PH.D. PRELIMINARY EXAMS
DAY 1
September 4, 2001

Do as many problems as you can in whatever order you wish. Use a separate blue book for each problem. Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Find the Laurent expansion of \( \frac{1}{z^2 - 4} \) around \( z = 2 \). Where is this expansion valid?

2. Show that between any two finite dimensional normed vector spaces over \( \mathbb{R} \) of the same dimension there is a linear homeomorphism.

3. Let \((X, \chi, \mu)\) be a measure space, and denote by \( \lambda \) and \( \mathcal{L} \) respectively the Lebesgue measure and the Lebesgue measurable sets on \( \mathbb{R} \). Let \( f \) be a nonnegative integrable function on \( X \) and let

\[ E = \{(x, y) \in X \times \mathbb{R} : 0 \leq y < f(x)\} . \]

Show that \( E \) is \( \chi \times \mathcal{L} \) measurable, and that \( \mu \times \lambda(E) = \int f(x) d\mu \), i.e., the integral is the “area” under the graph.

4. Prove that every continuous map \( f : D^n \to D^n \) has a fixed point, where \( D^n \) is the closed ball of radius 1 in \( \mathbb{R}^n \).
5.  
(a) Suppose $H$ is a nonnormal subgroup of a finite group $G$ of prime index $p$. Show that $G$ contains $p$ different subgroups that are conjugate to $H$.

(b) Continue to let $H$ be as above. Deduce from part (a) that $H$ contains a strictly smaller subgroup $K$ that is normal in $G$ such that the index $[G : K]$ divides $p$!

(c) Deduce from part (b) that if $H$ is any subgroup of index $p$ in a finite group $G$ where $p$ is the smallest prime dividing the order of $G$, then $H$ must be a normal subgroup of $G$.

6. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n > 2$, and let $K$ be a splitting field for $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group for $f(x)$ over $\mathbb{Q}$ is isomorphic to $S_n$, the symmetric group on $n$ letters.

(a) Show that $f$ is irreducible over $\mathbb{Q}$.

(b) If $\alpha$ is a root of $f$, show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.

(c) If $n \geq 4$, show that $\alpha^n \not\in \mathbb{Q}$.  

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7. Suppose that $a > 1$. Show that
\[ \int_0^{2\pi} \frac{dx}{a + \cos x} = \frac{2\pi}{\sqrt{a^2 - 1}} \]
(HINT: Express $\cos x$ by Euler’s formula, then put $z = e^{ix}$ to transform the integral into a contour integral.)

8. Let $X$ be the space obtained by taking the intersection of the unit sphere in $\mathbb{R}^3$ with the union of the three coordinate planes $x = 0, y = 0, z = 0$. Calculate the integral homology group of $X$.

9. Suppose that $V$ is a Banach space and $\ell_n$ is a sequence of bounded linear functionals on $V$, such that for all $v, \ell_n(v)$ converges. Show that the limit $\ell(v)$ is a bounded linear functional. Show by example that this is not necessarily true for the space $V$ of continuous functions of compact support on $\mathbb{R}$, with norm $\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}$.

10. Let $X$ be a subset of a metric space $Y$.

(a) Show that $X$ is connected if and only if for every continuous real valued function $f$ on $Y$, $f(X)$ is connected.

(b) Show that $X$ is compact if and only if for every continuous real valued function $f$ on $X$, $f(X)$ is compact.
11. Let $A$ be a commutative ring with multiplicative identity which has the property that for each $x \in A$, there exists an integer $n \geq 2$ with the property that $x^n = x$, where the integer $n$ may vary with $x$.

(a) Show that every prime ideal in $A$ is a maximal ideal.

(b) If $p$ is a prime ideal in $A$, show that the ring $A_p$ is a field, where $A_p$ denotes the localization of $A$ with respect to the set $S = A - p$.

12. Let $K/k$ be a finite Galois extension with Galois group $G$. Let $F$ be an immediate field between $K$ and $k$.

(a) Let $H$ be the subgroup of $G$ consisting of all automorphisms that map $F$ into itself. Show that $H$ equals the normalizer of $\text{Gal}(K/F)$ in $\text{Gal}(K/k)$.

(b) Let $E$ equal the fixed field of $H$. Show that $F/E$ is a Galois extension and $E$ is the smallest subfield of $F$ containing $k$ that has this property.
PH.D. PRELIMINARY EXAMS
DAY 3
September 1, 2000

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

13. In the diagram below, the circles represent sequences of integrable functions on \([0, \infty)\). \(AE\) represents those that converge almost everywhere, \(M\) represents those that converge in measure, and \(L^1\) represents those that converge in the mean, i.e., in \(L^1\) norm. For each of the subsets a–f in the diagram, either give an example of a sequence in the subset, or explain why there are no such sequences. (A clearly labeled sketch is sufficient for the examples.) How do the answers change if \([0, \infty)\) is replaced by \([0, 1]\)?

14. Suppose \(f\) is an entire function satisfying \(|f(z)| \leq c|z|^M\) for some real \(M \geq 0\) and all \(z \in \mathbb{C}\). Prove that \(f\) is a polynomial.

15. Let \(H\) be a Hilbert Space, and let \(A\) be a compact self-adjoint operator on \(H\) satisfying \(\langle Av, v \rangle \geq 0\) for all \(v \in H\). For \(v \in H\), find \(\lim_{n \to \infty} w_n\), where \(w_n = A^nv/\|A^nv\|\).

16. Let \(X\) be the metric space

\[
\{(x, y) : 0 \leq x \leq 1, \text{ and if } x = 0, -1 \leq y \leq 1, \text{ otherwise, } y = \sin(1/x)\},
\]

17. 
(a) State and prove the Hilbert basis theorem for polynomials over Noetherian rings.
(b) Is this theorem still true if the word Noetherian is replaced by Artinian? Prove this or give a counterexample and justify your reasoning.

18. Let $x^4 + ax^2 + b$ be an irreducible polynomial over $\mathbb{Q}$ with roots $\pm \alpha, \pm \beta$ and splitting field $K$.
(a) Show that the Galois group of $K$ over $\mathbb{Q}$ is isomorphic to a subgroup of $D_8$ (the dihedral group that contains 8 elements) and hence must be isomorphic to one of the following:
   (i) $\mathbb{Z}/4\mathbb{Z}$
   (ii) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
   (iii) $D_8$

(b) Show that case (i) happens if and only if $\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbb{Q}$, case (ii) happens if and only if $\alpha \beta \in \mathbb{Q}$ or $\alpha^2 - \beta^2 \in \mathbb{Q}$, and case (iii) happens the rest of the time.

(Actually, the case $\alpha^2 - \beta^2$ is in $\mathbb{Q}$ cannot occur since it corresponds to the Galois group of $K/\mathbb{Q}$ being a subgroup of $S_4$ that does not permute the roots $\pm \alpha, \pm \beta$ transitively.)

(c) Find the splitting field in $\mathbb{C}$ of the polynomial $x^4 - 4x^2 - 1$. Determine the Galois group of this splitting field over $\mathbb{Q}$, and match up all subgroups with all intermediate fields.