PH.D. PRELIMINARY EXAMS

DAY 1

January 8, 1999

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

1. Let A be a commutative ring having only one maximal ideal m. Let M be a finitely generated A module such that $mM = M$. Prove that $M = 0$.

2. Let k be a field and E a Galois extension of k of degree 261.

(a) Show that there exists an intermediate field F of degree 9 over k that is Galois over k.

(b) Show that there exists an intermediate field F of degree 29 over k that is Galois over k.

(c) If Gal (E/k) has no element of order 9, how many intermediate fields are there having degree 3 over k? How many of these are Galois over k? Be sure to completely justify your answers.

3. Let H be a Hilbert space and

$$E = \{e_\alpha : \alpha \in \Lambda\}$$

an orthonormal basis for H. Show that

$$\lim_{n\to\infty} \langle h_n, h\rangle = 0 \qquad \forall h \in H$$

if and only if $\sup\{\|h_n\| : n \geq 1\} < \infty$ and

$$\lim_{n\to\infty} \langle h_n, e_\alpha\rangle = 0 \qquad \forall e_\alpha \in E\ .$$

4.

(a) Suppose f is holomorphic on $\{z : Re\, z < 1\}$ and

$$|f(z)| \leq M_0 \text{ for } Re\, z = 0\ .$$

Find upper bounds for $|f(-1)|$, $|f'(-1)|$, and $|f''(-1)|$.

(b) Now suppose also that $f \in L^\infty(\{Re(z) < 1\})$ and $|f(z)| \leq M_{-3}$ for $Re\, z = -3$. Find new upper bounds for $|f(-1)|$, $|f'(-1)|$, $|f''(-1)|$.

5. Show that the closed interval $[0, 1]$ is compact.

6. Compute the homology of the 1-skeleton of the n-simplex for $n \geq 1$.

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DAY 2

January 11, 1999

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7. Either prove or find a counterexample to the following conjecture:

Conjecture If $f(z)$ is an entire function such that the composite $f \circ f(z) = f(f(z))$ is bounded, then $f(z)$ is constant.

8. Let

$$c_0 = \Big\{ \{a_n\}_{n\geq 1} \;\; : a_n \in \mathbb{C} \text{ and } \lim_{n\to\infty} a_n = 0 \Big\}$$
$$\ell_1 = \left\{ \{a_n\}_{n\geq 1} \;\; : a_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |a_n| < \infty \right\} .$$

Show that the dual of c_0 is ℓ_1.

9.

(a) Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point (i.e. a point x in $[0,1]$ with $f(x) = x$).

(b) Show that any continuous function

$$f : D^2 \to D^2$$

has a fixed point, where D^2 is the 2-disk, the points in the plane of norm less than or equal to 1.

10.

(a) State the Riesz representation theorem (concerning linear functionals on $\mathcal{C}(X)$).

(b) Let m_n be a sequence of positive numbers. Show that there exists a finite positive measure μ on $[0,1]$ such that $m_n = \int_0^1 x^n d\mu(x)$ for $n = 0, 1, \ldots$ if and only if the following condition is satisfied.

(*) For every polynomial $p(x) = \sum_{k=0}^{n} \alpha_k x^k$ (with real coefficients) which is non-negative on $[0,1]$, we have $\sum_{k=0}^{n} \alpha_k m_k \geq 0$.

11.

(a) Prove that all prime ideals in Artin rings are maximal ideals.

(b) Prove that an Artin ring can have only a finite number of prime ideals.

12. Find the Galois group of the polynomial $x^3 - x - 1$ over each of the following fields. Be sure to completely justify your answers.

(a) $\mathbb{Q}$

(b) $\mathbb{R}$

(c) $\mathbb{Z}/2\,\mathbb{Z}$

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DAY 3

January 12, 1999

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

13.

(a) State the Radon-Nikodym theorem.

(b) Let μ_1, μ_2 be finite positive measures on $(X, \mathcal{M})$ and let $\lambda = \mu_1 + \mu_2$. Show that $\mu_1 \perp \mu_2$ if and only if

$$\frac{d\mu_1}{d\lambda} \cdot \frac{d\mu_2}{d\lambda} = 0 \qquad \lambda - \text{a.e.}$$

(c) Let f_n be a sequence of absolutely continuous functions on $[0,1]$ such that $f_n(0)$ converges and f_n' converges in the L^p norm for some $p \geq 1$. Show that f_n converges uniformly.

14. Let H be a Hilbert space and $P : H \to H$ a projection operator, i.e. $P^2 = I$ and $P = P^*$. Prove that P is a compact operator if and only if $\dim(\text{Range}(P)) < \infty$.

15. Determine the fundamental group of the torus $S' \times S'$.

16. Prove that the unit interval $[0,1]$ is connected.

17.

(a) Let G be a group with a subgroup H. Then H acts on the set of left cosets $S = \{gH | g \in G\}$ by left translation. This induces a homomorphism φ from H into the group of permutations of S. What is the kernel of φ?

(b) Describe the set of all fixed points in S under the above action of G. (Do this by finding all elements g with the property that gH is a fixed point under the above action.)

(c) If H is a p-group, show that $[N_G(H) : H] \equiv [G : H]$ mod p, and deduce from this that if H is not a Sylow p-subgroup of G, then $N_G(H) \neq H$.

18.

(a) Let F be a field and α an algebraic element of F. Suppose that Gal $(F(\alpha)/F)$ is generated by an automorphism that takes α to $\alpha + 1$. Show that F has characteristic $p > 0$, and that $\alpha^p - \alpha \in F$.

(b) Conversely, if F is a field of characteristic $p > 0$, show that any Galois extension of degree p over F arises in the above way (Hint: consider an element B in the extension field having trace 1 (explain why such an element must exist) and construct α from the conjugates of B.)