Analysis Prelim Questions
Day 1—August 29, 2006

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. **TIME LIMIT:** 3 hours

1. Let \( \{ \nu_i, \mu_i : i = 1, \ldots n \} \) be a family of positive, \( \sigma \)-finite measures on \( \mathbb{R} \) with \( \nu_i \ll \mu_i \), and let \( f_i \) denote the Radon-Nikodym derivative \( d\mu_i/d\nu_i \). Show that the product measures satisfy \( \nu \ll \mu \). State and prove a formula for the Radon-Nikodym derivative \( f = d\mu/d\nu \) in terms of the Radon-Nikodym derivatives \( \{ f_i \} \).

2. (a) give an example of functions \( f_n, f \) belonging to both \( L^3(\mathbb{R}) \) and \( L^4(\mathbb{R}) \) such that \( f_n \to f \) in \( L^3 \) but not in \( L^4 \).
    (b) Give a similar example where \( f_n \to f \) in \( L^4 \) but not in \( L^3 \).
    (c) If we are working on \([0, 1]\) instead of \( \mathbb{R} \), show that if \( f_n \to f \) in \( L^4 \) then the convergence also holds in \( L^3 \).

3. Let \( f, g \) be absolutely continuous functions on \([0, 1]\). Show that the derivative \((fg)'\) exists a.e. on \([0, 1]\) and that
   \[
   \int_a^b (f(x)g(x))' \, dx = f(b)g(b) - f(a)g(a)
   \]
   for \([a, b] \subset [0, 1]\).

4. (Two unrelated problems)
   (a) For the function \( g(z) = \frac{1}{z} + \frac{1}{z-1} \), write down its Laurent expansion of the form
   \[
   \sum_{n=-\infty}^{\infty} a_n (z - \frac{1}{2})^n
   \]
   which is convergent for \( z \) with \( |z| \) large enough. Where exactly does the expansion converge? What is the type of convergence?
   (b) Prove that the equation \( \sin z = 3z^2 \) has at least two solutions in \( \{ z : |z| < 1\} \).

5. Using the theory of residues evaluate the integral
   \[
   \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)(9 + x^2)}.
   \]

6. Let \( \Re \) denote the real part of a complex number. Find a conformal mapping that maps the set \( D = \{ z : -\pi < \Re z < \pi \} \) onto the set \( \Omega = \{ w : 0 < \arg w < \pi \} \). Choose a direction on each part of the boundary \( \partial D \) of \( D \). Let a point \( z \) travel in that direction along \( \partial D \). What is the direction in which the image of \( z \) travels along \( \partial \Omega \)?
Algebra Prelim Questions  
Day 2—August 30, 2006

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. **TIME LIMIT: 3 hours**

7.  
(a) Let \( P \) be a \( p \)-group which acts on a set \( X \). If \( X^P = \{ x \in X | gx = x \text{ for all } g \in P \} \), show that \( |X| \equiv |X^P| \mod p \).

(b) Show that if \( G \) acts on \( X \) transitively, \( |X| > 1 \), then there exists \( g \in G \) such that \( gx \neq x \) for all \( x \in X \).

(c) Let \( \Sigma_5 \) be the symmetric group on 5 letters. Determine a complete list of the conjugacy classes of \( \Sigma_5 \) giving a representative for each and listing the size of each conjugacy class.

8.  
(a) Give a precise definition of what it means for a group \( \Gamma \) to be solvable. Show that in any nontrivial solvable group, there exists a nontrivial normal Abelian subgroup.

(b) Let \( G = GL_2(\mathbb{F}_5) \) be the general linear group of invertible \( 2 \times 2 \) matrices over the field \( \mathbb{F}_5 \). [Recall \( A \) is invertible if and only if \( \det A \neq 0 \).] Let \( H = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} | * \in \mathbb{F}_5 \right\} \). Show that \( H \) is a Sylow-5 subgroup of \( G \) and find \( N_G(H) \). Use this to determine the number of Sylow-5 subgroups of \( G \).

9.  
(a) Let \( \mathbb{Z}[i] = \{ a+bi | a, b \in \mathbb{Z} \} \) be the Gaussian integers which is a subring of the complex numbers \( \mathbb{C} \). Show that \( \mathbb{Z}[i] \) is a Euclidean domain with norm/degree \( D(a + bi) = a^2 + b^2 \).

(b) Determine the units of the ring \( \mathbb{Z}[i] \).

(c) Let \( p \in \mathbb{Z} \) be a prime number. Show that

\[
(p \text{ is reducible in } \mathbb{Z}[i]) \iff (p \equiv 1, 2 \mod 4) \\
\iff p = a^2 + b^2, a, b \in \mathbb{Z}
\]

(d) Let \( n \in \mathbb{N} \) have prime factorization \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) with the property that if \( p_j \equiv 3 \mod 4 \) then \( \alpha_j \) is even. Show that \( n \) is the sum of two integer squares.
10.
(a) Show that any group of order 225 is solvable.
(b) Let $G$ be a finite group. Let $p$ be a prime divisor of $|G|$. Show that if $H$ is a subgroup of $G$ that is contained in every $p$-Sylow subgroup of $G$, then $H$ must be a normal subgroup of $G$.

11.
(a) Let $L$ be the splitting field of the polynomial $f(x) = x^3 - 5$ over the field $\mathbb{Q}$. What is the Galois group of $L$ over $\mathbb{Q}$?
(b) How many subfields $E$ of $L$ have the property that $[E : \mathbb{Q}] = 3$?
(c) Let $M$ be the splitting field of the polynomial $f(x) = x^8 - 1$ over the field $\mathbb{Q}$. What is the Galois group of $M$ over $\mathbb{Q}$?
(d) How many subfields $K$ of $M$ have the property that $[K : \mathbb{Q}] = 2$?

12.
(a) Let $E$ and $L$ be Galois extensions of a field $K$. Suppose that $E$ and $L$ are each contained in a larger field $M$. Show that the compositum $EL$ is also a Galois extension of $K$.
(b) Suppose that $[E : K] = 11$ and $[L : K] = 5$. Show that the Galois group of $EL$ over $K$ is cyclic.
(c) Suppose that $[E : K] = 7$ and $[L : K] = 7$. Show that the Galois group of $EL$ over $K$ is not cyclic.
Prelim Questions:
A. General Topology, B. Algebraic Topology, and C. Functional Analysis
Day 3 — August 31, 2006

Directions: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C. Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

A. GENERAL TOPOLOGY

13. Prove one of the following four theorems from the basic definitions.
   (a) (Tychonoff Theorem) If $X$ and $Y$ are compact, then the product $X \times Y$ is compact.
   (b) (Heine-Borel Theorem) A subspace $X$ of $\mathbb{R}^n$ (with the Euclidean topology) is compact if and only if $X$ is closed and bounded.
   (c) The unit interval $[0, 1]$ is connected (with the natural Euclidean topology).
   (d) If $X$ is compact and Hausdorff, then $X$ is regular.

14. Give $\mathbb{R}$ the standard Euclidean topology. Define a subspace $X$ of $\mathbb{R}$ (with the subspace topology) by
   \[ X = (-\infty, -1) \cup [0, \infty). \]

   Define
   \[ f : X \to \mathbb{R} \]
   by the formula $f(x) = x$ for $x < -1$ and $f(x) = x - 1$ for $0 \leq x$.
   (a) Show that $f$ is a continuous bijection $x \to x$.
   (b) Is $f$ a homeomorphism? Justify your answer.

15. Sketch a computation of the fundamental group of one of the following two spaces.
   (a) the circle $S^1$ or
   (b) the figure 8 regarded as a subspace of the plane.
B. ALGEBRAIC TOPOLOGY

16. Let $X = \mathbb{R}^2 - \{(1, 0), (-1, 0)\}$.

(a) Find $H_1(X)$ and $H_2(X)$.

(b) Is $\pi_1(X)$ abelian? Justify your answer.

17. 
(a) Give an example of a nonempty topological space $X$ which is homeomorphic to $Y = X \sqcup X$ (the disjoint union of two copies of $X$), where each copy of $X$ is an open subset of $Y$.

(b) Let $M$ be the Möbius band and $D$ the 2-dimensional disk. Let $X$ be the quotient of $M \cup D$ obtained by identifying the bounding circles of $M$ and $D$ via a homeomorphism between them. Find $\pi_1(X)$.

18. Let $X$ be a polyhedral surface without boundary having 5 vertices, 10 edges and 5 square faces.

(a) Describe the topology of $X$. Is it uniquely determined by this information?

(b) Draw a picture of such an $X$.

C. FUNCTIONAL ANALYSIS

19. 
(a) Show that if $K$ is a closed convex subset of a Hilbert space $\mathcal{H}$, and $h$ is a vector in $\mathcal{H}$ outside $K$, there is a unique $k_0 \in K$ such that for all $k \in K$, $\|h - k_0\| \leq \|h - k\|$.

(b) Give an example of a closed convex subset of a Banach space where this uniqueness fails.

20. Suppose that $A = (a_{ij})$ is an infinite matrix such that for any square summable sequence $x = x_i$,

$$ (Ax)_i = \sum_{j=1}^{\infty} a_{ij}x_j $$

is also square summable. Show that the linear transformation $A : \ell^2 \to \ell^2$ is a bounded.

21. Let $X$ be a Banach space and let $X^*$ be its dual space. Suppose that $A : X \to X$ is a bounded linear transformation. Define $A^* : X^* \to X^*$ by $A^*x^*(y) = x^*(Ay)$ for all $x^*$ in $X^*$ and $y$ in $X$. Show that $\|A^*\| = \|A\|$. 
