Analysis Prelim Questions
Day 1, January 16, 2006

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. (Two unrelated problems)

(a) Prove that the equation $e^z = \pi z^2$ has at least two solutions in $\{z : |z| < 1\}$.

(b) Write down the Laurent expansion of the form $\sum_{n=-\infty}^{\infty} a_n(z-1)^n$ which is convergent on an unbounded domain for the function $g(z) = \frac{1}{z}$. Where exactly does it converge? What is the type of convergence?

2. Using the theory of residues, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(4+x^2)^2} \, .$$

3. Find a conformal mapping that maps the set $D = \{z : 0 < \arg z < 2\pi\}$ on the set $\Omega = \{w : 0 < \Im w < \pi\}$ where $\Im$ denotes the imaginary part. Choose a direction on each part of the boundary ∂D of D. Let a point z travel in that direction along ∂D. What is the direction in which the image of z travels along $\partial\Omega$?

4.

(a) Suppose that f is in $L^1([0,1])$ (with Lebesgue measure) and continuous on $(0,1]$. Show that the improper Riemann integral

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1} f(x)dx$$

equals the Lebesgue integral of f.

(b) Give an example of a function continuous on $(0,1]$ for which the above improper Riemann integral exists, but which is not in $L^1([0,1])$, and justify this.

5. Suppose that f is a nonnegative function in $L^1([0,1])$. Let $A(f) = \{(x,y) : 0 \le y \le f(x)\}$. Show that $A(f)$ is a measurable set in $\mathbb{R}^2$ and its product measure equals the integral of f.

6. Suppose that $1 \leq p < q \leq \infty$.

(a) Show that $L^p([0,1]) \supset L^q([0,1])$ and the inclusion is proper.

(b) Show that $\ell^p \subset \ell^q$ and the inclusion is proper.

$(\ell^p = \{(a_1, a_2 \ldots) : \sum_{j=1}^{\infty} |a_j|^p < \infty\})$

DIDN"T USE DAY 2...no one took these exams.

Prelim Questions:
A. General Topology, B. Algebraic Topology, and C. Functional Analysis
Day 3—January 18, 2006

Directions: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

A. GENERAL TOPOLOGY

13. (Heine-Borel Theorem) Prove the following theorem.

A subspace X of $\mathbb{R}^n$ (with the Euclidean topology) is compact if and only if X is closed and bounded.

14. (Connected spaces)

(a) Give the definition of the statement that "a topological space X is connected".

(b) Give an example of a space which is not connected. Prove that your example is not connected.

(c) Let X denote the subspace of the plane $\mathbb{R}^2$ given by the points (x, y) where either (i) $x = 0$ with $-1 < y < 1$ or (ii) $0 \leq x < 1$ with $y = 0$. Use properties of spaces that are not connected to show that if $f : X \to X$ is a homeomorphism, then $f(0, 0) = (0, 0)$.

15. (Fundamental groups)

Sketch a computation of the fundamental group of one of the following two spaces giving as much detail as you can: (1) the circle S^1, or (2) the plane punctured twice.

B. ALGEBRAIC TOPOLOGY

16. Let K_5 be the complete graph on 5 points, that is a graph with 5 vertices and 10 edges, one joining each pair of points. Use an Euler characteristic argument to show that K_5 cannot be embedded in the plane.

17. Which of the following spaces are homotopy equivalent. Prove your answer.

(i) The Mobius band with an interior point removed.

(ii) The unit disk with an interior point removed.

(iii) The unit disk with two interior points removed.

18. Give an example of a space X that is homeomorphic to itself with a single point removed and construct the homeomorphism.

C. FUNCTIONAL ANALYSIS

19. Let $\mathcal{X}$ be a Banach space, and let $\mathcal{B}(\mathcal{X})$ be the set of bounded linear transformations $T\mathcal{X} \to \mathcal{X}$, with $\|T\| = \inf\{C > 0 : \|Tx\| \leq C\|x\| \text{ for all } x \text{ in } X\}$.

(a) Show that $\| \quad \|$ is a norm on $\mathcal{B}(\mathcal{X})$.

(b) Show that $\mathcal{B}(\mathcal{X})$ is a Banach space.

(c) Show that if $\|T\| < 1$, then $(I - T)$ has an inverse in $\mathcal{B}(\mathcal{X})$.

20.

(a) Let $\mathcal{H}$ be a Hilbert space, and let h be a vector in $\mathcal{H}$. Find a linear functional f on $\mathcal{H}$ such that $f(h) = \|h\|$ and $\|f\| = 1$, and show that it is the only one with this property.

(b) Let $\mathcal{X}$ be a Banach space, and let x be a vector in $\mathcal{X}$. Show that there exists a linear functional g on $\mathcal{X}$ such that $g(x) = \|x\|$ and $\|g\| = 1$. Find a Banach space $\mathcal{X}$ and a vector x for which such g is not unique.

21. Suppose that K is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$. Show that the range of K cannot be all of $\mathcal{H}$.