Analysis Prelim Questions
Day 1, August 29, 2005

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Let $1 < q < p < \infty$ and suppose that $f \in L^q(X, \mu)$, where $\mu(X) < \infty$. Show that $\|f\|_q \leq \mu(X)^{1/q-1/p}\|f\|_p$. [Hint: $|f|^q = |f|^q \cdot 1$.]

2. Suppose that f and g are in $L^1(a, b])$ with Lebesgue measure on $[a, b]$. Let

$$f(x) = \int_{[a,x]} f(y)dy \text{ and } G(x) = \int_{[a,x]} g(y)dy \ .$$

(Then we know that $F' = f$ and $G' = g$ a.e.) Prove that the integration by parts formula

$$\int_{[a,b]} F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_{[a,b]} G(x)f(x)dx$$

holds. Note that the result from calculus cannot be used, since it was not proved in this generality (if it was proved at all.)

3. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$, and suppose that for each $y \in [0, 1]$, $f(\cdot\,, y)$ is continuous, and for each $x \in [0, 1]$, $f(x, \cdot)$ is Lebesgue measurable, and f is bounded on $[0, 1] \times 0, 1]$. Show that

$$F(x) = \int_{0,1]} f(x, y)dy$$

is continuous for x in $[0, 1]$.

4. Let $h : \mathbb{C} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Prove that h is entire.

5. Define $g(z) = \int_0^1 t^z(1 - t)^{1-z}dt$.

(a) Show that g is holomorphic on the strip $\Omega_0 = \{z : -1 < Re(z) < 2\}$.

(b) Construct a meromorphic continuation of g to the strip $\Omega_1 = \{z : -2 < Re(z) < 3\}$, finding all poles and their residues.

6. Let $\Omega \subset \mathbb{C}$ be connected and simply connected. Suppose $f : \Omega \to \mathbb{C}\backslash\{0\}$ is holomorphic.

(a) Show that f has a holomorphic logarithm, i.e., $\exists F : \Omega \to \mathbb{C}$ holomorphic s.t. $e^{F(z)} = f(z)$.

(b) Find all such functions F.

Algebra Prelim Questions
Day 2—August 30, 2005

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

7.
8.
9.
10.
11.
12.

Prelim Questions:
A. General Topology, B. Algebraic Topology,
and C. Functional Analysis
Day 3—August 31, 2005

Directions: This test has 3 sections: A. General Topology, B. Algebraic Topology, and C. Functional Analysis. Answer questions from A and EITHER B or C.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

A. GENERAL TOPOLOGY

13. Let A be a closed subspace of a compact, Hausdorff space X. Let p be a point in X not in A. Show that there exist disjoint open sets U, and V with A contained in U, and p contained in V. (In this problem, you must use the definitions rather than just stating a theorem.)

14. Give $X \times Y$ the product topology. Consider the projection map

$$p : X \times Y \to X.$$

(a) Prove that p is an open map.
(b) Prove that p is a quotient map.

15.

(a) Give the definition of a covering space

$$p : E \to B.$$

(b) Let the circle S^1 be the subspace of the complex numbers of length one. Show that the exponential map

$$\exp : \mathbb{R} \to S^1$$

from the real numbers to S^1 defined by the formula

$$\exp(z) = e^{iz} = cos(z) + isin(z)$$

gives a covering space.

(c) Outline a calculation of the fundamental group of a circle.

B. Algebraic Topology

16.

17.

18.

C. Functional Analysis

19. Let $\mathcal{Y}$ be a Banach space with norm $\|\cdots\|_{\mathcal{Y}}$ and $\mathcal{X}$ a proper linear subspace of $\mathcal{Y}$. Suppose that with a different norm $\|\cdots\|_{\mathcal{X}}$, $\mathcal{X}$ is a Banach space. Show that there is a constant C such that for all x in $\mathcal{X}$, $\|x\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$.

20. Let T be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Show that there is a compact self-adjoint operator S such that $S^3 = T$.

21. For f in $\mathcal{C}([0,1])$(= continuous functions on $[0,1]$), let $L(f) = f(1/2)$. For each of the following spaces $\mathcal{X}$ contained in $\mathcal{C}[0,1])$ and $\mathcal{Y}$ containing $\mathcal{C}([0,1])$, tell whether L extends from $\mathcal{X}$ to a continuous linear functional on $\mathcal{Y}$, and explain why.

(a) $\mathcal{X}$ = polynomials of degree n; $\mathcal{Y} = L^p([0,1])$, $1 \leq p < \infty$.

(b) $\mathcal{X} = \mathcal{C}([0,1]); \mathcal{Y} = L^p([0,1])$, $1 \leq p < \infty$.

(c) $\mathcal{X} = \mathcal{C}([0,1]); \mathcal{Y} = L^\infty([0,1])$. $T\cap$