Analysis Prelim Questions

Day 1 – January 12, 2004

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. (a) State the Lebesgue dominated convergence theorem for a measure space (Ω, F, μ).

(b) Consider the case $\Omega = \mathbb{N}, F = 2^{\mathbb{N}}$, and $\mu(E) =$ the number of elements in E. What sufficient condition does the Lebesgue dominated convergence theorem provide for the validity of

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_{nk} \ ?$$

(b) Show that

$$\lim_{n\to\infty} \int_0^{\pi} \sum_{k=0}^{\infty} a_k \sin^n kx \, dx = 0 \text{ if } \sum_{k=0}^{\infty} |a_k| < \infty \, .$$

2. (a) State Hölder's inequality for real valued functions f, g on $\mathbf{R}$.

(b) Given real valued functions f_1, f_2, f_3 on $\mathbf{R}$, show that if $p_1, p_2, p_3 > 1$ and

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

then

$$\int_{\mathbf{R}} |f_1(x) f_2(x) f_3(x)| dx \le \prod_{k=1}^{3} \left(\int_{\mathbf{R}} |f_k(x)|^{p_k} \right)^{1/p_k}$$

(c) Generalize part (b) to constants $p_1, \ldots, p_n$ and functions $f_1(x), \ldots, f_n(x)$.

(d) Take logarithms in part (c), replace $f_k(x)$ by $\exp(g(x,y))$, and replace p_k by $p(y)$. Replace the sum over k by an integral over y. What upper bound does this suggest for

$$\log\left[\int_{\mathbf{R}} \exp\left(\int_{\mathbf{R}} g(x,y)dy\right) dx\right]$$

You need not prove your result.

3. Consider

$$f(x) = x \log\left(1 + x^{-1}\right)$$

for $x > 0$.

(a) Show that $f(x)$ is strictly increasing on $(0, \infty)$.

(b) Compute the limit of $f(x)$ as $x \to 0$ and as $x \to \infty$.

4. Let $f(z)$ be a bounded analytic function in the right half-plane $Rez \geq 0$. Prove that if $f(n) = 0$ for $n = 1, 2, 3, \ldots$, then f is identically 0 in this half-plane.

5. Evaluate

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)^2}\, dx.$$

by the theory of residues. Justify all steps.

6. Construct a Moebius transformation (linear fractional map), T, that maps the inside of the circle $|z| = 2$ onto the outside of the circle $|z + 1| = 1$ and takes the point -2 to 0 and the point 0 to 1. Justify your construction, that is, prove that the map does what it is supposed to.

Algebra Prelim Questions
Day 2 – January 13, 2004

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1.

(a) Let S_n denote the symmetric group on n letters. Find all Sylow-2 and Sylow-3 subgroups of S_n for $n = 3, 4, 5$.

Note: You don't have to write out all the elements of each such Sylow subgroup if you don't wan to , but for each S_n, you should explicitly describe one Sylow-2 (resp. Sylow-3) subgroup (by writing out all its elements, for example), and also you should state exactly how many Sylow-2 (resp. Sylow-3) subgroups there are in that S_n, and explain how you get the other ones from the one you explicitly described.

(b) Let G a finite group, and P be a Sylow-p subgroup of G. Prove that

$$N_G(N_G(P)) = N_G(P)$$

where $N_G(P)$ in G.

2. Let A be an Abelian group containing elements a and b of orders m and n respectively.

(a) If m and n are relatively prime, show that ab has order mn.

(b) Show that even if m and n are not relatively prime, that A must contain some element of order equal to the least common multiple of m and n respectively.

(c) Give an example to show that ab does not have order $lcm(m, n)$ in general.

3. Let R be a commutative ring, S a multiplicatively closed subset that does not contain 0. Prove that there is a one-to-one correspondence between prime ideals in $S^{-1}(R)$ and prime ideals in R that do not meet S.

Note: While the result holds for arbitrary commutataive rings R, you may restrict to the case where R is an inegral domain if you wish.

4. Let R be a ring and let M', M and M'' be R modules, and suppose that

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of R-modules.

(a) Show that M is Noetherian if and only if M' and M'' are.

(b) Let M be an R-module and N_1, N_2 be R-submodules of M. Suppose that M/N_1 and M/N_2 are Noetherian R-modules. Prove that $M/(N_1 \cap N_2)$ is also a Noetherian R-module.

5.

(a) Let K be a field with no nontrivial Abelian Galois extensions. Suppose that n is a positive integer and either $char(K) = 0$ or n is relatively prime to $char(K)$. Prove that every element of K is an n-th power in K.

(b) Is the statement still true if n is not relatively prime to $char(K)$? Prove it or give a counterexample.

6.

(a) Let K be a finite (not necessarily separable) extension of F. Define the Norm from K to G of an element $\alpha \in K$, denoted by $N_{K/F}(\alpha)$, and define the Trace from K to F of α, denoted by $Tr_{K/F}(\alpha)$.

(b) Let ζ be a primitive nth root of unity, and let $K = \mathbb{Q}(\zeta)$.

(i) If $n = p^r$ for some prime p and $r \geq 1$, show that

$$N_{K/F}(1 - \zeta) = p \ .$$

(ii) If n is divisible by at least 2 distinct primes, show that

$$N_{K/F}(1 - \zeta) = 1 \ .$$

Prelim Questions:
A. General Topology, B. Algebraic Topology, and C. Functional Analysis
Day 3 – January 14, 2004

Directions: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

PART A

1. Let X be a complete metric space and $\{U_n\}$ a countable collection of open sets, each of which is dense in X (this means that the closure of each U_n equals X).

(a) Prove that the intersection $\bigcap_n U_n$ is dense in X. (Do not quote the Baire Category Theorem as this is a form of that theorem.)

(b) Give an example of a metric space X and $\{U_n\}$ with each U_n dense in X, but the intersection $\bigcap_n U_n$ not dense in X.

2. Let A, B be compact subspaces of X, Y respectively and N an open set in $X \times Y$ containing $A \times B$. Prove there exist open sets U, V in X and Y respectively such that

$$A \times B \subset U \times V \subset N \ .$$

3. Let $L \subset R^2$ be a homeomorphic image of the unit interval $[0, 1]$. Prove that L is a closed subset of R^2. Use the Tietze extension theorem to prove that L is a rectract of R^2; there is a continuous $f : R^2 \to L$ such that the restriction of f to L is the identity map.

PART B

4. Prove or disprove the Borsuk-Ulam theorem for the torus, which says the following. For every map $f : S^1 \times S^1 \to \mathbb{R}^2$, there is a point (x, y) in $S^1 \times S^1$ such that $f(-x, -y) = f(x, y)$. Here we regard S^1 as the unit circle in the complex numbers in order to define $-z$ for $z \in S^1$.

5. Recall that for topological spaces $A \subset X$, a retraction $X \to A$ is a continuous map such that the composite

$$A \to X \to A$$

is homotopic to the identity. Show that no retraction exists in the following cases:

(a) X is the Möbius band and A is its boundary circle.

(b) A is a figure eight ($S^1 \vee S^1$) embedded in the plane $\mathbb{R}^2$ in the usual way, and $X = D^2 \vee D^2$ is the region in the plane bounded by A.

6. Let M_g be a closed oriented surface of genus g. Its homology is as follows.

$$H_i(M_g) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}^{2g} & \text{for } i = 1 \\ \mathbb{Z} & \text{for } i = 2 \\ 0 & \text{for } i > 2 \end{cases}$$

Let $M_{g,k}$ be M_g with k disjoint open disks removed. Compute $H_*(M_{g,k})$ for $k > 0$ and prove your answer.

PART C

7.

(a) State the closed graph theorem.

(b) Let $e_j, j = 1, 2, \ldots$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and let v_j, $j = 1, 2, \ldots$ be a sequence of vectors in $\mathcal{H}$ such that for all h in $\mathcal{H}$,

$$\sum_{j=1}^{\infty} |\langle v_j, h \rangle|^2 < \infty \, .$$

Show that there is a unique bounded linear operator A in $\mathcal{H}$ such that for all h in $\mathcal{H}$, $\langle e_j, Ah \rangle = \langle a_j, h \rangle$.

8. Suppose that K is a compact operator on a Hilbert space $\mathcal{H}$, and u_n a sequence that converges weakly to 0, i.e., for all h in $\mathcal{H}$, $\langle u_n, h\rangle \to 0$ as $n \to \infty$. Show that $\|Ku_n\| \to 0$ as $n \to \infty$.

9. For what values of $\alpha > 0$ is

$$\sum_{k=1}^{\infty} \left| \int_0^{\pi} \frac{\sin kx}{x^{\alpha}} \, dx \right|^2 < \infty \ ?$$

(Prove your answer.)