Introduction to Stochastic Processes (MTH 202)

Spring 2016

Midterm solutions.

Problem A. A fair die is thrown repeatedly until we obtain the same number twice in a row. Compute the expected number of throws.

Solution. The problem can be modeled by a Markov chain \( X_0, X_1, \ldots \) with 2 states, say \( N \) for a non-repeated value of the last 2 throws, and \( R \) for a repeated one. The transition matrix \( P \) is given by

\[
P_{NN} = \frac{5}{6}, P_{NR} = \frac{1}{6}, P_{RN} = 0, \text{ and } P_{RR} = 1.
\]

After throwing the die twice, we are in state \( N \) with probability \( \frac{5}{6} \) and with probability \( \frac{1}{6} \) in state \( R \). So, if we denote by \( T \) the absorbing time and \( \tau_N = \mathbb{E}[T \mid X_0 = N] \), then we have

\[
\mathbb{E}[T] = \frac{5}{6} \times \tau_N + \frac{1}{6} \times 0.
\]

A first step analysis shows that

\[
\tau_N = 1 + \frac{5}{6} \tau_N,
\]

so \( \tau_N = 6 \), and therefore \( \mathbb{E}[T] = 5 \). Noting that at time \( n \) in the Markov chain we have thrown the die \( n + 2 \), the answer is \( \mathbb{E}[T] + 2 = 7 \). \( \diamond \)

Problem B. Consider all possible orderings of the numbers from 1 to 10. For such an ordering, a number is lucky if it appears in the same position as in the usual order. Assuming all orderings have the same probability, compute the expected number of lucky numbers in a random ordering.

Solution. An ordering is a bijective function \( \sigma : \{1, \ldots, 10\} \to \{1, \ldots, 10\} \). Denote by \( S \) the set of all such functions, and note that \#\( S \) = 10!. For \( \sigma \) in \( S \), denote by \( N(\sigma) \) the numbers of numbers that are lucky for \( \sigma \). Noting that number \( k \) in \( \{1, \ldots, 10\} \) is lucky if \( \sigma(k) = k \), we have

\[
\mathbb{E}[N] = \frac{1}{10!} \sum_{\sigma \in S} N(\sigma) = \frac{1}{10!} \sum_{\sigma \in S} \sum_{k \in \{1, \ldots, 10\}, \sigma(k) = k} 1 = \frac{1}{10!} \sum_{k \in \{1, \ldots, 10\}} \sum_{\sigma \in S, \sigma(k) = k} 1
\]

\[
= \frac{1}{10!} \sum_{k \in \{1, \ldots, 10\}} 9! = 1. \quad \diamond
\]
**Problem E.** Consider configurations of +’s or −’s placed at each vertex of a regular hexagon, not all being equal. Such a configuration is said to be polarized, if there are exactly 2 edges with different signs at its vertex. If at a given time \( t \) the configuration is polarized, then at time \( t + 1 \) the configuration is the same as in time \( t \). If the configuration is not polarized, then an edge having different signs at its vertex is chosen randomly with equal probabilities, and at time \( t + 1 \) the configuration is the same as in time \( t \) except that the signs at each vertex of the chosen edge have changed. Starting with a configuration having exactly 3 signs −’s forming an equilateral triangle, compute the expected time it take the configuration to become polarized.

**Solution.** Up to rotation and reflection there are 3 possible configurations, according to the number of edges with different signs at its vertex (6, 4, or 2 in the polarized case), as pictured below.

These configurations are also distinguished by the number of groups of −’s which are separated by +’s (3, 2, or 1, respectively).

The problem can be modeled by a Markov chain with states \( A \), \( B \), and \( C \). The transition matrix \( P \) is given by

\[
P = \begin{pmatrix}
0 & 1 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 0 & 1
\end{pmatrix}.
\]

Denoting as usual the absorbing time by \( T \), and

\[
\tau_A := E[T \mid X_0 = A] \quad \text{and} \quad \tau_B := E[T \mid X_0 = B],
\]

a first step analysis shows that

\[
\tau_A = 1 + \tau_B, \quad \text{and} \quad \tau_B = 1 + \frac{1}{4}\tau_A + \frac{1}{2}\tau_B.
\]

Solving, we have \( \tau_B = 5 \) and \( \tau_A = 6 \). Since in the problem we start with the configuration \( A \), the answer is \( \tau_A = 6 \). ⊗
Problem F (Bonus). There are 3 particles in the vertex of a cube. If at a given time \( t \) all 3 particles are in the same face of the cube, then they are in the same position at time \( t + 1 \). Otherwise one of particles is chosen at random, with uniform probability, and the particle is moved to one of the free neighboring vertex that is chosen randomly with uniform probability. Suppose at time \( t = 0 \) there are 2 particles in opposite vertex in the cube, and compute the expected time it takes the particles to reach the same face.

Solution. Up to rotation, the particles can be 3 positions:

A: No 2 being in the same vertex;
B: Exactly 1 pair being in the same vertex;
C: Exactly 2 pairs being in the same vertex.

Case B corresponds to having 2 particles in opposite vertex of the cube, and case C corresponds to the particles being in the same face.

The problem can be modeled by a Markov chain with states A, B, and C. State C is absorbing, and the transition matrix \( P \) is given by

\[
P = \begin{pmatrix}
0 & 2/3 & 1/3 \\
1/3 & 4/9 & 2/9 \\
0 & 0 & 1
\end{pmatrix}
\]

Denoting as usual the absorbing time by \( T \), and

\[
\tau_A := E[T \mid X_0 = A] \quad \text{and} \quad \tau_B := E[T \mid X_0 = B],
\]

a first step analysis shows that

\[
\tau_A = 1 + \frac{2}{3} \tau_B, \quad \text{and} \quad \tau_B = 1 + \frac{1}{3} \tau_A + \frac{4}{9} \tau_B.
\]

Solving, we have \( \tau_B = 4 \) and \( \tau_A = \frac{11}{3} \). Since in the problem we start with the configuration B, the answer is \( \tau_B = 4 \). \( \diamond \)