PH.D. PRELIMINARY EXAMS
DAY 1
August 25, 1999

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Suppose that $f$ is a bounded measurable function on $\mathbb{R}$ which is differentiable everywhere, and that $f'$ is bounded as well. Suppose also that $g$ is in $L^1(\mathbb{R})$.
   (a) Show that $f'$ is measurable.
   (b) Let $f \ast g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Show that this function is bounded and differentiable.
   (c) Suppose that $h$ is in $L^1(\mathbb{R})$, and $\int_{\mathbb{R}} h(x)dx = 0$. Show that $h \ast g(x)$ is defined for almost every $x$, is in $L^1(\mathbb{R})$, and has integral 0.

2. Let $H$ be a Hilbert space and let $A : H \rightarrow H$ be a linear operator such that for all $x, y \in H$,
   \[ \langle Ax, y \rangle = \langle x, Ay \rangle. \]
   Prove that $A$ is bounded.
3. Let \( f(z) = z^4 + iz^3 + 4z^2 + 3iz + 3 \).

(a) Show that all the roots of \( f \) lie in the annulus \( \{ \frac{1}{2} < |z| < 3 \} \).

(b) How many of the roots are in the upper half plane \( \{ \text{Im} z > 0 \} \)?

4. Let \( X \) be a path-connected space.

(a) What is the relationship between the fundamental group for \( X \) and the first homology group of \( X \)?

(b) The fundamental group of the Klein bottle \( K \) is generated by two elements, \( x \) and \( y \), subject to the single relation \( xyx^{-1} = y^{-1} \). Use part (a) to calculate the first homology group of \( K \).

5. Let \( G \) be a group.

(a) Define what it means for \( G \) to be solvable without using commutator subgroups.

(b) Let \( N \) be a normal subgroup of \( G \). Prove that \( G \) is solvable if and only if both \( N \) and \( G/N \) are solvable. Make sure to justify your argument completely. If you appeal to any elementary homomorphism theorems, be sure to include full statements of what you are using.

6.

(a) Let \( k \) be a field of characteristic 0. Show that any irreducible polynomial in \( k[x] \) is separable.

(b) Let \( f(x) = x^4 + 5x^2 + 1 \). Compute the degree of the splitting field \( E \) of \( f(x) \) over \( \mathbb{Q} \), find the Galois group \( \text{Gal}(E/\mathbb{Q}) \), and identify \( \text{Gal}(E/\mathbb{Q}) \) with a well-known group. Explain all of your reasoning.
Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. **TIME LIMIT:** 3 hours

7. Show that $L^2([0,1])$ is of first category in $L^1([0,1])$ by showing that

$$\left\{ f : \int_0^1 |f|^2 dx \leq n \right\}$$

is closed in $L^1([0,1])$ but has empty interior.

8. Compute the homology of the one-skeleton of the $n$-simplex.

9. Prove that if $X$ and $Y$ are compact, then so is their product. (In this problem, you may not assume the Tychonoff theorem; the problem is to prove this theorem for a product of two spaces.)

10. Let $F(z)$ be an entire function, not identically zero, satisfying

$$|F(z)| \leq ae^{b \text{Re}(z)} \quad \forall z \in \mathbb{C},$$

for some $a > 0, b \in \mathbb{R}$.

Find the image $F(S)$ under $F$ of a vertical strip $S = \{ z : x_1 < \text{Re}(z) < x_2 \}$.

11.

(a) Let $G$ be a finite group, let $N$ be a normal subgroup of $G$, and let $H$ be a normal subgroup of $N$. Suppose that $H$ is a Sylow $p$-subgroup of $N$. Show that $H$ is normal in $G$.

(b) Let $F$ be a field and $E$ a finite Galois extension of $F$. Let $K, L$ be intermediate fields with $F \subset K \subset L \subset E$, and suppose that $K/F$ and $L/K$ are Galois extensions. Moreover,
suppose that $[E : L]$ is a power of a prime $p$ and that $p \nmid [L : K]$. Show that $L/F$ is Galois. (Hint: use part (a).)

12. Let $A$ be an integral domain and let $M$ be an $A$-module. Prove that the following statements are equivalent:

(i) $M$ is torsion free.

(ii) $M_p$ is torsion free for all prime ideals $p \subset A$.

(iii) $M_m$ is torsion free for all maximal ideals $m \subset A$.

(Recall that if $p \subset A$ is a prime ideal, $M_p$ is the localization of $M$ with respect to the multiplicative set $A - p$.)


PH.D. PRELIMINARY EXAMS
DAY 3
August 27, 1999

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

13. Let \( f \) be a bounded measurable function on \([0, 1]\). Define the multiplication operator \( M_f \) on \( L^2([0, 1], dx) \) by

\[
(M_f \phi)(x) = f(x) \phi(x) \text{ for all } \phi \text{ in } L^2([0, 1]) .
\]

Suppose that \( f_n \) is a sequence of bounded measurable functions on \([0, 1]\). Show that \( \lim_{n \to \infty} \langle M_{f_n} \phi, \psi \rangle = 0 \) if and only if \( \| f_n \|_{\infty} \) is bounded and

\[
\lim_{n \to \infty} \int_{a}^{b} f_n(x) dx = 0, \quad \forall a, b \text{ with } 0 \leq a < b \leq 1 .
\]

14. Find a conformal mapping taking the punctured first quadrant \( \{ z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0, z \neq e^{i\pi/4} \} \) onto the complement of the closed unit disc.

15. Let \( \mathbb{R} \) denote the real numbers with the Euclidean topology. Define a function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by the formula

\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0) 
\end{cases}
\]

Show that \( f \) is continuous in \( x \) for each \( y \) and continuous in \( y \) for each \( x \), but \( f \) is not a continuous function.
16.  
(a) Give the definition of the singular homology groups for a space $X$.  
(b) Using this definition, find the zeroth homology group of the space $X$ given by the union of the two intervals $(0, 1)$ and $(2, 3)$ regarded as a subspace of the real numbers.

17.  Let $p > q > r$ be prime numbers. Let $G$ be a group of order $pqr$.  
(a) Show that if a Sylow $p$-subgroup of $G$ is not normal then any Sylow $q$-subgroup of $G$ is normal.  
(b) Use (a) to show that any Sylow $p$-subgroup of $G$ is normal.

18.  Let $n$ be a positive integer and let $k$ be a field such that char($k$) = 0 or char($k$) is a prime not dividing $n$. Suppose $a \in k$ and let $\alpha$ be a root of $x^n - a$ in a fixed algebraic closure $\bar{k}$ of $k$. Let $\zeta$ be a primitive $n$th root of unity in $\bar{k}$.  
(a) Show that if $\zeta \in k$, then $k(\alpha)$ is Galois over $k$ and, furthermore, prove that Gal($k(\alpha)/k$) is cyclic of order dividing $n$.  
(b) If $x^n - a$ is irreducible, show that the converse of the first part of (a) is true, i.e., if $k(\alpha)/k$ is Galois then $\zeta \in k$.  

6