6,Prop. 1.15]).

Let us now denote by $p(\rho, \eta)$ the probability that the Brownian motion sent from origin at time $t = 0$ leaves the origin centered ball with radius ρ before time η. Because of the scale-invariance of Brownian motion, $p(s\rho, s^2\eta) = p(\rho, \eta)$ for $s > 0$. (Indeed, let us consider reparametrized Brownian motion $\tilde{B}_t = sB_{s^{-2}t}$. As the probability densities of $(\tilde{B}_{t_1}, \tilde{B}_{t_2}, \dots, \tilde{B}_{t_m})$ coincide to those of $(B_{t_1}, B_{t_2}, \dots, B_{t_m})$ we see that we see that $\tilde{B}_t$ is Brownian motion, too.)

Let $A_{\rho,\eta} = \{|B_t - B_{\tau_H}| < \rho \text{ for } \tau_H \le t < \tau_H + \eta\}$. This set is measurable in the probability space and the probability of $A_{\rho,\eta}$ is $P(A_{\rho,\eta}) = 1 - p(\rho, \eta)$.

Let $m > 1$ and $\eta = \eta(m)$ be such that $p(1, \eta) \ge \frac{m-1}{m}$. Now, q is non-positive and by (4.4) $q_n(x) < \max(-c_1 n^{-\mu}, q(x))$ in some neighborhood V of H. When s is so small that the s-neighborhood of H is in V, we have by (4.2) that

$$|E(\exp\left(\int_0^\tau q(B_t)dt\right) f(B_\tau))| \le E(\exp\left(\int_{\tau_H}^{\tau_H + s^2\eta} q(B_t)dt\right) ||f||_{L^\infty})$$

$$\le (1 - P(A_{s\rho, s^2\eta(m)}))||f||_\infty + P(A_{s\rho, s^2\eta(m)})\exp\bigg(- s^2\eta(m)\min(C_0 s^\mu, c_1 n^{-\mu})\bigg)||f||_\infty.$$

Thus, choosing $s = n^{2/\mu}$ we see that for sufficiently large n

$$(4.8) \qquad ||u_n||_{L^\infty(\Omega_0)} \le (\frac{1}{m} + \frac{m-1}{m}\exp\left(-\eta(m) c_1 n^{2(2+\mu)/\mu}\right) ||f||_{L^\infty}.$$

As $u_n \to \tilde{u}$ in norm in $L^2(\Omega_0)$, $||\tilde{u}||_{L^2(\Omega_0)} \le \frac{1}{m}||f||_{L^\infty} \operatorname{vol}(\Omega_0)^{1/2}$ for any m. Thus we see that $\tilde{u} = 0$ in Ω_0.

Next we consider the case when $E \in \mathbb{R}$ and $f \in H^{1/2}(\partial\Omega)$. First, let $H_r = \{x \in \Omega : \text{ dist } (x, H) < r\}$ and let r be so small that $q(x) + E < 0$ for $x \in H_r$. If u is the solution of (4.1) in Ω, then its restriction $\tilde{u} = u|_{H_r}$ is the solution of boundary value problem

$$(4.9) \qquad (\Delta + q + E)\tilde{u} = 0 \quad \text{in} \quad H_r, \quad \tilde{u}|_{\partial H_r} = \tilde{f},$$

where $\tilde{f} = u|_{\partial H_r} \in C^\infty(\partial H_r)$, that is, $\tilde{u}$ is the solution of minimization problem (4.3) in domain H_r. Let q_n approximate q in H_r as above and $\tilde{u}_n$ be the corresponding solutions of problem (4.9) with q replaced with q_n. As above, we see that problem (4.9) is uniquely solvable, $\tilde{u}_n \to \tilde{u}$ in $L^2(H_r)$, and that $\tilde{u}_n(x)$ can be represented by the Feynman-Kac formula. Let $x \in \Omega_0 \cap H_r$, $\tilde{\tau}$ be the first time when the Brownian motion sent from x at $t = 0$ hits ∂H_r, and $\tilde{A} = \{B_t \in \Omega_0 \cap H_r \text{ for } 0 \le t < \tilde{\tau}\}$. Let us denote $\tilde{f} = \tilde{f}_+ + \tilde{f}_-$, where $\tilde{f}_+$ vanishes on $\Omega_0 \cap \partial H_r$ and $\tilde{f}_-$ vanishes on $(\Omega \setminus \Omega_0) \cap \partial H_r$. Then we see that

$$\tilde{u}_n(x) = P(\tilde{A})E(\exp\left(\int_0^{\tilde{\tau}} q_n(B_t)dt\right) \tilde{f}_-(B_{\tilde{\tau}})|\tilde{A}) +$$

$$(4.10) \qquad +(1-P(\tilde{A}))E(\exp\left(\int_0^{\tilde{\tau}} q_n(B_t)dt\right)\tilde{f}(B_{\tilde{\tau}})|\tilde{A}^c)$$

where $E(\cdot|\tilde{A})$ is conditional expectation with condition $\tilde{A}$ and $\tilde{A}^c$ denotes the complement of $\tilde{A}$. Note that in the case of $\tilde{A}^c$, the process B_t hits H at least once. Analyzing how long Brownian motion is near H as above, we see that when $n \to \infty$, the second term on the right hand side of (4.10) goes to zero. Thus $\tilde{u}(x)$, for $x \in H_r \cap \Omega_0$, depends only on q in $H_r \cap \Omega_0$ and f_-. Similarly, we see that $\tilde{u}(x)$, $x \in H_r \setminus \overline{\Omega}_0$, depends only on q in $H_r \setminus \overline{\Omega}_0$ and f_+. Moreover, analogously to (4.8) we see that

$$\lim_{n\to\infty} ||\tilde{u}_n|_H||_{L^\infty(H)} = 0.$$

Choosing a subsequence, we can assume that $\tilde{u}_n \to \tilde{u}$ weakly in $H^1(H_r)$ and thus in norm in $H^{3/4}(H_r)$. Hence, by taking the trace $H^{3/4}(H_r) \to L^2(H)$ we see that $\tilde{u}|_H = 0$.

In conclusion, for the boundary value problem (4.9) there are well defined maps

$$T_+ : f_+ \mapsto \tilde{u}|_{H_r\setminus\overline{\Omega}_0} \in \{v \in H^1(H_r \setminus \overline{\Omega}_0) : \ v|_H = 0\},$$

$$T_- : f_- \mapsto \tilde{u}|_{H_r\cap\Omega_0} \in \{v \in H^1(H_r \cap \Omega_0) : \ v|_H = 0\}$$

where T_+ depends only on q in $H_r \setminus \overline{\Omega}_0$ and T_- on q in $H_r \cap \Omega_0$.

In particular, on the boundaries $\partial H_r \cap \Omega_0$ and $\partial H_r \cap (\Omega \setminus \Omega_0)$ we have "independent" Dirichlet-to-Neumann maps

$$\Lambda_+ : f_+ \mapsto \partial_n \tilde{u}|_{\partial H_r\setminus\overline{\Omega}_0}, \quad \Lambda_- : f_- \mapsto \partial_n \tilde{u}|_{\partial H_r\cap\Omega_0},$$

where n is the exterior normal of H_r.

Next, if u is a solution of boundary value problem (4.1) we denote $u_+ = u|_{\Omega\setminus\overline{\Omega}_0}$ and $u_- = u|_{\Omega_0}$. To motivate the next step, we observe that u_+ and u_- satisfy "independent" boundary value problems in $\Omega \setminus (H_r \cup \Omega_0)$

$$(\Delta + q + E)u_+ = 0, \quad u_+|_{\partial H_r} = \tilde{f}, \quad \partial_n u_+|_{\partial H_r\setminus\Omega_0} = \Lambda_+(u_+|_{\partial H_r\setminus\Omega_0})$$

and in $\Omega_0 \setminus H_r$

$$(\Delta + q + E)u_- = 0, \quad \partial_n u_-|_{\partial H_r\cap\Omega_0} = \Lambda_-(u_-|_{\partial H_r\cap\Omega_0}).$$

Now, considering the form of G and the fact that the solution u of boundary value problem (4.1) satisfies $u|_H = 0$, we see that $u_+ = u|_{\Omega\setminus\overline{\Omega}_0}$ is a minimizer of G in the set $\{v \in H^1(\Omega \setminus \overline{\Omega}_0) : \ v|_{\partial\Omega} = f, \ v|_H = 0\}$ and $u_- = u|_{\Omega_0}$ is a minimizer of G in the set $\{v \in H^1(\Omega_0) : \ v|_{\partial\Omega_0} = 0\}$.

Conversely, if $U = v_+$ in $\Omega \setminus \overline{\Omega}_0$ and $U = v_-$ in Ω_0 where v_+ and v_- are any minimizers of G in the sets $\{v \in H^1(\Omega \setminus \overline{\Omega}_0) : \ v|_{\partial\Omega} = f, \ v|_H = 0\}$ and $\{v \in H^1(\Omega_0) : \ v|_{\partial\Omega_0} = 0\}$, respectively, then U is solution of (4.1).

In particular, we see that the Cauchy data of solutions u of (4.1) on $\partial\Omega$ are independent of $u|_{\Omega_0}$ and thus of q inside H. This finishes the proof of Prop. 4.1. As a concluding remark we note that by using the Courant-Hilbert min-max principle, we see that there always are values of E such that minimization problem for v_- has non-zero solutions, that is, there are eigenstates U which have vanishing Cauchy data on $\partial\Omega$. $\Box$

Physically, this example has the following interpretation: In theory it is possible to construct a potential wall $q(x)$ such that no particles can "tunnel" through it, using an analogy with quantum mechanics. Thus exterior observers can make no conclusions about the existence of objects or structures inside this wall. Moreover, inside H the solution can be in an eigenstate and its Cauchy data vanishes on the boundary of Ω. Thus, making another analogy with quantum mechanics, in this nest the Schrödinger cat could live happily forever.

References

[1]. R. Beals and R. Coifman, *The spectral problem for the Davey-Stewarson and Ishimori hierarchies*, in Nonlinear evolution equations: Integrability and spectral methods, Manchester Univ. Press, 1988, 15–23.

[2]. R. Brown, *Global uniqueness in the impedance-imaging problem for less regular conductivities*, SIAM J. Math. Anal. **27** (1996), 1049–1056.

[3]. R. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions.*, Comm. PDE **22** (1997), 1009–1027.

[4]. S. Chanillo, *A problem in electrical prospection and an n-dimensional Borg-Levinson theorem*, Proc. Amer. Math. Soc. **108** (1990), 761–767.

[5]. J. Chazarain and A. Piriou, *Introduction to the theory of linear partial differential equations. (Translated from the French.) Studies in Mathematics and its Applications*, vol. 14, North-Holland, 1982.

[6]. K.L. Chung and Z.X. Zhao, *From Brownian motion to Schrödinger's equation, Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, 1995.

[7]. R. Coifman, A. McIntosh and Y. Meyer, *L'integrale de Cauchy sur les courbes lipschitziennes*, Ann. of Math. **116** (1982), 361–387.

[8]. D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory. Second edition. Applied Mathematical Sciences*, vol. 93, Springer-Verlag, 1998.

[9]. J.J. Duistermaat and L. Hörmander, *Fourier integral operators,II*, Acta math. **128** (1972), 183–269.

[10].L.D. Faddeev, *Growing solutions of the Schrödinger equation*, (in Russian), Dokl. Akad. Nauk SSR **165**, 514–517.

[11].N. Garofalo and F.-H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35**, 245–268.

[12].A. Greenleaf and G. Uhlmann, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, Jour.Func.Anal. **89** (1990), 202–232.

[13]______, *Recovering singularities of a potential from singularities of scattering data*, Comm. Math. Phys. **157** (1993), 549–572.

[14].V. Guillemin and G. Uhlmann, *Oscillatory integrals with singular symbols*, Duke Math. Jour. **48** (1981), 251–267.

[15].L. Hörmander, *Fourier integral operators,I*, Acta math. **127** (1971), 79–183.

[16]______, *The Analysis of Linear Partial Differential Operators,I-IV*, Springer-Verlag, 1985.

[17].V. Isakov, *On uniqueness of recovery of a discontinuous conductivity coefficient*, Comm. Pure Appl. Math. **41** (1988), 865–877.

[19].K. Knudsen and A. Tamasan, *Reconstruction of less regular conductivities in the plane*, arXiv:math.FA/0110298 (Oct. 2001).

[18].C. Kenig and N. Nadirashvili, *A counterexample in unique continuation*, Math. Res. Lett. **7** (2000), 625–630.

[20].R. Kohn and R. Vogelius, *Determining conductivity by boundary measurements, II: Interior results*, Comm. Pure Appl. Math. **38** (1985), 643-667.

[21].P. Levy, *Processus stochastiques et mouvement brownien. (French)*, Gauthier-Villars, 1965.

[22].R. Melrose, *Marked lagrangians*, Unpublished notes, MIT.

[23].R. Melrose and N. Ritter, *Interaction of nonlinear progressing waves for semilinear wave equations*, Ann. of Math. **121** (1985), 187–213.

[24].R. Melrose and G. Uhlmann, *Lagrangian intersection and the Cauchy problem*, Comm. Pure Appl. Math. **32** (1979), 482–512.

[25].A. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. **128** (1988), 531–576.

[26].A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math. **143** (1996), 71–96.

[27].A. Nachman, J. Sylvester and G. Uhlmann, *An n-dimensional Borg-Levinson theorem*, Comm. Math. Phys. **115** (1988), 595–605.

[28].L. Päivärinta, A. Panchencko and G. Uhlmann, *Complex geometrical optics solutions for Lipschitz conductivities*, , to appear, Revista Matematica Iberoamericana.

[29].A. Piriou, *Calcul symbolique non lineare pour une onde conormale simple*, Ann. Inst. Four. **38** (1988), 173–188.

[30].R.T. Rockafellar and R. Wets, *Variational analysis. Grundlehren der Mathematischen Wissenschaften 317*, Springer-Verlag, 1998.

[31].C. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press, New York, 1993.

[32].J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125** (1987), 153–169.

[33].M. Taylor, *Pseudodifferential operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.

Department of Mathematics, University of Rochester, Rochester, NY 14627

Rolf Nevanlinna Institute, P.O. Box 4, 00014 University of Helsinki, Finland

Department of Mathematics, University of Washington, Seattle, WA 98195