Algebra Prelim- Fall 2012

Problems

1. Prove that a Euclidean domain is a Principal Ideal Domain.

2. (a) Let $G$ be an abelian group with only finitely many subgroups. Prove that $G$ is a finite group.
(b) Let $P$ be a $p$-Sylow subgroup of a finite group $G$. Let $H \leq G$ such that the normalizer $N_G(P) \leq H$.
   i. For $g \in N_G(H)$, prove that $P$ and $gPg^{-1}$ are $p$-Sylow subgroups of $H$, and hence are conjugate in $H$.
   ii. Prove that $N_G(H) = H$.

3. Let $E/F$ be a finite separable extension of degree $n$. Denote by $E$ an algebraic closure of $E$. Prove that there are precisely $n$ homomorphisms $\sigma : E \rightarrow E$ such that $\sigma|_F = id$. (Hint: one approach is to induct on $n$.)

4. Let $F(\alpha)$ be a Galois extension of $F$, and suppose there exists $\sigma \in \text{Gal}(F(\alpha)/F)$ such that $\sigma(\alpha) = \alpha^{-1}$. Prove that $[F(\alpha) : F]$ is even, and that $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[F(\alpha) : F]$.

5. Suppose

$$0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$$

is an exact sequence of $R$-modules. Prove that $M$ is Noetherian if and only if $N_1$ and $N_2$ are Noetherian.

6. Let $K$ be a field. A discrete valuation on $K$ is a function $\nu : K \rightarrow \mathbb{Z} \cup \{+\infty\}$ satisfying three properties:
   (a) $\nu(0) := +\infty$, and $\nu : K^\times \rightarrow \mathbb{Z}$ is surjective
   (b) $\nu(ab) = \nu(a) + \nu(b)$
   (c) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for every nonzero $a$ and $b$ with $a + b \neq 0$.

The set $R := \{a \in K \mid \nu(a) \geq 0\} \cup \{0\}$ is called the discrete valuation ring of $\nu$. The following will show that $R$ is a Dedekind domain.
   (a) Prove that every non-zero ideal of $R$ is of the form $m_k := \{x \in R \mid \nu(x) \geq k\}$.
   (b) Prove that $R$ has only one maximal ideal $m := m_1$, and that every ideal in $R$ is of the form $m^k$ for some positive integer $k$.
   (c) Prove that $m$ is principal, and thus all ideals are principal.
   (d) Prove that all prime ideals are maximal.
   (e) Prove that $R$ is integrally closed in its field of fractions.