PH.D. PRELIMINARY EXAMS
DAY 1
January 8, 1999

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Let $A$ be a commutative ring having only one maximal ideal $m$. Let $M$ be a finitely generated $A$ module such that $mM = M$. Prove that $M = 0$.

2. Let $k$ be a field and $E$ a Galois extension of $k$ of degree 261.
   (a) Show that there exists an intermediate field $F$ of degree 9 over $k$ that is Galois over $k$.
   (b) Show that there exists an intermediate field $F$ of degree 29 over $k$ that is Galois over $k$.
   (c) If $\text{Gal}(E/k)$ has no element of order 9, how many intermediate fields are there having degree 3 over $k$? How many of these are Galois over $k$? Be sure to completely justify your answers.

3. Let $H$ be a Hilbert space and

   $E = \{e_\alpha : \alpha \in \Lambda\}$

   an orthonormal basis for $H$. Show that

   $$\lim_{n \to \infty} \langle h_n, h \rangle = 0 \quad \forall h \in H$$

   if and only if $\sup\{\|h_n\| : n \geq 1\} < \infty$ and

   $$\lim_{n \to \infty} \langle h_n, e_\alpha \rangle = 0 \quad \forall e_\alpha \in E .$$

4.
(a) Suppose $f$ is holomorphic on $\{ z : Re z < 1 \}$ and

$$|f(z)| \leq M_0 \text{ for } Re z = 0.$$

Find upper bounds for $|f(-1)|$, $|f'(-1)|$, and $|f''(-1)|$.

(b) Now suppose also that $f \in L^\infty(\{ Re(z) < 1 \})$ and $|f(z)| \leq M_{-3}$ for $Re z = -3$. Find new upper bounds for $|f(-1)|$, $|f'(-1)|$, $|f''(-1)|$.

5. Show that the closed interval $[0,1]$ is compact.

6. Compute the homology of the 1-skeleton of the $n$-simplex for $n \geq 1$. 
PH.D. PRELIMINARY EXAMS
DAY 2
January 11, 1999

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7. Either prove or find a counterexample to the following conjecture:

**Conjecture** If $f(z)$ is an entire function such that the composite $f \circ f(z) = f(f(z))$ is bounded, then $f(z)$ is constant.

8. Let

$$
\ell_1 = \left\{ \{a_n\}_{n \geq 1} : a_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |a_n| < \infty \right\}
$$

Show that the dual of $c_0$ is $\ell_1$.

9.

(a) Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point (i.e. a point $x$ in $[0,1]$ with $f(x) = x$).

(b) Show that any continuous function

$$
f : D^2 \to D^2
$$

has a fixed point, where $D^2$ is the 2-disk, the points in the plane of norm less than or equal to 1.
10.

(a) State the Riesz representation theorem (concerning linear functionals on $C(X)$).

(b) Let $m_n$ be a sequence of positive numbers. Show that there exists a finite positive measure $\mu$ on $[0,1]$ such that $m_n = \int_0^1 x^n d\mu(x)$ for $n = 0, 1, \ldots$ if and only if the following condition is satisfied.

$$ (*) \text{ For every polynomial } p(x) = \sum_{k=0}^n a_k x^k \text{ (with real coefficients) which is non-negative on } [0,1], \text{ we have } \sum_{k=0}^n a_k m_k \geq 0. $$

11.

(a) Prove that all prime ideals in Artin rings are maximal ideals.

(b) Prove that an Artin ring can have only a finite number of prime ideals.

12. Find the Galois group of the polynomial $x^3 - x - 1$ over each of the following fields. Be sure to completely justify your answers.

(a) $\mathbb{Q}$

(b) $\mathbb{R}$

(c) $\mathbb{Z}/2\mathbb{Z}$
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13.

(a) State the Radon-Nikodym theorem.

(b) Let $\mu_1$, $\mu_2$ be finite positive measures on $(X, \mathcal{M})$ and let $\lambda = \mu_1 + \mu_2$. Show that $\mu_1 \perp \mu_2$ if and only if
$$\frac{d\mu_1}{d\lambda} \cdot \frac{d\mu_2}{d\lambda} = 0 \quad \lambda - \text{a.e.}$$

(c) Let $f_n$ be a sequence of absolutely continuous functions on $[0,1]$ such that $f_n(0)$ converges and $f_n'$ converges in the $L^p$ norm for some $p \geq 1$. Show that $f_n$ converges uniformly.

14. Let $H$ be a Hilbert space and $P : H \rightarrow H$ a projection operator, i.e. $P^2 = I$ and $P = P^*$. Prove that $P$ is a compact operator if and only if $\dim(\text{Range}(P)) < \infty$.

15. Determine the fundamental group of the torus $S' \times S'$.

16. Prove that the unit interval $[0,1]$ is connected.
17.

(a) Let $G$ be a group with a subgroup $H$. Then $H$ acts on the set of left cosets $S = \{gH|g \in G\}$ by left translation. This induces a homomorphism $\varphi$ from $H$ into the group of permutations of $S$. What is the kernel of $\varphi$?

(b) Describe the set of all fixed points in $S$ under the above action of $G$. (Do this by finding all elements $g$ with the property that $gH$ is a fixed point under the above action.)

(c) If $H$ is a $p$-group, show that $[N_G(H) : H] \equiv [G : H] \mod p$, and deduce from this that if $H$ is not a Sylow $p$-subgroup of $G$, then $N_G(H) \neq H$.

18.

(a) Let $F$ be a field and $\alpha$ an algebraic element of $F$. Suppose that $\text{Gal}(F(\alpha)/F)$ is generated by an automorphism that takes $\alpha$ to $\alpha + 1$. Show that $F$ has characteristic $p > 0$, and that $\alpha^p - \alpha \in F$.

(b) Conversely, if $F$ is a field of characteristic $p > 0$, show that any Galois extension of degree $p$ over $F$ arises in the above way (Hint: consider an element $B$ in the extension field having trace 1 (explain why such an element must exist) and construct $\alpha$ from the conjugates of $B$.)