Local uniqueness for the Dirichlet-to-Neumann map via the two-plane transform

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Revised July 20, 2000

Abstract

We consider the Dirichlet-to-Neumann map associated to the Schrödinger equation with a potential on a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$. We show that the integral of the potential over a two-plane Π is determined by the restriction of the Dirichlet-to-Neumann map to any open subset $\mathcal{U} \subset \partial\Omega$ which contains $\Pi \cap \partial\Omega$.

0 Introduction

For Ω a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $\partial\Omega$, and $q(x) \in L^\infty(\Omega)$, let

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega) \tag{0.1}$$

be the Dirichlet-to-Neumann map associated with the operator $\Delta + q$ on Ω. (We assume throughout that $\lambda = 0$ is not a Dirichlet eigenvalue for $\Delta + q$ on Ω). If Ω and $q(x)$ are C^∞, then Λ_q is a first order ΨDO, with an integral kernel K_q:

$$\Lambda_q f(x) = \int_{\partial\Omega} K_q(x,y) f(y) d\sigma(y), x \in \partial\Omega. \tag{0.2}$$

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[*] Partially supported by an NSF grant.

[†] Partially supported by an NSF grant and the Royal Research Fund at the University of Washington

This paper is concerned with the problem of obtaining partial knowledge of $q(x)$ from partial knowledge of Λ_q (or K_q), namely its restriction to certain open subsets of the boundary. The approach taken here is to use concentrated, exponentially growing, approximate solutions to relate Λ_q on an open $\mathcal{U} \subset \partial\Omega$ to the two-plane transform of the potential $q(x)$ on two-planes whose intersections with $\partial\Omega$ are contained in $\mathcal{U}$. If $\mathcal{U} \subset \partial\Omega$ is open, then we will say that $\Lambda_{q_1} = \Lambda_{q_2}$ on $\mathcal{U}$ if

$$(0.3) \qquad (\Lambda_{q_1} f, g) = (\Lambda_{q_2} f, g), \quad \forall f, g \in H^{\frac{1}{2}}(\partial\Omega), \quad supp(f), supp(g) \subset \mathcal{U}.$$

Now let $M_{2,n}$ denote the $(3n-6)$-dimensional Grassmannian of all affine two-planes $\Pi \subset \mathbb{R}^n$, and

$$(0.4) \qquad R_{2,n} f(\Pi) = \int_\Pi f(y) d\lambda_\Pi(y), f \in L^2(\mathbb{R}^n),$$

the two-plane transform on $\mathbb{R}^n$ [H65, H80]. Here, $d\lambda_\Pi$ denotes two-dimensional Lebesgue measure on $\Pi \in M_{2,n}$, which can be defined by

$$(0.5) \qquad < f, d\lambda_\Pi >= \lim_{\epsilon\to 0} \frac{1}{|B^{n-2}(0;\epsilon)|} \int_{\{dist(x,\Pi)<\epsilon\}} f(x) dx.$$

(Note that for $n = 3$, $R_{2,3}$ is just the usual Radon transform on $\mathbb{R}^3$.) We will also need the variant of $d\lambda_\Pi$ defined relative to Ω:

$$(0.6) \qquad < f, d\lambda_\Pi^\Omega >= \lim_{\epsilon\to 0} \frac{1}{|B^{n-2}(0;\epsilon)|} \int_{\Omega\cap\{dist(x,\Pi)<\epsilon\}} f(x) dx,$$

which gives rise to a two-plane transform relative to Ω,

$$(0.7) \qquad R_{2,n}^\Omega f(\Pi) = \int_\Pi f(y) d\lambda_\Pi^\Omega(y).$$

Note that if $\partial\Omega$ is C^1 and $\Pi \cap \partial\Omega$ transversally, then $< d\lambda_\Pi^\Omega, f >=< d\lambda_\Pi, f \cdot \chi_\Omega >$ and $R_{2,n}^\Omega f(\Pi) = R_{2,n}(f \cdot \chi_\Omega)(\Pi)$.

For each $\Pi \in M_{2,n}$, let $\gamma_\Pi = \Pi \cap \partial\Omega \subset \partial\Omega$. The main result proved here is

Theorem 1 *Let $n \geq 3$. Assume $\partial\Omega$ is $C^{2,1}$ and potentials $q_1(x)$ and $q_2(x)$ are in the Sobolev space $H^s(\Omega)$, for $s > \frac{n}{2}$ if $n \geq 4$ and $s = 2$ if $n = 3$. Let $\Pi \in M_{2,n}$. If $\Lambda_{q_1} = \Lambda_{q_2}$ on some neighborhood of γ_Π in $\partial\Omega$, then*

$$(0.8) \qquad R_{2,n}^\Omega (q_1 - q_2)(\Pi) = 0,$$

i.e., $\int q_1(y) d\lambda_\Pi^\Omega(y) = \int q_2(y) d\lambda_\Pi^\Omega(y)$.

If $\Lambda_{q_1} = \Lambda_{q_2}$ on all of $\partial\Omega$, then, at least for a $C^{2,1}$ boundary, this implies that $R_{2,n}((q_1 - q_2)\chi_\Omega)(\Pi) = 0$, $\forall\, \Pi \in M_{2,n}$, which by the uniqueness theorem for $R_{2,n}$ yields that $q_1 - q_2 \equiv 0$ on Ω, providing a new proof of the global uniqueness theorem of [SU87a], under the regularity assumptions of Thm. 1, for $n \geq 3$. (We note that our technique is limited to three or more dimensions and says nothing in the case $n = 2$ [N96].) One is able to obtain local results by replacing the uniqueness theorem for the two-plane transform with Helgason's support theorem [H80, Cor. 2.8]: if $C \subset \mathbb{R}^n$ is a closed, convex set and $f(x)$ a function[1] such that $R_{2,n}f(\Pi) = 0$ for all Π disjoint from C, then $\text{supp}(f) \subset C$. We then immediately obtain the following two results.

Theorem 2 *Suppose $\partial\Omega$ is $C^{2,1}$ and $C \subset \Omega$ is a closed, convex set and potentials q_1, q_2 are as in Thm. 1. If $\Lambda_{q_1} = \Lambda_{q_2}$ on a neighborhood of γ_Π for all $\Pi \in M_{2,n}$ such that $\Pi \cap C = \phi$, then* $\text{supp}(q_1 - q_2) \subseteq C$, *i.e., $q_1 = q_2$ on $\Omega \backslash C$.*

Theorem 3 *Suppose $\partial\Omega$ is $C^{2,1}$ and strictly convex, and potentials q_1, q_2 are as in Thm. 1. If, for some $r > 0$, $\Lambda_{q_1} = \Lambda_{q_2}$ on all balls $B^n(x_0; r) \cap \partial\Omega \subset \partial\Omega$, then*

$$\text{dist}(\text{supp}(q_1 - q_2), \partial\Omega) \geq Cr^2,$$

i.e., $q_1 = q_2$ on the tubular neighborhood $\{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) \leq Cr^2\}$ of $\partial\Omega$ in $\overline{\Omega}$.

Remarks

(i) The conclusions of Thms. 2 and 3 can be strengthened by combining them with a result in Isakov [Is]. Namely, if either $C \subset\subset \Omega$ in Thm. 2, or the assumption of Thm. 3 holds for some $r > 0$, we can conclude from Thm. 2 or 3 that *supp*$(q_1 - q_2) \subset\subset \Omega$. By Ex. 5.7.4 in [Is], based on a technique of Kohn and Vogelius[KV85] , this, together with the condition that $\Lambda_{q_1} = \Lambda_{q_2}$ on a neighborhood of some point $x_0 \in \partial\Omega$ (which is weaker than the assumptions in Thms. 2 and 3), implies that $q_1 \equiv q_2$ everywhere on Ω. We are indebted to Adrian Nachman for pointing this out to us.

(ii) Under the same conditions on Ω we can consider an electrical conductivity on Ω, which we denote by γ and assume is in $H^{s+2}(\Omega)$ with s as in

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[1] The support and uniqueness theorems are usually stated under the assumption that $f(x)$ is continuous, of rapid decay in the case of the support theorem, but the proofs in [H80] are easily seen to extend to the case where $f(x) = q(x)\chi_\Omega(x)$ with $\Omega \subset \mathbb{R}^n$ bounded, $q \in C(\overline{\Omega})$.

Thm. 1. We can define the Dirichlet-to-Neumann map Λ_γ associated with the operator $L_\gamma :=$ div $(\gamma\nabla)$. We denote the integral kernel of Λ_γ by K_γ. Since

$$L_\gamma(\gamma^{-1/2}u) = \gamma^{1/2}(\Delta - q)(u)$$

with $q = \frac{\Delta(\sqrt{\gamma})}{\sqrt{\gamma}}$, we have that

$$\Lambda_q f = (\gamma^{-1/2}|_{\partial\Omega})\Lambda_\gamma(\gamma^{-1/2}f) + \Big(\frac{1}{2}\gamma^{-1/2}\frac{\partial\gamma}{\partial n}\Big)|_{\partial\Omega} \cdot f,$$

where n denotes the unit outer normal to $\partial\Omega$. It was proven by Kohn and Vogelius [KV84] that if we know K_γ on $\mathcal{U} \times \mathcal{U}$ where $\mathcal{U}$ is an open subset of $\partial\Omega$ then we can recover $\partial^\alpha\gamma, \alpha \leq 1$ on $\mathcal{U}$. (See [SU88] for another proof; in [N88] this is also considered for the case of Lipschitz boundaries.) Therefore the study of the local Dirichlet-to-Neumann map for the conductivity equation is reduced to the study of the local Dirichlet-to-Neumann map for the Schrödinger equation and Theorems 1,2,3 and the remarks following are valid with q_i replaced by $\frac{\Delta(\sqrt{\gamma}_i)}{\sqrt{\gamma}_i}, i = 1, 2$.

The authors would like to thank Alexander Bukhgeim for pointing out an error in an earlier version of this paper.

1 Approximate solutions

To prove Thm. 1, we make use of exponentially growing approximate solutions. As in [C, SU86, SU87a], let

$$\mathcal{V} = \{\rho \in \mathbb{C}^n : \rho \cdot \rho = 0\}$$

be the (complex) characteristic variety of Δ. Each $\rho \in \mathcal{V}$ can be written as $\rho = |\rho|\frac{\rho}{|\rho|} = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in \mathbb{R} \cdot (S^{n-1} + iS^{n-1})$, with $\omega_R \cdot \omega_I = 0$. For $\rho \in \mathcal{V}$, let $\Delta_\rho = \Delta + 2\rho \cdot \nabla$. Then

$$\Delta_\rho + q(x) = e^{-\rho\cdot x}(\Delta + q(x))e^{\rho\cdot x}, \tag{1.1}$$

so that, with $v(x) = e^{\rho\cdot x}u(x)$,

$$(\Delta_\rho + q(x))u(x) = w(x) \Leftrightarrow (\Delta + q(x))v(x) = e^{\rho\cdot x}w(x) \tag{1.2}$$

and, in particular, $(\Delta_\rho + q(x))u(x) = 0 \Leftrightarrow (\Delta + q(x))v(x) = 0$. If $\rho_1, \rho_2 \in \mathcal{V}$ and $(\Delta_{\rho_j} + q_j(x))u_j(x) = w_j(x)$, $v_j(x) = e^{\rho_j\cdot x}u_j(x)$ as above, we modify the identity of Alessandrini[A] as follows. Let v_j^* be the (exact) solution of

$$(\Delta + q_j)v_j^* = 0 \text{ on } \Omega,\ v_j^*|_{\partial\Omega} = v_j|_{\partial\Omega},$$

and $u_j^* = e^{-\rho_j \cdot x} v_j^*$. Then, if $\Lambda_{q_1} = \Lambda_{q_2}$ on an open set $\mathcal{U} \subset \partial\Omega$ containing $supp(v_j|_{\partial\Omega}) = supp(v_j^*|_{\partial\Omega}), j = 1, 2$, one has

$$\begin{aligned}
0 &= \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2})(v_1) v_2 d\sigma \\
&= \int_{\partial\Omega} v_1 \Lambda_{q_2}(v_2) - \Lambda_{q_1}(v_1) v_2 d\sigma \text{ since } \Lambda_{q_2} = \Lambda_{q_2}^* \\
&= \int_{\partial\Omega} v_1^* \Lambda_{q_2}(v_2^*) - \Lambda_{q_1}(v_1^*) v_2^* d\sigma \\
&= \int_{\partial\Omega} v_1^* \frac{\partial v_2^*}{\partial n} - \frac{\partial v_1^*}{\partial n} v_2^* d\sigma \text{ since } v_1^*, v_2^* \text{ are actual solutions} \\
&= \int_{\Omega} v_1^* (\Delta v_2^*) - (\Delta v_1^*) v_2^* dx \text{ by Green's Theorem} \\
&= \int_{\Omega} (q_1(x) - q_2(x)) v_1^*(x) v_2^*(x) dx \\
&= \int_{\Omega} (q_1 - q_2) \big(v_1 v_2 + (v_1^* - v_1) v_2 + v_1 (v_2^* - v_2) + (v_1^* - v_1)(v_2^* - v_2) \big) dx.
\end{aligned}$$

Thus, if $\rho_2 = -\rho_1$, the exponentials cancel and we have

(1.3)
$$0 = \int_{\Omega} (q_1 - q_2) u_1 u_2 dx + \int_{\Omega} (q_1 - q_2) \big[(u_1^* - u_1) u_2 + u_1 (u_2^* - u_2) + (u_1^* - u_1)(u_2^* - u_2) \big] dx.$$

To use this identity to recover $R_{2,n}^{\Omega}(q_2 - q_1)(\Pi)$ for $\Pi \in M_{2,n}$, we will construct a family of approximate solutions to $(\Delta_\rho + q(x))u = 0$, denoted $u(x, \rho, \Pi)$, which are concentrated on tubular neighborhoods of Π shrinking to Π as $|\rho| \to \infty$. Here, $\rho \in \mathcal{V}$ is such that Π is parallel to $\text{span}\{\omega_R, \omega_I\}$. Furthermore, for u_2 another approximate solution associated with $\rho_2 = e^{i\theta} \cdot \rho_1$, for any $\theta \in \mathbb{R}$, and another potential q_2, we will have that $u_1 u_2 \to d\lambda_{\Pi}^{\Omega}$, so that the first term in (1.3) converges to $R_{2,n}^{\Omega}(q_2 - q_1)(\Pi)$ as $|\rho| \to \infty$. More precisely, we show

Theorem 4 *Let* Ω *be Lipschitz and* $q(x) \in H^s(\Omega)$ *for some* $s > \frac{n}{2}$. *Then,* $\exists\ \beta_0 = \beta_0(n,s) > 0$ *such that for any* $0 < \beta < \beta_0$ *fixed, the following holds:* $\exists\ \epsilon > 0$ *such that, for any* $\rho = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in \mathcal{V}$ *and any two-plane* Π *parallel to* $\mathrm{span}\{\omega_R, \omega_I\}$, *we can find an approximate solution* $u = u(x, \rho, \Pi)$ *to* $(\Delta_\rho + q(x))u = 0$ *satisfying*

$$\|u\|_{L^2(\Omega)} \simeq [\lambda_\Pi^\Omega(\Pi \cap \Omega)]^{\frac{1}{2}} \tag{1.5}$$

$$\mathrm{supp}(u) \subset \left\{x \in \mathbb{R}^n : \mathrm{dist}(x, \Pi) \leq \frac{2}{|\rho|^\beta}\right\} \tag{1.6}$$

and

$$\|(\Delta_\rho + q)u\|_{L^2(\Omega)} \leq \frac{C_\epsilon}{|\rho|^\epsilon}. \tag{1.7}$$

Furthermore, for any two such solutions, u_1, u_2, *associated with possibly different potentials* $q_1(x), q_2(x)$ *and with* $\rho_1, \rho_2 = e^{i\theta}\rho_1 \in \mathcal{V}$,

$$u_1(\cdot, \rho_1, \Pi)u_2(\cdot, \rho_2, \Pi) \to d\lambda_\Pi^\Omega \ \textit{weakly as}\ |\rho_1| \to \infty. \tag{1.8}$$

To see that Thm. 1 is a consequence of Thm. 4, we need some facts about the relevant Green's operators. The Faddeev Green's function, $G_\rho \in \mathcal{S}'(\mathbb{R}^n)$, essentially introduced in [F], is the fundamental solution of Δ_ρ defined by $\hat{G}_\rho(\xi) = [\sigma_\rho(\xi)]^{-1}$, where $\sigma_\rho(\xi) = -\left[(|\xi|^2 - 2|\rho|\omega_I \cdot \xi) + i2|\rho|\omega_R \cdot \xi\right]$ is the full symbol of Δ_ρ, which will be discussed in more detail below. One has the local estimates [SU86, SU87a,Ha96]

$$\|\phi_2(G_\rho * \phi_1 w)\|_{H^{s+1}(\mathbb{R}^n)} \leq C\|w\|_{H^s(\mathbb{R}^n)}$$

and

$$\|\phi_2(G_\rho * \phi_1 w)\|_{H^s(\mathbb{R}^n)} \leq \frac{C}{|\rho|}\|w\|_{H^s(\mathbb{R}^n)},$$

$\forall s \in \mathbb{R}$, uniformly in $|\rho|$ whenever $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^n)$. A right inverse on Ω for the perturbed operator $\Delta_\rho + q$ (with no boundary condition), which we shall denote $(\Delta_\rho + q)^{-1}$, was constructed in [SU86,SU87a] and shown to satisfy

$$\|(\Delta_\rho + q)^{-1}\|_{H^s(\Omega) \to H^{s+1}(\Omega)} \leq C$$

and

$$\|(\Delta_\rho + q)^{-1}\|_{H^s(\Omega)\to H^s(\Omega)} \leq \frac{C}{|\rho|}, \forall s \in \mathbb{R}.$$

The statements in [SU86,SU87a] are for $q \in C^\infty$, but the proofs are easily seen to hold if $q \in H^s(\Omega)$ with $s > \frac{n}{2}$.

Finally, we will need the Dirichlet Green's operators for both Δ and $\Delta_\rho + q$ on Ω; it is only at this point that we need to impose more than Lipschitz (i.e., $C^{0,1}$) regularity on $\partial\Omega$. Recall that Δ is an isomorphism $H^2_{0,D}(\Omega) \to L^2(\Omega)$, where $H^2_{0,D}(\Omega) = \{w \in H^2(\Omega) : w|_{\partial\Omega} = 0\}$, and thus admits an inverse, $\Delta_D^{-1} : L^2(\Omega) \to H^2_{0,D}(\Omega)$; if $\partial\Omega \in C^{2,1}$, then one also has $\Delta_D^{-1} : H^s(\Omega) \to H^{s+2}(\Omega)$ for $0 \leq s \leq 1$ [G,Thm. 2.5.1.1]. If $q(x) \in L^\infty(\Omega)$ and $\lambda = 0$ is not a Dirichlet eigenvalue for $\Delta + q$ on Ω (as we are assuming in order to define Λ_q), then it is also not a Dirichlet eigenvalue for $\Delta_\rho + q$, and $\Delta_\rho + q : H^2_{0,D}(\Omega) \to L^2(\Omega)$ is an isomorphism, with an inverse which we denote by $(\Delta_\rho + q)^{-1}_D$. We will show that this is a bounded operator from $H^2(\Omega) \to H^2_{0,D}(\Omega)$ uniformly in $|\rho|$.

Theorem 5 *If $\partial\Omega \in C^{2,1}$ and $q(x) \in H^s(\Omega)$ with $s \geq 2$ and $s > \frac{n}{2}$, then*

$$\|(\Delta_\rho + q)^{-1}_D\|_{H^2(\Omega)\to H^2_{0,D}(\Omega)} \leq C$$

uniformly in $|\rho|$.

To prove Thm. 5, we first note that, since Ω is Lipschitz, there exists an extension operator $E_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^n)$ which also maps $H^2 \to H^2$; see, e.g., [St70, Ch.7]. Let R_Ω denote restriction to Ω, which maps $H^s(\mathbb{R}^n) \to H^s(\Omega), \forall s \in \mathbb{R}$. Define $G^\Omega_\rho w = R_\Omega(G_\rho * E_\Omega w)$ and note that

$$\|G^\Omega_\rho\|_{H^s(\Omega)\to H^{s+1}(\Omega)} \leq C \text{ and } \|G^\Omega_\rho\|_{H^s(\Omega\to H^s(\Omega)} \leq \frac{C}{|\rho|}, \quad s \geq 0.$$

Define an approximate Dirichlet Green's operator for $\Delta_\rho + q(x)$ on Ω by

$$F_\rho = \Delta_D^{-1}(G^\Omega_\rho + L_\rho)\Delta,$$

where Δ_D^{-1} is the Dirichlet Green's operator for the standard Laplacian Δ on Ω and L_ρ, uniformly bounded in $|\rho|$ as an operator on $L^2(\Omega)$, will be chosen below. Because of the final Δ_D^{-1}, F_ρ will map into $H^2_{0,D}(\Omega)$. We calculate, acting on functions in $H^2(\Omega)$ and using $\Delta\Delta_D^{-1} = I$,

$$\begin{aligned}
(\Delta_\rho + q(x))F_\rho &= (\Delta + 2i\rho\cdot\nabla + q)\Delta_D^{-1}\big(G_\rho^\Omega + L_\rho\big)\Delta \\
&= (\Delta + 2i\rho\cdot\nabla + q)\Big(\Delta_D^{-1}\big(\Delta G_\rho^\Omega + [G_\rho^\Omega,\Delta]\big) + \Delta_D^{-1}L_\rho\Delta\Big) \\
&= (\Delta + 2i\rho\cdot\nabla + q)\big(G_\rho^\Omega + [\Delta_D^{-1},\Delta]G_\rho^\Omega + \Delta_D^{-1}[G_\rho^\Omega,\Delta]\big) + (\Delta_\rho + q)\Delta_D^{-1}L_\rho\Delta \\
&= I + qG_\rho^\Omega + [G_\rho^\Omega,\Delta] + (\Delta_\rho + q)[\Delta_D^{-1},\Delta]G_\rho^\Omega \\
&\quad + (2i\rho\cdot\nabla + q)\Delta_D^{-1}[G_\rho^\Omega,\Delta] + (\Delta_\rho + q)\Delta_D^{-1}L_\rho\Delta.
\end{aligned}$$

Under the assumptions of Thm. 5, multiplication by $q(x)$ is bounded on both H^2 and H^{-2}, which we will use several times below. In particular, $\|qG_\rho^\Omega\|_{H^2(\Omega)\to H^2(\Omega)} \le \frac{c}{|\rho|}$. We deal with the other terms using the following lemma, which will be proved later.

Lemma 6

$$\|[G_\rho^\Omega,\Delta]\|_{H^s(\Omega)\to H^{s-1}(\Omega)} \le C, \quad \forall s\ge 1$$

and

$$\|[G_\rho^\Omega,\Delta]\|_{H^s(\Omega)\to H^{s-2}(\Omega)} \le \frac{C}{|\rho|}, \quad \forall s\ge 2.$$

Using this, we see that $\|q\Delta_D^{-1}[G_\rho^\Omega,\Delta]\|_{H^2\to H^2} \le \frac{C}{|\rho|}$. We now choose $L_\rho = \sum_{j=1}^3 L_\rho^{(j)}$ with

$$L_\rho^{(1)} = -\Delta(\Delta_\rho + q)^{-1}[G_\rho^\Omega,\Delta]\Delta_D^{-1},$$

$$L_\rho^{(2)} = -\Delta(\Delta_\rho + q)^{-1}(\Delta_\rho + q)[\Delta_D^{-1},\Delta]G_\rho^\Omega\Delta_D^{-1}$$

and

$$L_\rho^{(3)} = -2i\Delta(\Delta_\rho + q)^{-1}(\rho\cdot\nabla)\Delta_D^{-1}[G_\rho^\Omega,\Delta]\Delta_D^{-1}$$

We then have that

$$\begin{aligned}
\|L_\rho^{(1)}w\|_{L^2} &\le c\|(\Delta_\rho + q)^{-1}[G_\rho^\Omega,\Delta]\Delta_D^{-1}w\|_{H^2} \\
&\le c \quad \|[G_\rho^\Omega,\Delta]\Delta_D^{-1}w\|_{H^1} \\
&\le c \quad \|\Delta_D^{-1}w\|_{H^2} \le c\|w\|_{L^2}
\end{aligned}$$

and

$$\begin{array}{rll} \|L_\rho^{(3)} w\|_{L^2} & \leq c & \|(\Delta_\rho + q)^{-1}(\rho\cdot\nabla)\Delta_D^{-1}[G_\rho^\Omega, \Delta]\Delta_D^{-1} w\|_{H^2} \\ & \leq c & \|(\rho\cdot\nabla)\Delta_D^{-1}[G_\rho^\Omega, \Delta]\Delta_D^{-1} w\|_{H^1} \\ & \leq c|\rho| & \|\Delta_D^{-1}[G_\rho^\Omega, \Delta]\Delta_D^{-1} w\|_{H^2} \\ & \leq c|\rho| & \|[G_\rho^\Omega, \Delta]\Delta_D^{-1} w\|_{L^2} \\ & \leq c|\rho|\cdot\frac{1}{|\rho|} & \|\Delta_D^{-1} w\|_{H^2} \leq c\|w\|_{L^2} \end{array}$$

uniformly in $|\rho|$. On the other hand, noting that $[\Delta_D^{-1}, \Delta] = \Delta_D^{-1}\Delta - I : H^3 \to H^3$ if $\partial\Omega \in C^{2,1}$, we see that $L_\rho^{(2)}$ is bounded on $L^2(\Omega)$ uniformly in $|\rho|$. With these choices of $L_\rho^{(j)}$, we have that $F_\rho : H^2(\Omega) \to H^2_{0,D}(\Omega)$ is uniformly bounded in $|\rho|$ and $(\Delta_\rho + q)F_\rho - I$ is an operator on H^2 of norm $\leq \frac{c}{|\rho|}$. Thus, for $|\rho|$ sufficiently large, we may compose F_ρ on the right with an invertible operator and obtain the exact Dirichlet Green's operator $\tilde{F}_\rho : H^2(\Omega) \to H^2_{0,D}(\Omega)$.

To prove Lemma 6, we note that

$$\begin{aligned} [G_\rho^\Omega, \Delta] &= R_\Omega G_\rho E_\Omega \Delta - \Delta R_\Omega G_\rho E_\Omega \\ &= R_\Omega G_\rho [E_\Omega, \Delta] + R_\Omega [G_\rho, \Delta] E_\Omega. \end{aligned}$$

Since G_ρ and Δ are translation–invariant on $\mathbb{R}^n$, the second term is $\equiv 0$. As for the first, we have that both $E_\Omega\Delta$ and ΔE_Ω map $H^s(\Omega) \to H^{s-2}(\mathbb{R}^n), \forall s \geq 2$ and then use the boundedness of G_ρ from $H^{s-2}(\mathbb{R}^n)$ to either H^{s-1} uniformly in $|\rho|$ or to H^{s-2} with decay $\frac{c}{|\rho|}$. This finishes the proof of the Lemma and thus also the proof of Thm. 5.

We may now return to the proof of Thm. 1, following (1.3). With $\Pi \in M_{2,n}$ specified, we pick any $\rho \in \mathcal{V}$ such that Π is a translate of $\text{span}\{\omega_R, \omega_I\}$ and let $\rho_1 = \rho, \rho_2 = -\rho$. Construct approximate solutions $u_j(x, \rho_j, \Pi)$ as in Thm. 4 for $\Delta_{\rho_j} + q_j(x)$, $j = 1, 2$, and let $w_j = (\Delta_{\rho_j} + q_j)u_j$. If $\Lambda_{q_1} = \Lambda_{q_2}$ on a neighborhood $\mathcal{U}$ of γ_Π in $\partial\Omega$, then by (1.6), for $|\rho|$ large enough, the calculation above (1.3) applies. Note that $u_j^* - u_j$ satisfies

$$(\Delta_{\rho_j} + q_j)(u_j^* - u_j) = -w_j \text{ on } \Omega,\ (u_j^* - u_j)|_{\partial\Omega} = 0,$$

and thus $u_j^* - u_j = -(\Delta_{\rho_j} + q_j)_D^{-1}(w_j)$. It will follow from the proof of Thm. 4 that the natural Sobolev versions of (1.5) and (1.7),

(1.9)
$$||u_j||_{H^{\pm 2}(\Omega)} \leq c|\rho|^{\pm 2\beta} \text{ and } ||(\Delta_{\rho_j} + q_j)u_j||_{H^{\pm 2}(\Omega)} \leq c|\rho|^{\pm 2\beta - \epsilon}, \quad j = 1, 2,$$

hold, so that the first term in the second integral in (1.3) can be estimated by

$$\begin{aligned} \left|\int (q_1-q_2)(u_1^*-u_1)u_2dx\right| &\leq \|(\Delta_{\rho_1}+q_1)_D^{-1}(w_1)\|_{H^2}\cdot\|(q_1-q_2)u_2\|_{H^{-2}} \\ &\leq c\|w_1\|_{H^2}\cdot\|u_2\|_{H^{-2}}\leq c|\rho|^{2\beta-\epsilon}\cdot|\rho|^{-2\beta}=c|\rho|^{-\epsilon} \end{aligned}$$

and similarly for $\int(q_1-q_2)u_1(u_2^*-u_2)dx$; here we are using Thm. 5 and the fact that multiplication by q_j preserves H^{-2}. The final term in (1.3) can be written as

$$\begin{aligned} &\left|\int(\Delta_{\rho_1}+q_1)_D^{-1}(w_1)\cdot\big((q_1-q_2)\overline{(u_2^*-u_2)}\big)dx\right| \\ &=\left|\int w_1\cdot(\Delta_{-\overline{\rho_1}}+q_1)_D^{-1}\big((q_1-q_2)\overline{(u_2^*-u_2)}\big)dx\right| \\ &\leq\|w_1\|_{H^{-2}}\cdot\|(\Delta_{-\overline{\rho_1}}+q_1)_D^{-1}\big((q_1-q_2)\overline{(u_2^*-u_2)}\big)\|_{H^2} \\ &\leq c|\rho|^{-2\beta-\epsilon}\cdot\|(q_1-q_2)\overline{(u_2^*-u_2)}\|_{H^2}\leq c|\rho|^{-2\beta-\epsilon}|\rho|^{2\beta-\epsilon}=c|\rho|^{-2\epsilon}. \end{aligned}$$

Hence, (1.3) yields that

$$0=\int_\Omega(q_1-q_2)u_1u_2dx+O(|\rho|^{-\epsilon});$$

taking $|\rho|\to\infty$, by (1.8) we obtain $R^\Omega_{2,n}(q_1-q_2)(\Pi)=0$, yielding Thm. 1.

Now, to prove Thm. 4 we may use the rotation invariance of Δ and the invariance of $\mathcal{V}$ under $S^1=\{e^{i\theta}\}$, and note that it suffices to treat the case[2] $\rho=|\rho|(\vec{e_1}+i\vec{e_2})$, where $\{\vec{e_1},\dots,\vec{e_n}\}$ is the standard orthonormal basis for $\mathbb{R}^n$. Write $x\in\mathbb{R}^n$ as $x=(x',x'')\in\mathbb{R}^2\times\mathbb{R}^{n-2}$ and similarly $\xi=(\xi',\xi'')$.

If $\Pi\in M_{2,n}$ is parallel to $\text{span}\{\omega_R,\omega_I\}=\text{span}\{\vec{e_1},\vec{e_2}\}=\mathbb{R}^2\times\{0\}$, then $\Pi=\text{span}\{\vec{e_1},\vec{e_2}\}+(0,x_0'')$ for some $x_0''\in\mathbb{R}^{n-2}$. Given $|\rho|>1$ and $x_0''\in\mathbb{R}^{n-2}$, we will define an approximate solution $u(x,\rho,\Pi)$ to $(\Delta_\rho+q(x))u=0$ on $\mathbb{R}^n$, of the form $u(x,\rho,\Pi)=u_0(x,\rho,\Pi)+u_1(x,\rho,\Pi)$.

For notational convenience, we will usually suppress the dependence on ρ and Π and simply write $u(x)=u_0(x)+u_1(x)$. We will use various cutoff functions χ_j; for j even or odd, χ_j will always denote a function of x' or x'',

---

[2] Of course, the length of this element of $\mathcal{V}$ is $\sqrt{2}|\rho|$, but this is irrelevant for the proofs and is notationally convenient.

respectively. Also, $B^m(a;r)$ and $S^{m-1}(a;r)$ will denote the closed ball and sphere of radius r centered at a point $a \in \mathbb{R}^m$.

To define u_0, first fix $\chi_0 \in C_0^\infty(\mathbb{R}^2)$ with $\chi_0 \equiv 1$ on $B^2(0;R)$ for any $R > \sup\{|x'| : (x',x'') \in \Omega \text{ for some } x'' \in \mathbb{R}^{n-2}\}$; let $C_0 = \|\chi_0\|_{L^2(\mathbb{R}^2)}$. Secondly, let $\psi_1 \in C_0^\infty(\mathbb{R}^{n-2})$ be radial, nonnegative, supported in the unit ball, and satisfy

$$\int_{\mathbb{R}^{n-2}} (\psi_1(x''))^2 dx'' = 1.$$

Now, for $\beta > 0$ to be fixed later, we let δ be the small parameter $\delta = |\rho|^{-\beta}$ and define

$$\chi_1(x'') = \delta^{-\frac{n-2}{2}} \psi_1\left(\frac{x' - x_0''}{\delta}\right),$$

so that

$$\|\chi_1\|_{L^2(\mathbb{R}^{n-2})} = \|\psi_1\|_{L^2(\mathbb{R}^{n-2})} = 1, \ \forall\, \delta > 0. \tag{1.10}$$

Set $u_0(x) = u_0(x',x'') = \chi_0(x')\chi_1(x'')$; then $\|u_0\|_{L^2(\mathbb{R}^n)} = C_0$ and $\|u_0\|_{L^2(\Omega)} \to [\lambda_\Pi(\Pi \cap \Omega)]^{\frac{1}{2}}$ as $\delta \to 0^+$, i.e., as $|\rho| \to \infty$. Apropos of (1.9), note also that $||u_0||_{H^{\pm 2}} \leq c\delta^{\mp 2} = c|\rho|^{\pm 2\beta}$. Since $\Delta_\rho = \Delta + 2\rho \cdot \nabla = \Delta + 2|\rho|(\vec{e_1} + i\vec{e_2}) \cdot \nabla = \Delta + 4|\rho|\bar{\partial}_{x'}$ and $\rho \perp \mathbb{R}^{n-2}$,

$$\begin{aligned}(\Delta_\rho + q(x))u_0 &= (\Delta\chi_0) \cdot \chi_1 + 2(\nabla\chi_0) \cdot (\nabla\chi_1) + \chi_0(\Delta\chi_1) \\ &\qquad +2(\rho \cdot \nabla)(\chi_0)\chi_1 + 2\chi_0(\rho \cdot \nabla)(\chi_1) + q\chi_0\chi_1 \\ &= \chi_0(x')(\Delta_{x''} + q)(\chi_1)(x'') \text{ on } B^2(0;R) \times \mathbb{R}^{n-2},\end{aligned}$$

since the first and fourth terms after the first equality vanish because $(\rho \cdot \nabla)(\chi_0) = 2\bar{\partial}\chi_0 \equiv 0$ on $B^2(0;R)$, and the second and fifth because $\nabla\chi_1 \perp \mathbb{R}^2$.

To define the second term in the approximate solution, $u_1(x)$, we make use of a truncated form of the Faddeev Green's function, G_ρ, and an associated projection operator. The operator Δ_ρ has, for $\rho \in \mathcal{V}$, (full) symbol

$$\sigma(\xi) = -[(|\xi|^2 - 2|\rho|\omega_I \cdot \xi) + i2|\rho|(\omega_R \cdot \xi)], \tag{1.11}$$

and so for $\frac{\rho}{|\rho|} = e_1 + ie_2$, we have

$$\sigma(\xi) = -[(|\xi - |\rho|\vec{e_2}|^2 - |\rho|^2) + i(2|\rho|\xi_1)],$$

which has (full) characteristic variety

$$\begin{aligned}(1.12)\qquad \Sigma_\rho &= \{\xi\in\mathbb{R}^n : \xi_1=0, |\xi-|\rho|e_2|=|\rho|\}\\ &= \{0\}\times S^{n-2}((|\rho|,0,\dots,0);|\rho|)\subset \mathbb{R}_{\xi_1}\times\mathbb{R}^{n-1}_{\xi_2,\xi''}.\end{aligned}$$

As described earlier, the Faddeev Green's function $G_\rho(x)$ is defined by $G_\rho = (-\sigma(\xi)^{-1})^\vee \in \mathcal{S}'(\mathbb{R}^n)$. We now introduce, for an $\epsilon_0>0$ to be fixed later, a tubular neighborhood of Σ_ρ,

$$(1.13)\qquad T_\rho = \{\xi : \mathrm{dist}(\xi,\Sigma_\rho) < |\rho|^{-\frac{1}{2}-\epsilon_0}\},$$

as well as its complement, T_ρ^C, and let χ_{T_ρ}, $\chi_{T_\rho^C}$ be their characteristic functions. Define a projection operator, P_ρ, and a truncated Green's function, $\widetilde{G}_\rho$, by

$$\begin{aligned}(1.14)\qquad \widehat{P_\rho f}(\xi) &= \chi_{T_\rho}(\xi)\cdot\widehat{f}(\xi) \text{ and}\\ (1.15)\qquad (\widetilde{G}_\rho f)^\wedge(\xi) &= \chi_{T_\rho^C}(\xi)\cdot[-\sigma(\xi)]^{-1}\widehat{f}(\xi)\end{aligned}$$

for $f\in\mathcal{S}(\mathbb{R}^n)$. Note that $\Delta_\rho\widetilde{G}_\rho = I-P_\rho$.

Choose a $\psi_3\in C_0^\infty(\mathbb{R}^{n-2})$, supported in $B^{n-2}(0;2)$, radial and with $\psi_3\equiv 1$ on $\mathrm{supp}(\psi_1)$, and set $\chi_3(x'')=\psi_3(\frac{x''-x_0''}{\delta})$. We now define the second term, $u_1(x,\rho,\Pi)$ in the approximate solution by

$$(1.16)\qquad u_1(x) = -\chi_3(x'')\widetilde{G}_\rho((\Delta_\rho+q(x))u_0(x))$$

and set $u(x)=u_0(x)+u_1(x)$. Then u_1 (as well as u_0) is supported in $\{x:\mathrm{dist}(x,\Pi)\le 2\delta\}$, yielding (1.6). We will see below that $\|u_1\|_{L^2(\Omega)}\le C|\rho|^{-\epsilon}$ as $|\rho|\to\infty$, so that (1.5) holds as well, and $\|u_1\|_{H^{\pm 2}}\le C|\rho|^{\pm 2\beta-\epsilon}$, so that the first part of (1.9) holds as well. To start the proof of (1.7), note that

$$\begin{aligned}(\Delta_\rho+q)(u_0+u_1) &= (\Delta_\rho+q)u_0-(\Delta_\rho+q)\chi_3\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\ &= (\Delta_\rho+q)u_0-\chi_3(\Delta_\rho+q)\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\ &\quad -[\Delta_\rho+q,\chi_3]\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\ &= (\Delta_\rho+q)u_0-\chi_3(I-P_\rho)(\Delta_\rho+q)u_0-\chi_3 q\widetilde{G}_\rho(\Delta_\rho+q)u_0\\ &\quad -2(\nabla\chi_3\cdot\nabla_{x''})\widetilde{G}_\rho(\Delta_\rho+q)u_0-(\Delta_{x''}\chi_3)\widetilde{G}_\rho(\Delta_\rho+q)u_0\\ &= \chi_3P_\rho(\Delta_\rho+q)u_0\\ &\quad -[q\chi_3+2(\nabla\chi_3\cdot\nabla_{x''})-(\Delta_{\chi''}\chi_3)]\widetilde{G}_\rho(\Delta_\rho+q)u_0\end{aligned}$$

on Ω, since $\chi_3 \equiv 1$ on $\mathrm{supp}(\chi_1)$. Now, since $q_1\chi_3 \in L^\infty$, $|\nabla\chi_3| \leq C\delta^{-1} = c|\rho|^\beta$ and $|\Delta_{x''}\chi_3| \leq C\delta^{-2} = c|\rho|^{2\beta}$, (1.7) will follow if we can show that for some $\epsilon > 0$,

$$\begin{aligned} &(1.17) && \|P_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} && \leq C|\rho|^{-\epsilon}, \\ &(1.18) && \|\,|D''|\widetilde{G}_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} && \leq C|\rho|^{-\beta-\epsilon}, \text{ and} \\ &(1.19) && \|\widetilde{G}_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} && \leq C|\rho|^{-2\beta-\epsilon}, \end{aligned}$$

with C independent of $|\rho| > 1$. Furthermore, the second part of (1.9) will follow if the $H^{\pm 2}$ norms are controlled by $|\rho|^{\pm 2\beta}$ times the same expressions. Before proceeding to prove these, we note that for any $u^{(1)}, u^{(2)}$ constructed in this way for the same two-plane Π,

$$u_0^{(1)}(x)u_0^{(2)}(x) = \chi_0^2(x')\delta^{-(n-2)}\psi_1^2\left(\frac{x'' - x_0''}{\delta}\right) \to d\lambda_\Pi^\Omega \text{ in } \Omega$$

as $\delta \to 0$ by (1.11), while $u_1^{(1)}u_0^{(2)} + u_0^{(1)}u_1^{(2)} + u_1^{(1)}u_1^{(2)} \to 0$ in $L^2(\Omega)$, yielding (1.8). Thus, we are reduced to establishing (1.17–1.19).

2 L^2 estimates

We will first prove (1.17)–(1.19) under the simplifying assumption that $q_1, q_2 \in C^{n-1+\sigma}(\overline{\Omega})$ for some $\sigma > 0$, turning to the Sobolev space case in Section 3. Start by noting that the desired estimates (1.17)–(1.19) cannot be simply obtained from operator norms; for example, $\|P_\rho\|_{L^2\to L^2} = 1$ for all ρ. One needs to make use of the special structure of $(\Delta_\rho + q)u_0$; we first deal with $\Delta_\rho u_0$, leaving $q(x)\cdot u_0$ for the end. So, we will show that $\|P_\rho\Delta_\rho u_0\|_{L^2(\Omega)} \leq C|\rho|^{-\epsilon}$, etc. Since $\nabla\chi_0 \cdot \nabla\chi_1 \equiv 0$,

$$\Delta_\rho u_0 = \chi_0\Delta_{x''}\chi_1 + (\Delta_{x'} + 4|\rho|\overline{\partial}_{x'})(\chi_0)\cdot\chi_1. \tag{2.1}$$

The second term is supported on Ω^c, but P_ρ and $\widetilde{G}_\rho$ are nonlocal operators and we need to control the contribution from this term (on Ω). However, because $\Delta_{x'}(\chi_0)$ is a fixed, δ-independent element of $C_0^\infty(\mathbb{R}^2)$, this can be handled in the same way as the $q(x)\cdot u_0$ terms of (1.17–1.19), which will be dealt with later. The contribution from $4|\rho|\overline{\partial}\chi_0\cdot\chi_1$ will be handled at the end.

So, for the time being, we are interested in estimating $\|P_\rho(\chi_0(x')\Delta_{x''}\chi_1(x''))\|_{L^2(\Omega)}$, etc. Now, $\Delta_{x''}\chi_1(x'') = \delta^{-2}\chi_5(x'')$, where $\chi_5(x'') = \delta^{-\frac{n-2}{2}}\psi_5\left(\frac{x''-x_0''}{\delta}\right)$ is associated with the radial function $\psi_5 = \Delta_{x''}\psi_1$ as χ_1 is associated with ψ_1. Note for future use that $\widehat{\psi}_5$ vanishes to second order at 0. Of course, $\chi_0 \in C_0^\infty \Rightarrow \widehat{\chi}_0 \in \mathcal{S}(\mathbb{R}^n)$, but looking ahead to estimating the terms involving $q(x)\cdot u_0(x)$, we will now prove the analogues of (1.17–1.19) where P_ρ and $\widetilde{G}_\rho$ act on $\chi_2(x')\Delta\chi_1(x'')$, under the weaker assumption that χ_2 is radial and satisfies the uniform decay estimate

$$|\widehat{\chi}_2(\xi)| \le C(1+|\xi|)^{-\alpha} \tag*{$(2.2)_\alpha$}$$

for some $\alpha > 0$.

Now, by (1.15) and Plancherel,

$$\begin{aligned}
\|P_\rho(\chi_2\Delta\chi_1)\|_{L^2(\Omega)} &\le \|(P_\rho(\chi_2\Delta\chi_1))^\wedge\|_{L^2(\mathbb{R}^n)} \\
&= \|\delta^{-2}|\widehat{\chi}_2(\xi')|\delta^{\frac{n-2}{2}}|\widehat{\psi}_5(\delta\xi'')|\,\|_{L^2(T_\rho)}.
\end{aligned}$$

The characteristic variety Σ_ρ, of which T_ρ is a tubular neighborhood, passes through the origin, and we may represent Σ_ρ near O as a graph over the ξ''-plane: $\Sigma_\rho = \Sigma_\rho^s \cup \Sigma_\rho^n \cup \Sigma_\rho^e$, with

$$\begin{aligned}
\Sigma_\rho^s &= \left\{\xi_1 = 0, \xi_2 = |\rho| - (|\rho|^2 - |\xi''|^2)^{\frac{1}{2}}, |\xi''| \le \frac{|\rho|}{2}\right\} \\
&\simeq \left\{\xi_1 = 0, \xi_2 = \frac{|\xi''|}{2|\rho|}, |\xi''| \le \frac{|\rho|}{2}\right\}
\end{aligned} \tag{2.3}$$

a neighborhood of the south pole O,

$$\begin{aligned}
\Sigma_\rho^n &= \left\{\xi_1 = 0, \xi_2 = |\rho| + (|\rho|^2 - |\xi''|^2)^{\frac{1}{2}}, |\xi''| \le \frac{|\rho|}{2}\right\} \\
&\simeq \left\{\xi_1 = 0, \xi_2 = 2|\rho| - \frac{|\xi''|^2}{2|\rho|}, |\xi''| \le \frac{|\rho|}{2}\right\}
\end{aligned} \tag{2.4}$$

a neighborhood of the north pole $(0, 2|\rho|, 0, \dots, 0)$, and Σ_ρ^e a neighborhood of the equator $\{\xi \in \Sigma_\rho : \xi_2 = |\rho|\}$. We have a corresponding decomposition $T_\rho = T_\rho^s \cup T_\rho^n \cup T_\rho^e$, where, e.g.,

$$T_\rho^s \simeq \left\{(\xi', \xi'') : \xi' \in B^2\left(\left(0, \frac{|\xi''|^2}{2|\rho|}\right); |\rho|^{-\frac{1}{2}-\epsilon_0}\right), |\xi''| \le \frac{|\rho|}{2}\right\}. \tag{2.5}$$

Recalling that χ_2 and ψ_3 are radial, so are $\widehat{\chi}_2$ and $\widehat{\chi}_3$, and by abuse of notation we consider these as functions of one variable satisfying $(2.2)_\alpha$ and rapidly decreasing, respectively. Thus, using polar coordinates in ξ'',

$$
\begin{aligned}
\|\widehat{\chi_2\Delta\chi_1}\|^2_{L^2(T^s_\rho)} &\simeq \int_0^{\frac{|\rho|}{2}}\int_{B^2\left(\left(0,\frac{r^2}{2|\rho|}\right);|\rho|^{-\frac{1}{2}-\epsilon_0}\right)} |\widehat{\chi}_2(\xi')|^2 d\xi' \delta^{n-6}|\widehat{\psi}_5(\delta r)|^2 r^{n-3}dr \\
\text{(2.6)} \qquad &\simeq \int_0^{\sqrt{2}|\rho|^{\frac{1}{4}}}\int_{B^2((0,0);|\rho|^{-\frac{1}{2}-\epsilon_0})} |\widehat{\chi}_2|^2 d\xi' \delta^{n-6}|\widehat{\psi}_5(\delta r)|^2 r^{n-2}\frac{dr}{r} \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{4}}}^{\frac{|\rho|}{2}} \left|\widehat{\chi}_2\left(\frac{r^2}{2|\rho|}\right)\right|^2 \cdot |B^2((0,0);|\rho|^{-\frac{1}{2}})|\delta^{n-6}|\widehat{\psi}_5(\delta r)|^2 r^{n-2}\frac{dr}{r}.
\end{aligned}
$$

Since we will be taking $\delta = |\rho|^{-\beta}$ with $\beta < \frac{1}{4}$, if we choose $0 < \epsilon_0 < 2(\frac{1}{4} - \beta)$, then the quantity $|\rho|^{\frac{1}{4}}\delta \to \infty$ as $|\rho| \to \infty$ and so

$$
\begin{aligned}
\text{(2.7)} \qquad \|\widehat{\chi_2\Delta\chi_1}\|^2_{L^2(T^s_\rho)} &\leq c\frac{\delta^{-4}}{|\rho|^{1+2\epsilon_0}}\left(\int_0^{\sqrt{2}|\rho|^{\frac{1}{4}}\delta} |\widehat{\psi}_5(r)|^2 r^{n-2}\frac{dr}{r}\right) \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{4}}\delta}^{\frac{|\rho|}{2}\delta} \left|\widehat{\chi}_2\left(\frac{r^2}{2\delta^2|\rho|}\right)\right|^2 |\widehat{\psi}_5(r)|^2 r^{n-2}\frac{dr}{r} \\
&\leq c(\delta^4|\rho|)^{-1},
\end{aligned}
$$

which is $\leq c|\rho|^{-2\epsilon}$ with $\epsilon = \frac{1}{2}(1-4\beta) > 0$.

In order to establish the second part of (1.9), we also need to estimate the $H^{\pm 2}(\Omega)$ norms. Since $\overline{\Omega}$ is compact, we may work instead with the usual norms on the homogeneous Sobolev spaces $\dot{H}^{\pm 2}(\mathbb{R}^n)$. The contribution from T^s_ρ simply involves replacing the factor r^{n-2} in the integrand of (2.7) with $r^{n-2\pm 4}$, which results in the entire expression being multiplied by $\delta^{\mp 4} = |\rho|^{\pm 4\beta}$. Similar considerations hold for all of the estimates below.

The other contributions to $\|P_\rho\chi_2\Delta\chi_1\|_{L^2}$, coming from T^n_ρ and T^e_ρ are handled similarly and are even smaller, due to the decrease of $\widehat{\chi}_2$ and $\widehat{\psi}_5$.

We next turn to estimating $\|\,|D''|\widetilde{G}_\rho\Delta_\rho u_0\|_{L^2}$; by the remark above, we may concentrate on the $\chi_2\Delta\chi_1$ term of $\Delta_\rho u_0$. Then

$$
\text{(2.8)} \qquad \|\,|D''|\widetilde{G}_\rho(\chi_2\Delta\chi_1)\|^2_{L^2(\Omega)} \leq \|\,|\xi''|(\sigma(\xi))^{-1}(\chi_2\Delta\chi_1)^\wedge(\xi)\|^2_{L^2(T^C_\rho)}.
$$

We may cover T^C_ρ by $T^{C,s}_\rho \cup T^{C,n}_\rho \cup T^{C,e}_\rho \cup T^{C,\infty}_\rho$, where

(2.9)

$$
T^{C,s}_\rho = \left\{\xi : \xi' \in B^2\left(\left(0,\frac{|\xi''|^2}{2|\rho|}\right);|\rho|^{-\frac{1}{2}-\epsilon_0}\right)^C \cap B^2\left(\left(0,2|\rho|-\frac{|\xi''|^2}{2|\rho|}\right);\frac{1}{4}|\rho|\right)^C, |\xi''| \leq \frac{|\rho|}{2}\right\},
$$

$T_\rho^{C,n}$ is defined similarly,

$$T_\rho^{C,e} = \left\{ \xi : \frac{|\rho|}{4} < \xi_2 < \frac{7|\rho|}{4}, |\rho|^{-\frac{1}{2}} < \mathrm{dist}(\xi, \Sigma_\rho) < |\rho|, |\xi''| < 2|\rho| \right\} \tag{2.10}$$

and

$$T_\rho^{C,\infty} = \left\{ \xi : |\xi| \geq 3|\rho|, |\xi''| \geq \frac{3}{2}|\rho| \right\}. \tag{2.11}$$

One has the lower bounds on σ,

$$|\sigma(\xi)| \geq \begin{cases} C|\rho|\mathrm{dist}(\xi, \Sigma_\rho), & |\xi| \leq 3|\rho| \\ C|\xi|^2, & |\xi| \geq 3|\rho| \end{cases} \tag{2.12}$$

with C (as always) uniform in $|\rho|$. The first inequality in (2.12) follows from noting that $\frac{1}{2}\nabla\sigma(\xi) = (\xi - |\rho|\vec{e_2}) + i(|\rho|\vec{e_1})$, so that $|\nabla\sigma(\xi)| = 2\sqrt{2}|\rho|$ on Σ_ρ, while the second follows from $Re(\sigma(\xi)) = \mathrm{dist}(\xi, |\rho|\vec{e_2})^2 - |\rho|^2$. Using the first estimate in (2.12), we can then dominate the contribution to the right side of (2.8) from the region $T_\rho^{C,s}$ by

(2.13)

$$\delta^{n-6} \int_{|\xi''| \leq \frac{|\rho|}{2}} \int_{B^2\left(\left(0, \frac{|\xi''|^2}{2|\rho|}\right); |\rho|^{-\frac{1}{2}-\epsilon_0}\right)^C} |\rho|^{-2} \left| \xi' - \frac{|\xi''|^2}{2|\rho|}\vec{e_2} \right|^{-2} |\widehat{\chi}_2(\xi')|^2 d\xi' |\xi''|^2 |\widehat{\psi}_5(\delta\xi'')|^2 d\xi''.$$

The inner integral is the convolution

$$|\rho|^{-2} \left(|\widehat{\chi}_2|^2 *_{\mathbb{R}^2} \frac{\chi\{|\xi'| \geq |\rho|^{-\frac{1}{2}-\epsilon_0}\}}{|\xi'|^2} \right) \Bigg|_{\xi' = \frac{|\xi''|^2}{2|\rho|}\vec{e_2}}.$$

An elementary calculation shows that, for $\widehat{\chi}_2$ satisfying $(2.2)_\alpha$ for some $0 < \alpha < 1$, and any $0 < a < 1$,

$$|\widehat{\chi}_2|^2 *_{\mathbb{R}^2} \frac{\chi\{|\xi'| \geq a\}}{|\xi'|^2} \leq \begin{cases} C_1(1 + \log(a^{-1})), & |\xi'| \leq 1 \\ C_2|\xi'|^{-2} + C_3|\xi'|^{-2\alpha} \log\left(\frac{|\xi'|}{a}\right), & |\xi'| \geq 1, \end{cases} \tag{2.14}$$

so that, taking $a = |\rho|^{-\frac{1}{2}-\epsilon_0}$ and $|\xi'| = \frac{|\xi''|^2}{2|\rho|}$, the inner integral in (2.13) is

$$\leq \begin{cases} C_1|\rho|^{-2}\log|\rho|, & 0 < |\xi''| \leq \sqrt{2}|\rho|^{\frac{1}{2}} \\ C_2|\xi''|^{-4} + C_3|\rho|^{2\alpha-2}|\xi''|^{-4\alpha} \log\left(\frac{|\xi''|^2}{2|\rho|^{\frac{1}{2}-\epsilon_0}}\right), & \sqrt{2}|\rho|^{\frac{1}{2}} \leq |\xi''| \leq \frac{|\rho|}{2}. \end{cases}$$

Employing polar coordinates in ξ'' and rescaling by δ, we see that (2.13) is

$$\begin{aligned}
\leq\ & C_1\delta^{-6}|\rho|^{-2}\log|\rho|\int_0^{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}|\widehat{\psi}_5(r)|^2 r^n\frac{dr}{r}\\
&+C_2\delta^{-2}\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta}|\widehat{\psi}_5(r)|^2 r^{n-4}\frac{dr}{r}\\
&+C_3\delta^{4\alpha-4}|\rho|^{2\alpha-2}\log|\rho|\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta}|\widehat{\psi}_5(r)|^2 r^{n-2-4\alpha}\frac{dr}{r}.
\end{aligned}$$

With $\delta=|\rho|^{-\beta}$, $\beta<\frac{1}{4}$, $|\rho|^{\frac{1}{2}}\delta\to\infty$ as $|\rho|\to\infty$, and thus we estimate this for any $N>0$ (using the rapid decay of $\widehat{\psi}_5$) by

$$C_1|\rho|^{6\beta-2}\log|\rho|+C_2\delta^{-2}(|\rho|^{\frac{1}{2}}\delta)^{-N}+C_3|\rho|^{(4-4\alpha)\beta+2\alpha-2}\log|\rho|(|\rho|^{\frac{1}{2}}\delta)^{-N},$$

the terms of which will be less than the desired $|\rho|^{-2\beta-\epsilon'}$, for any $\alpha>0$, since $\beta<\frac{1}{4}$, $\beta<\frac{1}{2}$ and $\beta<\frac{1}{2}$, respectively.

Moving ahead for the moment to (1.19), the contribution to $\|\widetilde{G}_\rho\chi_2\Delta\chi_1\|_{L^2}^2$ (which we want $\leq C|\rho|^{-4\beta-\epsilon'}$) from $T_\rho^{C,s}$ is handled in the same fashion, the only differences being the absence of the multiplier $|D''|^\wedge=|\xi''|$ on the left and the improved gain we are demanding on the right. Taking these into account, we need to control

$$\begin{aligned}
(2.15)\quad & C_1\delta^{-4}|\rho|^{-2}\log|\rho|\int_0^{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}|\widehat{\psi}_5(r)|^2 r^{n-2}\frac{dr}{r}\\
&+C_2\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{1}{2}|\rho|\delta}|\widehat{\psi}_5(r)|^2 r^{n-6}\frac{dr}{r}\\
&+C_3\delta^{4\alpha-2}|\rho|^{2\alpha-2}\log|\rho|\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{1}{2}|\rho|\delta}|\widehat{\psi}_5(r)|^2 r^{n-4-4\alpha}\frac{dr}{r}\\
&\leq C_1\delta^{-4}|\rho|^{-2}\log|\rho|+C_2(|\rho|^{\frac{1}{2}}\delta)^{-N}+C_N\delta^{4\alpha-2}|\rho|^{2\alpha-2}\log|\rho|(|\rho|^{\frac{1}{2}}\delta)^{-N},
\end{aligned}$$

and this is $\leq C|\rho|^{-4\beta-\epsilon}$ provided $\beta<\frac{1}{4}$ and N is sufficiently large.

The contributions to (1.19) and (1.19) from $T_\rho^{C,n}$ and $T_\rho^{C,e}$ are handled similarly. To treat the contribution from $T_\rho^{C,\infty}$, we use the second estimate in (2.12) and calculate (for (1.19))

$$
\begin{aligned}
(2.16)\quad & \| \, |\xi''| (\sigma(\xi))^{-1} (\chi_2 \Delta \chi_1)^\wedge(\xi) \|^2_{L^2(T_\rho^{C,\infty})} \\
& \le C \iint_{|\xi| \ge 3|\rho|} \delta^{n-6} |\widehat{\chi}_2(\xi')|^2 |\widehat{\psi}_5(\delta\xi'')|^2 \frac{|\xi''|^2 d\xi' d\xi''}{|\xi|^4} \\
& \le C \left(\int_{|\xi''| \le |\rho|} \delta^{n-6} |\rho|^{-2\alpha-2} |\widehat{\psi}_5(\delta\xi'')|^2 |\xi''|^2 d\xi'' \right. \\
& \quad \left. + \int_{|\xi''| \ge |\rho|} \delta^{n-6} |\widehat{\psi}_5(\delta\chi'')|^2 |\xi''|^{-2\alpha} d\xi'' \right) \\
& = C \left(\delta^{-6} |\rho|^{-2\alpha-2} \int_0^{|\rho|\delta} |\widehat{\psi}_5(r)|^2 r^n \frac{dr}{r} \right. \\
& \quad \left. + \delta^{2\alpha-4} \int_{|\rho|\delta}^\infty |\widehat{\psi}_5(r)|^2 r^{n-2-2\alpha} \frac{dr}{r} \right) \\
& \le C(\delta^{-6} |\rho|^{-2\alpha-2} + \delta^{2\alpha-4} (|\rho|\delta)^{-N}), \qquad \forall\, N > 0,
\end{aligned}
$$

which, for $\delta = |\rho|^{-\beta}$ and N large is $\le C|\rho|^{-2\beta-\epsilon}$ provided $\beta < \frac{1}{4}$. A similar analysis holds for the $T_\rho^{C,\infty}$ contribution to (1.19).

We now turn to controlling the $q(x)u_0(x)$ terms in (1.17)–(1.19), as well as the contributions from the $\Delta(\chi_0) \cdot \chi_1$ term in (2.1). Note that since $q(x)$ is $C^{n-1+\sigma}$ (for some $\sigma > 0$), $q(x)$ has an extension (see, e.g., [St70,Ch.6]) to a $C^{n-1+\sigma}$ function of compact support on $\mathbb{R}^n$, which we also denote by q. The restriction of q to any $\Pi \in M_{2,n}$ is still $C^{n-1+\sigma}$.

Let $\{D_t : 0 < t < \infty\}$ be the one-parameter group of partial dilations on $\mathcal{S}'(\mathbb{R}^{n^*})$,

$$(D_t f)(\xi', \xi'') = t^{n-2} f(\xi', t\xi''),$$

which, for $f, g \in L^1$, satisfy $\int_{\mathbb{R}^n} D_t f \, d\xi = \int_{\mathbb{R}^n} f \, d\xi$ and $D_t(f * g) = D_t f * D_t g$. Then

$$
\begin{aligned}
(2.17)\widehat{qu_0}(\xi) \quad &= \quad \widehat{q} * \widehat{u}_0(\xi) = D_\delta(D_{\delta^{-1}} \widehat{q}) * \delta^{-\frac{n-2}{2}} D_\delta(\widehat{\chi}_0(\xi') \widehat{\psi}_1(\xi'') e^{i x_0'' \cdot \xi''}) \\
&= \quad D_\delta(D_{\delta^{-1}}(\widehat{q}) * \delta^{-\frac{n-2}{2}} \widehat{\chi}_0 \widehat{\psi}_1 e^{i x_0'' \cdot \xi''}).
\end{aligned}
$$

Now, as $\delta = |\rho|^{-\beta} \to 0$, $D_{\delta^{-1}}(\widehat{q}) = \delta^{-(n-2)} \widehat{q}(\xi', \delta\xi'')$ converges weakly to the singular measure

$$(2.18) \qquad Q(\xi') \otimes \delta(\xi'') = Q(\xi') d\xi',$$

where $Q(\xi') = \int_{\mathbb{R}^{n-2}} \widehat{q}(\xi',\xi'')d\xi''$; note that $q \in C^{n-1+\gamma}$ implies that the integral defining Q converges and Q satisfies $(2.2)_{1+\gamma}$. Letting $F(\xi) = \widehat{\chi}_0(\xi')\widehat{\psi}_1(\xi'')e^{ix_0''\cdot\xi''}$, it follows from (2.17) that

$$
\begin{aligned}
(2.19)\widehat{qu_0}(\xi) &= D_\delta(D_{\delta^{-1}}(\widehat{q}) * \delta^{-\frac{n-2}{2}}F) \\
&= D_\delta((Qd\xi') * \delta^{-\frac{n-2}{2}}F) + D_\delta((D_{\delta^{-1}}\widehat{q} - Qd\xi') * \delta^{-\frac{n-2}{2}}F).
\end{aligned}
$$

If we define $\widehat{\chi}_4(\xi') = Q *_{\mathbb{R}^2} \widehat{\chi}_0(\xi')$, then $\widehat{\chi}_4$ also satisfies condition $(2.2)_{1+\gamma}$ (and thus $(2.2)_{\alpha'}$ for $0 < \alpha' < 1$, so that (2.14) can be applied), and the first term in (2.19) is

$$
D_\delta((Qd\xi') * \delta^{-\frac{n-2}{2}}F) = \widehat{\chi}_4(\xi')\delta^{\frac{n-2}{2}}\widehat{\psi}_1(\delta\xi'')e^{i\delta x_0''\cdot\xi''}. \tag{2.20}
$$

Thus, the contributions to $\|P_\delta(qu_0)\|_{L^2}$, $\| \, |D''|\widetilde{G}_\rho(qu_0)\|_{L^2}$ and $\|\widetilde{G}_\rho(qu_0)\|_{L^2}$ from the first term in (2.19) may be handled as the main $\chi_2\Delta\chi_1$ term was earlier, with the obvious absence of the factor δ^{-2}. To control the contributions from the second term in (2.19), we use the elementary

Lemma 7 *Let $\varphi(x)$, $f(x)$ be functions on $\mathbb{R}^m$ such that $\varphi(x)$, $|x|\varphi(x)$, $f(x)$ and $|\nabla f(x)|$ are in $L^1(\mathbb{R}^m)$. Then, $\forall \ \epsilon > 0$*

$$
\begin{aligned}
\left|\left(\epsilon^{-m}\varphi\left(\frac{x}{\epsilon}\right)\right.\right. &- \left.\left.\left(\int_{\mathbb{R}^m}\varphi dy\right)\delta(x)\right) * f(x)\right| \\
&\leq C_m(\|\varphi\|_{L^1} + \| \, |x|\varphi\|_{L^1}) \cdot (\|f\|_{L^\infty(B(0;|x|-1))} + \|\nabla f\|_{L^\infty(B(x;1))}) \cdot \epsilon.
\end{aligned}
$$

Applying this for $\epsilon = \delta$, $\xi' \in \mathbb{R}^2$ fixed, and using $F \in \mathcal{S}$, $|\widehat{q}(\xi)| \leq C(1+|\xi|)^{-(n-1+\gamma)}$, we find that, $\forall \ N > 0$

$$
|(D_{\delta-1}(\widehat{q}) - Qd\xi') * F(\xi)| \leq C_N(1+|\xi'|)^{-\gamma}(1+|\xi''|)^{-N}\delta. \tag{2.21}
$$

Hence, the second term in (2.19) is $\leq C_N\delta^{\frac{n}{2}}(1+|\xi'|)^{-\gamma}(1+|\delta\xi''|)^{-N}$ and this allows the contributions to (1.17)–(1.19) to be dealt with as the $\chi_2\Delta_{x''}\chi_1$ term was before.

Finally, we need to establish the estimates (1.17–1.19) for the $4|\rho|\overline{\partial}\chi_0$ term in 2.1; thus, we need to show

$$
\begin{aligned}
\|P_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2(\Omega)} &\leq C|\rho|^{-1-\epsilon}, && (2.22)\\
\| \, |D''|\widetilde{G}_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2(\Omega)} &\leq C|\rho|^{-1-\beta-\epsilon}, \text{ and} && (2.23)\\
\|\widetilde{G}_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2(\Omega)} &\leq C|\rho|^{-1-2\beta-\epsilon}, && (2.24)
\end{aligned}
$$

for some $\epsilon > 0$. Using the fact that $\widehat{\bar\partial\chi_0}(\xi')$ is rapidly decreasing and vanishes to first order at $\xi' = 0$, we may replace (2.6) with

$$
\begin{aligned}
\|\widehat{\bar\partial\chi_0\chi_1}\|^2_{L^2(T^s_\rho)} \quad &\simeq \quad \int_0^{\frac{|\rho|}{2}} \int_{B^2\left(\left(0,\frac{r^2}{2|\rho|}\right);|\rho|^{-\frac{1}{2}-\epsilon_0}\right)} |\widehat{\bar\partial\chi_0}(\xi')|^2 d\xi' \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-3} dr \\
&\le c_N\Big(\int_0^{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}} |\rho|^{-2-4\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r} \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}}^{\sqrt{2}|\rho|^{\frac{1}{2}}} (\frac{r^2}{2|\rho|})^2 |\rho|^{-1-2\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r} \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{2}}}^{\frac{|\rho|}{2}} (\frac{r^2}{2|\rho|})^{-N} |\rho|^{-1-2\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r}\Big) \\
&\le c_N\Big(|\rho|^{-2-4\epsilon_0} \int_0^{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}\delta} |\widehat{\psi}_1|^2 r^{n-2} \frac{dr}{r} \\
&\quad + |\rho|^{-3-2\epsilon_0} \delta^{-4} \int_{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}\delta}^{\sqrt{2}|\rho|^{\frac{1}{2}}} |\widehat{\psi}_1|^2 r^{n+2} \frac{dr}{r} \\
&\quad + |\rho|^{-1-2\epsilon_0+N} \delta^{2N} \int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta} |\widehat{\psi}_1|^2 r^{n-2-2N} \frac{dr}{r}\Big) \\
&\le c_N \quad \big(|\rho|^{-2-4\epsilon_0} + |\rho|^{-3-2\epsilon_0+4\beta}(|\rho|^{\frac{1-2\epsilon_0}{4}-\beta})^{-N'} \\
&\quad + |\rho|^{-1-2\epsilon_0+N-2N\beta-N'(\frac{1}{2}-\beta)}\big)
\end{aligned}
\tag{2.25}
$$

for any $N, N' \ge 0$. As before, the contributions from T^n_ρ and T^e_ρ are handled similarly. If N' is chosen large enough, this yields (2.23) with $\epsilon = 2\epsilon_0$.

The desired estimates (2.23),(2.24) are even easier and hold for any $\beta < \frac{1}{2}$. The contribution to (2.24) from $T^{C,s}_\rho$ is controlled as in (2.13), but with the factor δ^{n-2} and with the $\widehat{\chi_2}$ in the integrand replaced by $\widehat{\bar\partial\chi_0}$; this is then dominated in the same manner as below (2.14). The $T^{C,s}_\rho$ contribution to (2.25) is estimated as in (2.15), but with the absence of the δ^{-4}. All other contributions are dealt with similarly.

This concludes the proof of Thm.4 for the case of potentials in the Hölder class $C^{n-1+\sigma}(\overline{\Omega}), \sigma > 0$.

3 Remarks

(i) The proof of Thm. 4 needs to be slightly modified if we assume that the potential $q(x)$ belongs to the Sobolev space $H^{\frac{n}{2}+\sigma}(\Omega)$ for some $\sigma > 0$. Since $\partial\Omega$ is Lipschitz, such a $q(x)$ can, by the Calderón extension theorem, be extended to be in $H^{\frac{n}{2}+\sigma}(\mathbb{R}^n)$. Again denoting the extension by q, one has by Cauchy-Schwarz

$$\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^{n-2}} (1+|\xi''|)|\hat{q}(\xi',\xi'')|d\xi'' \Big)^2 (1+|\xi'|)^\sigma d\xi' \leq c\big(\|q\|_{\frac{n}{2}+\sigma}\big)^2 \tag{3.1}$$

Thus, Q as in (2.18) belongs to $L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')$, so that $\widehat{\chi}_4 = Q *_{\mathbb{R}^2} \widehat{\chi}_0 \in L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi') \cap L^\infty$. Replacing the uniform decay estimate $(2.2)_\alpha$ with

$$\widehat{\chi}_2 \in L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi') \tag*{$(3.2)_\sigma$}$$

will allow us to handle the first term in (2.19). Furthermore, if for ξ' fixed, we let $\phi(\cdot) = \widehat{q}(\xi',\cdot)$ in Lemma 5, then $\phi(\xi'')$ and $|\xi''|\phi(\xi'')$ are in $L^1(\mathbb{R}^{n-2})$ with norms (as functions of ξ') in $L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')$, and so the second term in (2.19) is $\leq c_N \widehat{\chi_6}(\xi')(1+|\delta\xi''|)^{-N}, \forall N$, with $\widehat{\chi_6}$ satisfying condition $(3.2)_\sigma$. So, we are reduced to repeating the analysis of Section 2 with $(2.2)_\alpha$ replaced by $(3.2)_\sigma$. The decay of $\widehat{\chi_2}$ was used in only two places in the argument. In (2.14), under $(3.2)_\sigma$, we have the same estimate except for the absence of $|\xi'|^{-2\alpha}$; however, this loss is absorbed into terms rapidly decreasing in $|\rho|^{\frac{1}{2}}\delta = |\rho|^{\frac{1}{2}-\beta}$ where (2.14) is used. On the other hand, in (2.16) we may estimate the inner integral by

$$\begin{aligned} \int_{|\xi'|\geq 2|\rho|} |\widehat{\chi_2}(\xi')|^2 \frac{d\xi'}{(|\xi'|^2+|\xi''|^2)^2} &\leq \textstyle\int_{\mathbb{R}^2} |\widehat{\chi_2}|^2 \frac{d\xi'}{(1+|\xi'|)^\sigma|\xi'|^4} \\ &\leq c|\rho|^{-4-\sigma} \text{ if } |\xi''| \leq \rho \end{aligned} \tag{3.3}$$

and

$$\int_{\mathbb{R}^2} |\widehat{\chi_2}(\xi')|^2 \frac{d\xi'}{(|\xi'|^2+|\xi''|^2)^2} \leq c|\xi''|^{-4} \text{ if } |\xi'| \geq \rho, \tag{3.4}$$

so that

$$
\begin{aligned}
(3.5) \qquad & \| \, |\xi''| (\sigma(\xi))^{-1} (\chi_2 \Delta \chi_1)^\wedge(\xi) \|^2_{L^2(T_\rho^{C,\infty})} \\
\leq C \Big(& \int_{|\xi''| \leq |\rho|} \delta^{n-6} |\rho|^{-4-\sigma} |\widehat{\psi}_5(\delta\xi'')|^2 |\xi''|^2 d\xi'' + \int_{|\xi''| \geq |\rho|} \delta^{n-6} |\widehat{\psi}_5(\delta\chi'')|^2 |\xi''|^{-2} d\xi'' \Big) \\
&= C \Big(\delta^{-6} |\rho|^{-4-\sigma} \int_0^{|\rho|\delta} |\widehat{\psi}_5(r)|^2 r^n \tfrac{dr}{r} \\
&\qquad + \delta^{-2} \int_{|\rho|\delta}^{\infty} |\widehat{\psi}_5(r)|^2 r^{n-4} \tfrac{dr}{r} \Big) \\
&\leq C_N (\delta^{-6} |\rho|^{-4-\sigma} + \delta^{-2} (|\rho|\delta)^{-N}) \\
&= C_N \Big(|\rho|^{6\beta-4-\sigma} + |\rho|^{2\beta} (|\rho|^{\beta - \frac{1}{2}})^N \Big), \qquad \forall N,
\end{aligned}
$$

which is $\leq c|\rho|^{-2\beta-\epsilon}$ for N sufficiently large, since $\beta < \frac{1}{2}$.

(ii) The construction of the approximate solutions given by Thm. 4 may be generalized by taking χ_0 to be an arbitrary analytic function of $z = x_1 + ix_2$, defined on a domain $\Pi \cap \Omega \subset\subset \Omega' \subset \Pi$. Since $\overline{\partial}\chi_0 = \Delta_{x'}\chi_0 \equiv 0$ on Ω, the resulting $u = u_0 + u_1$ is still an approximate solution in the sense of Thm. 4, except that (1.8) no longer applies. Thus, Thm. 1 can be strengthened to conclude that $(q_1 - q_2)|_\Pi$ is orthogonal in $L^2(\Pi \cap \Omega, d\lambda_\Pi)$ to the Bergman space $A^2(\Pi \cap \Omega)$ of square-integrable holomorphic functions on $\Pi \cap \Omega$. Furthermore, by repeating the construction using $\overline{\rho} = \frac{1}{\sqrt{2}}|\rho|(\omega_R - i\omega_I)$, which induces the conjugate complex structure on Π, for which the $\overline{\partial}$ operator equals the ∂ operator induced by ρ, we obtain that $(q_1 - q_2)|_\Pi$ is also orthogonal to the conjugate Bergman space $\overline{A}^2(\Pi \cap \Omega)$ of anti-holomorphic functions. (The analogue of this in two dimensions was obtained in [SU87b].) It would be interesting to make further use of this information.

(iii) To obtain variants of Thm. 1 establishing smaller sets of uniqueness in $\partial\Omega$, it might be useful to use approximate solutions associated to different two-planes. For this, it seems necessary to construct approximate solutions with much thinner supports, i.e., to overcome the restriction $\beta < \frac{1}{4}$ in Thm. 4. Such an improvement might also be useful in extending the results to $q_j \in L^\infty$.

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