PH.D. PRELIMINARY EXAMS

DAY 1: Analysis and General Topology

Friday, May 16, 2003

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours.

1.

(a) For measurable functions $f_n : [0,1] \to \mathbb{R}$, define

(i) Convergence in measure.

(ii) Convergence a.e.

(iii) Convergence in $\mathbf{L}^1$.

(b) Explain which types of convergence (i), (ii), (iii) imply which of the other types. Prove at least one of these statements.

(c) For one of the cases in which one type of convergence does not imply a second type, give a counterexample. (This question is only asking for one counterexample).

2. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of functions with domain $[0,1]$ and range $\mathbb{R}$. That is, there exists a constant $M < \infty$ such that for all n and for all $x \in [0,1]$, we have $|f_n(x)| \leq M$. Also, suppose that the family $\{f_n\}_{n=1}^{\infty}$ is equicontinuous. That is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all n and for $x, y \in [0,1]$ with $|x - y| \leq \delta$, we have $|f_n(x) - f_n(y)| \leq \varepsilon$. Let

$$F_n(x) = \max_{1 \leq k \leq n} f_k(x).$$

Show that F_n converges uniformly on $[0,1]$ as $n \to \infty$.

3. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if, for all $\lambda \in [0,1]$ and for all $\hat{x}, \hat{y} \in \mathbb{R}^n$, we have

$$f(\lambda \hat{x} + (1-\lambda)\hat{y}) \leq \lambda f(\hat{x}) + (1-\lambda) f(\hat{y}).$$

The Legendre transform $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ of f is defined by

$$f^*(\hat{x}) = \sup_{\hat{y} \in \mathbb{R}^n} \{\hat{x} \cdot \hat{y} - f(\hat{y})\}$$

Prove that if f is convex, then f^* is convex. (On regions where f^* does not have infinite value.)

4. Let S be an uncountable subset of $\mathbb{R}$. Prove that there exists a real number t such that $S \cap (-\infty, t)$ and $S \cap (t, \infty)$ are both uncountable sets.

5. Let $I_1, \ldots, I_n$ be pairwise disjoint closed subintervals of $\mathbb{R}$ of positive length.
(a) Prove that if $p(x)$ is a real polynomial of degree strictly less than n such that

$$\int_{I_j} p(x)dx = 0 \qquad \text{for } j = 1, \ldots, n \tag{1}$$

then $p = 0$.
(b) Prove that there exists a nonzero real polynomial p of degree n that satisfies (1).

6. Let X be a connected, normal (hence Hausdorff) space with at least two distinct points. Prove that X is uncountable.

PH.D. PRELIMINARY EXAMS

DAY 2: Algebra

Saturday, May 17, 2003

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours.

1.

(a) For any non-abelian group of order p^3, p a prime, show that the center and commutator subgroup are equal.

(b) Find the center and commutator subgroup of each of the following groups: Include brief reasons for your answers:

(i) S_n, the symmetric group on n letters where $n \geq 5$.

(ii) D_{2n}, the dihedral group with $2n$ elements. (Hint: Treat the case of even and odd n separately.)

(iii) Any nonabelian simple group G.

2.

(a) Let G be a group with a normal subgroup N of order d such that $\{g \in G | gn = ng \ \forall n \in N\} = \{e\}$. Prove that $|G|$ divides $d!$.

(b) Let P be a p-group, and A a normal subgroup of order p. Prove that A is contained in the center of P.

(c) Prove that any group of order p^2q where p and q are distinct primes has a normal Sylow subgroup. Deduce that it must have a normal subgroup of prime index.

3. Give an example of each of the following situations if possible. If not possible, briefly explain why not:

(a) A commutative ring A and a multiplicatively closed subset S which does not contain 0, such that $S^{-1}A$ is not a local ring.

(b) A commutative ring A and a multiplicatively closed subset S which does not contain 0, such that the canonical map $f : A \to S^{-1}A$ is not injective.

(c) A non-Noetherian ring that is a Noetherian $\mathbb{Z}$-module.

(d) A Noetherian ring that is a non Noetherian $\mathbb{Z}$-module.

(e) A commutative ring A and a submodule of a finitely generated module that is not finitely generated.

(f) A subring A of a Noetherian ring B that is not a Noetherian ring.

4. Let D_8 be the dihedral group of order 8 and C_6 the cyclic group of order 6.
(a) Give an example of a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $D_8 \times C_6$. Be precise when describing your answer.
(b) Prove that there exists an infinite number of such extensions over $\mathbb{Q}$.
(c) Give an example of a Galois extension over a finite field with Galois group isomorphic to D_8 if possible. If not possible, give a brief explanation why not.

5.
(a) Prove that the nth cyclotomic polynomial $\phi_n(x)$ is irreducible over $\mathbb{Q}$.
(b) Prove that if α is algebraic over a field k, then the multiplicity of α in its minimal polynomial over k must be a power of p if k has positive characteristic p, and must be 1 if k has characteristic zero.

6.
Recall that a field E is called algebraically closed if any nonconstant polynomial with coefficients in E, has at least one root in E.

Suppose that K is a field with an algebraic extension F with the property that any nonconstant polynomial with coefficients in K has at least one root in F. Prove that F is the algebraic closure of K, i.e., prove that F is algebraically closed.

Hint: Discuss the separable and purely inseparable cases separately, and use the Primitive Element Theorem.

PH.D. PRELIMINARY EXAMS

DAY 3: Complex Analysis and Topics in Topology.

Sunday, May 18, 2003

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours.

1. Give (with proofs) an example of a countable Hausdorff topological space which is not metrizable.

2. Prove that the 2-dimensional torus is not homeomorphic to the 2-dimensional sphere.

3. Let $Y = S^1 \bigvee S^1$ be the one point union of two circles (i.e., "the figure eight"). Find a covering space $X \xrightarrow{p} Y$ such that the image of $\pi_1(X)$ in $\pi_1(Y)$ is a subgroup of index three. Describe both the space X and the map p and prove that they have the desired property.

4. Suppose f is analytic in a deleted neighborhood D of z_0 except for poles at all points of a sequence $\{z_n\} \to z_0$. Note that z_0 is a limit point of poles and so <u>not</u> an isolated singularity. Nevertheless, show that the analogue of the Casorati-Weierstrass Theorem holds for z_0, namely, in D, f comes arbitrarily close to every complex value (i.e. $f(D)$ is dense in the complex plane).

5. Find

$$\int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\, dx \, .$$

6.

(a) Suppose R is a simply-connected region and not equal to all of $\mathbb{C}$. Show that there is <u>no</u> one-to-one analytic function mapping $\mathbb{C}$ onto R.

(b) Show that if $z_1, z_2 \in R$, then there exist a one-to-one analytic function mapping R to R and taking z_1 to z_2.

NOTE: You may cite the Riemann mapping theorem in your answers if you wish.