Analysis Prelim Questions  
Day 1 – August 30, 2004

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Suppose that $f$ and $g$ belong to $L^1(\mathbb{R})$. The convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$ 

A. Show that $f * g$ is defined almost everywhere on $\mathbb{R}$, and belongs to $L^1(\mathbb{R})$.

B. Show that if $f$ is bounded and continuous, then $f * g$ is also bounded and continuous.

2. 

A. Define “convergence in measure.”

B. Let $f_n$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$. Consider the following three statements:

1. $f_n$ converges in measure to 0 as $n \to \infty$.

2. $f_n$ converges almost everywhere to 0 as $n \to \infty$.

3. $f_n$ converges in $L^1$ norm to 0 as $n \to \infty$.

(The measure in each case is Lebesgue measure.)

For a given sequence $f_n$ there are 8 possibilities, depending on which of these 3 statements is true:

(a) (1), (2), and (3) are all false.

(b) (1), (2) are false and (3) is true.

(c) (1) is false, (2) is true, (3) is false.

(d) (1) is false, (2) and (3) are true.

(e) (1) is true, (2) and (3) are false.
(f) (1) is true, (2) is false, (3) is true.

(g) (1) is true, (2) is true, (3) is false.

(h) (1), (2), and (3) are all true.

For each of these, give an example, or prove that it is impossible.

3.
A. Show that if \( f \) is differentiable on \([0,1]\), then \( f \) is measurable.
B. Show that if \( f \) is in \( L^1([0,1]) \), then \([If](x) = \int_{0,x} f(t)dt \) is measurable.
C. Show that if \( f \) and \( g \) are in \( L^1([0,1]) \), then integration by parts holds in the sense that
\[
\int_{[0,1]} [If](x)g(x)dx = [If](1)[Ig](1) - \int_{[0,1]} f(x)[Ig](x)dx .
\]
[Here \([If] \) is as defined in B.]

4. Give the Laurent expansion of \( f(z) = \frac{1}{z(z-1)} \) in the region
A. \( \{ z : 0 < |z| < 1 \} \)
B. \( \{ z : 0 < |z - 1| < 1 \} \).

5. The Fourier transform of a complex-valued function \( f \) defined on \((-\infty, \infty)\) is
\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx .
\]
Suppose that \( f \) is continuous and has compact support. Show that its Fourier transform is an entire analytic function of \( k \). Is it possible for the Fourier transform of \( f \), restricted to the real line, to have compact support? (Explain.)

6. Use complex analysis to prove the fundamental theorem of algebra: if \( p \) is a polynomial, then there is a complex number \( z \) such that \( p(z) = 0 \).
Algebra Prelim Questions  
Day 2 – August 31, 2004

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. **TIME LIMIT:** 3 hours
Prelim Questions:
A. General Topology, B. Algebraic Topology, and C. Functional Analysis
Day 3 – September 1, 2004

Directions: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C.

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

A. General Topology:

13. (a) Give an example of a path connected subspace of a topological space whose closure is not path connected.
(b) If $X$ is a topological space and $A, B$ subspaces satisfying $A \subseteq B \subseteq \bar{A}$ then prove that if $A$ is connected then $B$ is also connected.

14. Let $X$ be a complete metric space and let $\{U_i\}_{i=1}^{\infty}$ be a countable collection of open subsets of $X$. Suppose each $U_i$ is dense in $X$ i.e., $\bar{U_i} = X$. Prove that $\bigcap_{i=1}^{\infty} U_i$ is dense in $X$. (Do not just quote the Baire Category Theorem but prove it in this setting.)

15. (a) Give an example of a complete metric space that is not compact.
(b) State a necessary and sufficient hypothesis on a complete metric space for it to be compact.
(c) Prove your statement in (b).

B. Algebraic Topology:

16. Consider the space $X_n$ which is obtained as a quotient space of a regular $n$-gon by identifying
all the edges of the n-gon to a single edge as indicated in the diagram below:

(a) Find a cellular or simplicial structure for $X_n$ and use it to compute all its integral homology groups $H_*(X_n)$.
(b) Recall given a space $X$, the (unreduced) suspension of $X$, denoted $\Sigma X$, is the quotient space of $X \times [0, 1]$ where $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for all $x_1, x_2 \in X$. We write inductively $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$ for $n \geq 2$. Find the integer cohomology groups of $\Sigma^4(X_5)$.
(c) Find the integral homology groups of $Y = X_3 \times X_3 \times X_5$.

17.
(a) Let $M$ be a compact, connected, orientable 8-dimensional manifold without boundary. Suppose you are given partial information on the integral homology of $M$:

$H_*(M) \cong \begin{cases} 
\mathbb{Z} & \text{if } * = 0 \\
\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} & \text{if } * = 1 \\
0 & \text{if } * = 2 \\
\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} & \text{if } * = 3 \\
0 & \text{if } * = 4 
\end{cases}$

Use this information to compute the integral cohomology groups $H^n(M)$ for all $n \in \mathbb{N}$.
(b) Let $f : S^1 \to S^3$ be a smooth embedding (e.g. a knot). (Thus in particular $f$ induces a homeomorphism of $S^1$ with $f(S^1)$. You may assume that the embedding is “tame”.) Let $X = S^3 - f(S^1)$. Compute $H^n(X)$ for all $n \in \mathbb{N}$.

18.
(a) Let $\mathbb{RP}^n$ denote $n$-dimensional real projective space. Describe the universal cover $X$ of $\mathbb{RP}^n$ for $n \geq 2$. What is $\pi_1(\mathbb{RP}^n)$? Let $g : X \to \mathbb{RP}^n$ be the covering map. Describe a path $\gamma$ in $X$ such that $g(\gamma)$ generates $\pi_1(\mathbb{RP}^n)$.
(b) Describe the cohomology ring $H^*(\mathbb{RP}^n; \mathbb{F})$ completely where $\mathbb{F}$ is the field with 2 ele-
ments.

(c) Recall $S^n = \{x \in \mathbb{R}^{n+1} ||x|| = 1\}$ is the $n$-sphere. Prove the Borsuk-Ulam theorem which says that there does not exist a continuous map $f : S^n \to S^m$ with $f(-x) = -f(x)$ if $n > m \geq 2$.

C. Functional Analysis:

19. Suppose that $K$ is a compact operator on a Hilbert space $\mathcal{H}$. If $f$ is a continuous function on $\mathbb{R}$, define the operator $f(K)$ on $\mathcal{H}$. Under what conditions on $f$ is $f(K)$ compact and self-adjoint?

20. Let $h$ and $k$ be linearly independent vectors in a Hilbert space $\mathcal{H}$. Find a linear functional $L$ on $\mathcal{H}$ such that $L(h) = ||h||$, $L(k) = 0$, and $||L|| = 1$. Must there be such a linear functional for a pair of linearly independent vectors in a Banach Space? (Explain.)

21. If $\mathcal{X}$ is a Banach space, $\mathcal{B}(\mathcal{X})$ is the set of bounded linear operators taking $\mathcal{X}$ into $\mathcal{X}$. The spectrum of an operator $T$ in $\mathcal{B}(\mathcal{X})$ is

$$\sigma(T) = \{z \in \mathbb{C} : T - zI \text{ does not have an inverse in } \mathcal{B}(\mathcal{X})\}.$$

A. Is it possible for $T - zI$ to have an inverse which is not in $\mathcal{B}(\mathcal{X})$?

B. Give examples of $T$ and $z$ such that $T - zI$ is

(i) neither injective nor surjective.

(ii) injective, but not surjective.

(iii) surjective, but not injective.