Prop: $\mathbb{R}^{n-1} \times \mathbb{R}^{n}$ closed, stable, minimal w/ trivial normal bundle. Then:

(A) If $\text{Ric}_m \geq 0$, $\Sigma$ is totally geodesic and $\text{Ric}_m(\mu \nu) = 0$ on $\Sigma$.

(B) $\text{Scal}_m > 0$, $n = 3$, then $\Sigma = S^2$ or $\mathbb{R}P^2$

Proof of (A): Stability $\Rightarrow \sum n^2 |A|^2 \leq -\frac{1}{2} \text{Ric}_m(\mu \nu) n^2 + 12 n^2$

$\Rightarrow |A| \equiv 0$, $\text{Ric}_m(\mu \nu) = 0$ along $\Sigma$

Proof of (B): Fix an orthonormal frame $E_1, E_2, E_3$ along $\Sigma$ with $N = E_3$, and set

$R_{i j k l} = \langle R_m(E_i, E_j) E_k, E_l \rangle$

Recall that $R_{i j k l}$ satisfies the relations

$R_{i j k l} = -R_{j i k l} = -R_{i j k l} = R_{k i j l}$

We then have that:

$\text{Ric}_m(\mu \nu) = R_{1313} + R_{2323}$

$\text{Ric}_m(E_i, E_i) = R_{1212} + R_{1313}$

$\text{Ric}_m(E_i, E_j) = R_{1212} + R_{2323}$
Gauss's Equation states:

\[ R_{ik}e - R_{ik}^\Sigma = A_{ik}A_{ik} - A_{ij}A_{ik} \]

\[ \Rightarrow \text{ since } k^2 = R_{i2}^2, \text{ we have } \]

\[ R_{i2}^2 = k^2 + A_{12}^2 - A_{11}A_{22} \Rightarrow \]

\[ k^2 = R_{i2}^2 - \text{det}(A). \]

Then:

\[ \text{scal}_M = \frac{1}{2} R_{ikm}(e_i f_i) = 2\left( R_{i2}^2 + R_{13}^2 + R_{23}^2 \right) \]

\[ = 2 \cdot \text{Ric}(M, v) + 2 \cdot R_{i2}^2 \]

\[ \Rightarrow \]

\[ \text{Ric}_M(v, u) = \frac{1}{2} \text{scal}_M - R_{i2}^2 \]

\[ \Rightarrow \frac{1}{2} \text{scal}_M + 1A^2 \]

\[ = \frac{1}{2} \text{scal}_M - \text{det}(A) - k^2 \]

\[ = \frac{1}{2} \text{scal}_M - \frac{1}{2} |A|^2 - k^2. \]

\[ \text{Stability} \Rightarrow \frac{1}{2} \sum \text{Ric}_M(v, u) + 1A^2 \leq 0 \]

\[ \Rightarrow \frac{1}{2} \sum \text{scal}_M + |A|^2 \leq \sum k^2 = 2 \chi(X) \]

\[ \Rightarrow \chi(X) \geq 0. \]
Stability and eigenvalues.

Set
\[ \lambda_1(n,2) = \inf \left\{ -\sum n^2 n^2 : n \in C^0_c(n), \sum n^2 = 1 \right\} \]

where \( C^0_c(n) \) is the set of smooth functions on \( n \) which vanish on \( \partial n \).

Stability \( \Rightarrow \lambda_1(n,2) > 0 \).

Generalization to Sobolev spaces.

Def: For \( u : \Sigma \to \mathbb{R} \) smooth, set
\[ \| u \|_{W^{1,2}(n)} = \sqrt{\sum n^2 + \sum 16 u^2 n^2} \]

we let \( W^{1,2}(n) \) denote the closure of \( C^0_c(n) \) with respect to the norm \( \| \cdot \|_{W^{1,2}(n)} \).

Proposition: Set \( \lambda_1 = \lambda_1(n,2) \). Then we have
\[ \lambda_1 = \inf \left\{ \frac{\sum 16 u^2}{\sum n^2} - 16 u^2 - \text{Ric}(u,u) u^2 \right\} \]
\[ = \left\{ \text{I}(u) \right\} \inf \mathbb{I} \]

Moreover, there exist \( u \in W^{1,2}_0(n) \) that achieves the infimum, and it holds that \( u \) is smooth and
\[ 2\lambda = \lambda_1 u. \]
Proof: set $v(x) = |A|^2(x) + \text{Ric}(x, n)(x)$. Observe that trivially we have

$$I \leq \lambda.$$ 

Now, choose a minimizing sequence $n_j$ w/ 

$$I(n_j) \leq I.$$ That is, we have 

$$\frac{\int \nabla n_j^2 - n_j^2}{\int n_j^2} \leq I + \frac{1}{j}.$$

Since $I(-)$ is homogeneous degree zero functional, we can assume that $\frac{\int n_j^2}{\int} = 1$. Moreover we have that 

$$\|n_j\|_{W^{1,2}(\mathbb{R}^2)} \leq I + \frac{1}{j} \leq I + 1,$$

so that the sequence is uniformly bounded in $W^{1,2}(\mathbb{R}^2)$. The Rellich compactness theorem gives a subsequence — still denoted $n_j$ — s.t.,

1. $n_j \rightharpoonup n$ weakly in $W^{1,2}$

2. $n_j \rightarrow n$ strongly in $L^2$

for some $n \in W^{1,2}(\mathbb{R}^2)$. Since the energy integral is lower semi-continuous,

$$\frac{\int n_j^2}{\int} \leq \liminf \frac{\int |A_n|^2}{\int},$$
\[ I(n) = \sum_i \nabla u_i^2 + V_n^2 \leq \lim_{\xi \to n} \sum_i \nabla u_i^2 + V_n^2 = \lim_{\xi \to n} I(n) = 0. \]

(above we used that \( \sum_n^2 = \lim_{\xi \to n} \sum_i^2 = 1 \))

Take \( \psi \in C_0^\infty (\Omega) \) s.t. \( \nabla n = 0 \). Then

\[ \frac{d}{dt} \int_\Omega (n+t\psi)^2 = 2 \int_\Omega \nabla(n+t\psi)^2. \]

\[ \Rightarrow \frac{d}{dt} \int_\Omega 2 \sum_i \nabla(n+t\psi) - Vn \psi. \]

For general \( \psi \in C_0^\infty (\Omega) \), set \( \psi = \phi - n \nabla n \). Then

\[ \nabla n = 0, \] so that

\[ \sum_i \nabla(n+t\psi) - Vn \psi = \sum_i (\nabla n^2 - Vn^2). \]

\[ = I \sum_i \nabla n. \]

Thus, \( n \) is a weak solution to the eigenvalue problem

\[ 9n + In = 0. \]

Elliptic Regularity Theory implies \( n \) is in fact smooth.
\[ \Rightarrow \mathcal{L} u \leq I. \]

**Proposition:** set \( \mathcal{L} u = \mathcal{L}(u, \varphi) \). If \( u \in C^0(\varphi) \) sat's

\[ \mathcal{L} u = -\mathcal{L} u \quad \text{in} \quad \varphi, \]

then \( u(p) \to 1 \quad \text{as} \quad p \to \varphi. \)

**Lemma: (Harnack's Inequality)** Suppose \( u : D \to \mathbb{R} \) sat's

\[ \mathcal{L} u = 0, \quad u \geq 0 \]

for an elliptic operator \( \mathcal{L} \). Then there are constants \( C_1, C_2 \), independent of \( u \) sat.

\[ \sup_{D} u < C \inf_{D} u \]

**Proof (Proof of Proposition):** let \( u \) be such a solution.

Observe that

\[ \mathcal{L}(u, \varphi) = I(u) = I(\varphi u) \]

This \( \varphi u \) is a smooth function satisfying

\[ \mathcal{L} u + 2 \varphi u = 0. \]

Harnack's Inequality \( \Rightarrow \) \( u \geq 0 \) in \( \varphi. \)
Proposition: Let $\Sigma$ be a minimal hypersurface with trivial normal bundle. Then, if there exists $u: \Sigma \rightarrow \mathbb{R}$ such that $Q u = 0$, $u > 0$ on $\Sigma$,

then $\Sigma$ is stable.

Proof: Set $w = \log(u)$. Then

$$\nabla w = \frac{\partial w}{\partial u}, \quad \Delta w = -\frac{\partial w}{u} + \frac{\partial^2 w}{u}$$

$$= -\frac{\partial w}{u} + 1A^2 - \text{Ric}(\mu, \nu)$$

Let $f$ be compactly supported in $\Sigma$. Then

$$\int_{\Sigma} f^2 (1A^2 + \text{Ric}(\mu, \nu)) + \int_{\Sigma} f^2 \Delta w^2 = -\int_{\Sigma} f^2 \partial w$$

$$= \int_{\Sigma} \int_{\Sigma} f \partial f \cdot \partial w$$

$$\leq \int_{\Sigma} f^2 \partial w^2 + \int_{\Sigma} f \partial f^2$$

$$\Rightarrow \int_{\Sigma} f^2 (1A^2 + \text{Ric}(\mu, \nu)) \leq \int_{\Sigma} f \partial f^2$$
Corollary: Minimal graphs are stable.

Proof: \( u = \langle \mathbb{V}, e^+ \rangle \) sats\( \lambda u = 0, \ u > 0. \)

Proposition: If \( \mathcal{I} \) is complete, non-compact minimal hypersurface \( \mathcal{M} \) with trivial normal bundle, then the following are equivalent:

1. \( \lambda_1(\mathcal{I}) > 0 \) for every bounded domain.
2. \( \lambda_1(A) > 0 \) for every bounded domain.
3. There exists a positive function \( u \) s.t. \( u > 0. \)

Proof: Clearly (2) \( \Rightarrow \) (1). To see that (1) \( \Rightarrow \) (3), fix a bounded domain \( \Omega \). Let \( \Omega_1 \) be a strictly larger domain. It then follows immediately that

\[ \lambda_1(\Omega_1, \Omega) > \lambda_1(\Omega_1, \Omega_1) \geq 0. \]

Let \( u_0 \) be the first eigenfunction on \( \Omega_0 \). Let \( u_1 : \Omega_1 \to \mathbb{R} \) by

\[ u_1(p) = \begin{cases} u_0(p), & p \in \Omega_0, \\ 0, & \text{otherwise}. \end{cases} \]
Assume now that $\lambda_{1}(\mathcal{N}, q) = \lambda_{2}(\mathcal{N}, q)$. Then $u_{1}$ satisfies

$$2u_{1} = \lambda_{1}(\mathcal{N}, q) u_{1}$$

Harnack's Inequality $\Rightarrow u_{1} > 0$ in $\mathcal{N}$, a contradiction. Thus, we have that

$$\lambda_{1}(\mathcal{N}, q) > \lambda_{2}(\mathcal{N}, q).$$

We now show that (2) $\Rightarrow$ (3). Then fix $p \in \mathcal{Z}$.

Set

$$\mathcal{B}_{r} = \mathcal{B}_{r}(p) = \{ q \in \mathcal{Z} \mid d_{\mathcal{Z}}(p, q) \leq r \}$$

Since $\lambda_{1}(\mathcal{B}_{r}, q) > 0$, the Fredholm Alternative gives a function $u_{r} : \mathcal{B}_{r} \to \mathbb{R}$ such that

$$2u_{r} = -\lambda_{1}^{2} - \text{Ric}_{\mathcal{N}}(N, N) \text{ on } \mathcal{B}_{r}$$

$$u_{r} = 0 \text{ on } \partial \mathcal{B}_{r}.$$  

Set $u_{r} = u_{r} + 1$, so that

$$2u_{r} = 0 \text{ on } \mathcal{B}_{r}, \quad u_{r} = 1 \text{ on } \partial \mathcal{B}_{r}.$$  

We claim $u_{r} > 0$ on $\mathcal{B}_{r}$. Suppose not. Then pick a connected component $\mathcal{C}$ of the set

$$\{ x \in \mathcal{B}_{r} \mid u_{r}(x) < 0 \}.  $$
By construction, we have that \( \lambda_1(n, \Omega) = 0 \), which contradicts (2). Thus, we have that
\[ u \geq 0 \]
in \( B_r \). We now renormalize the functions \( u_r \) by setting
\[ w_r = u_r(u) \]
so that \( w_r(\rho) = 1 \). The Harnack Inequality then gives
\[ \sup_{B_r} w_r \leq C(r) \]
For \( r' > r \), Schauder theory (what is commonly called "Standard elliptic Regularity") then gives
\[ |w_r|_{\mathcal{C}^1(B_{r'})} \leq C'(r) \]
Arzela-Ascoli then \( \Rightarrow \) \( \{ w_r \} \) is precompact on compact sets and converges to
\[ w \in \mathcal{C}^2, \mathcal{A} (\Omega) \]
satisfying \( w \equiv 0 \), \( w \geq 0 \).
\[ \square \]