Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function, and there exists \( k > 0 \) such that for every \( y \in \mathbb{R} \) there are at most \( k \) distinct points \( x \in \mathbb{R} \) with \( f(x) = y \). Prove that the derivative of \( f \) exists for a.e. \( x \in \mathbb{R} \).

2. Suppose that \((X, \mathcal{M}, \mu)\) is a finite measure space, and suppose that \( \lambda \) is a finitely additive, nonnegative, real valued set function on \( \mathcal{M} \) for which the following holds. For all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( E \in \mathcal{M} \),
\[
\mu(E) < \delta \quad \Rightarrow \quad \lambda(E) < \varepsilon.
\]
Prove that there exists \( g \in L^1(\mu) \) such that for all \( E \in \mathcal{M} \),
\[
\lambda(E) = \int_E gd\mu.
\]

3. Calculate
\[
\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx
\]
and justify your steps.

4. Let \( C = \{z : |z| = 1/2\} \) and \(|w| < 3/4\). Set
\[
f(w) = \frac{1}{2\pi i} \int_C \frac{z(z + 1)}{z^2 + 2z - w} dz.
\]

   a) Why is \( f(w) \) analytic in \(|w| < 3/4\)?
   b) Find \( f(0) \).
   c) Find \( f'(0) \).

5. Let \( D = \{z \in \mathbb{C} : |z| \leq 1\} \). Suppose that \( h(z) \) is analytic in a neighborhood of \( D \), \( h(0) = 0 \), and \(|h(z)| < 1\) for all \( z \in D \). Prove that the function \( f(z) = z + h(z) \) is one-to-one on \( \{z \in \mathbb{C} : |z| < 1\} \).

6. Let \( f(z) \) be an entire function satisfying
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|d\theta \leq r^{10/3}
\]
for all \( r > 0 \). Prove that \( f \) is the zero function.
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7. (a) Define the commutator subgroup $G^c$ of a group $G$, and prove that if $H$ is any subgroup of $G$ containing $G^c$, then $H$ is a normal subgroup of $G$.

(b) Show that if $N$ is a normal subgroup of $G$, then so is $N^c$.

8. If $G$ is a group of order 30, prove that both the Sylow 3 subgroup and the Sylow 5 subgroup must be normal. (Hint: Start by proving that at least one of them must be normal.)

(b) Find all groups of order 30 up to isomorphism. Justify all your reasoning.

9. Recall that an element of a ring is called nilpotent if $x^n = 0$ for some positive integer $n$.

(a) Suppose $A$ is a commutative ring with $1 \neq 0$ satisfying the property that the localization $A_m$ has no nonzero nilpotent elements for any maximal ideal $m$. Prove that $A$ has no nonzero nilpotent elements.

(b) Would the analogous result still be true if the phrase “no nonzero nilpotent elements” is replaced by the phrase “zero divisors” (where by definition a zero divisor can not equal 0)? Prove it or give a counterexample.

10. (a) Prove that the order of any finite field must be a power of a prime and that there is one and only one field of order $p^n$ within a fixed algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(b) Prove that all finite extensions of finite fields are both normal and separable.

(c) Prove that the Galois group of any finite extension of a finite field is cyclic, and give an explicit generator for such a Galois group, being sure to justify your answer.
11. Give a complete justification for all of your answers.

(a) Find the Galois group of \( x^4 - 2 \) over \( \mathbb{Q} \). Give each element of this Galois group a name, and tell its effect on the generators of the splitting field of the given polynomial.

(b) List all the subgroups along with the corresponding intermediate fields.

12. Let \( p \) be an arbitrary prime. In each of the following two cases, find a Galois extension of degree \( p \) of the given field \( k \) if such an extension exists. If no such extension exists, then explain why not. Be as precise as possible, and give complete proofs of all of your claims.

(a) \( k = \mathbb{Q}^a(\beta) \) where \( \mathbb{Q}^a \) denotes the algebraic closure of \( \mathbb{Q} \) and \( \beta \) is an element transcendental over \( \mathbb{Q} \).

(b) \( k = \mathbb{F}_p^a(\beta) \) where \( \mathbb{F}_p^a \) denotes the algebraic closure of the field \( \mathbb{F}_p \) with \( p \) elements and \( \beta \) is an element transcendental over \( \mathbb{F}_p \).
DIRECTIONS: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C.

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A. GENERAL TOPOLOGY

13. Let \((X_i)_{i \in I}\) be an indexed family of connected topological spaces and let \(a_i\) be an element of \(X_i\), for all \(i \in I\). Let \(X\) be the Cartesian product of the spaces \(X_i\), for \(i \in I\).
   
   (a) For every finite subset \(K\) of \(I\), let \(X_K\) be the subset of \(X\) consisting of all \((x_i)_{i \in I}\) such that \(x_i = a_i\) for all \(i\) not in \(K\). Then \(X_K\) is clearly homeomorphic to the finite Cartesian product of the \(X_i\) for \(i \in K\). Prove that \(X_K\) is connected.
   
   (b) Show that the union \(Y\) of the subspaces \(X_K\) of \(X\) is connected.
   
   (c) Show that \(X\) is the closure of \(Y\).
   
   (d) Prove that the closure of a connected subspace of a topological space is connected.
   
   (e) Use what you’ve proven above to show that \(X\) is connected.

14. Let \(A\) and \(B\) be disjoint compact subsets of a Hausdorff topological space \(X\). Then prove that there exist disjoint open subsets \(U\) and \(V\) of \(X\) such that \(A\) is contained in \(U\) and \(B\) is contained in \(V\).

15. Let \(G\) be a multiplicative topological group such that there is a countable open base for the neighborhood system at 1. Then
   
   (a) Show that \(G\) obeys the first Axiom of Countability (i.e., that there is a countable open base for the neighborhood system at \(x\), for all \(x \in G\)).
   
   (b) Suppose that \(G\) as above also has a countable dense subset – i.e., is separable. Then prove that \(G\) obeys the second Axiom of Countability (i.e., that there is a countable open base for the topology of \(G\)).
B. ALGEBRAIC TOPOLOGY

16. Compute the rational homology for the 3-skeleton of the 7-simplex. To start, find its Euler characteristic.

17. Prove that there is no closed orientable 2-manifold with integral homology in dimension 1 isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

18. Let $A$ be a real $n \times n$ matrix with positive entries. Prove that $A$ has a positive eigenvalue.

C. FUNCTIONAL ANALYSIS

19. 1. For $1 \leq p < \infty$ and $n$ positive integer, define the linear operators $S_n, T_n : l^p \rightarrow l^p$ by

\[ S_n(x) = \left( \frac{x_1}{n}, \frac{x_2}{2n}, \frac{x_3}{3n}, \ldots \right), \quad T_n(x) = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots), \quad (\forall) x = (x_n)_{n} \in l^p \]

Prove that:
   i) $\lim_{n \to \infty} \|S_n\| = 0$;
   ii) $\lim_{n \to \infty} T_n(x) = 0, (\forall) x \in l^p$;
   iii) $\lim_{n \to \infty} \|T_n\| \neq 0$.

20. Let $H$ be a complex Hilbert space and $G$ a closed linear subspace of $H$. Prove that for any $f \in G' = B(G, \mathbb{C})$, there exists a unique $F \in H'$ that extends $f$ to $H$ and $\|f\|_{G'} = \|F\|_{H'}$.

21. Let $H$ be a Hilbert space and $A, B \in B(H)$ two positive operators (i.e., $C \in B(H)$ is positive if it is self-adjoint and $<Cx, x> \geq 0, (\forall) x \in H$), such that $B$ is compact and $B - A$ is positive. Prove that $A$ is compact.