Problem A. A fair die is thrown 3 times. What is the expected value of the largest of the 3 outcomes?

Problem B. An unfair coin has probability $\frac{3}{7}$ for heads and $\frac{4}{7}$ for tails. What is the least amount of times we need to toss the coin so that with at least 50% chance we obtain 2 tails in a row?

Problem C. After 1 hour, the time elapsed marked by an inaccurate clock is 61 minutes with probability $\frac{1}{3}$, and 59 minutes with probability $\frac{2}{3}$. If the clock marks the exact time at 7am, what is the probability that it marks the exact time at 5pm?

Problem D. There are 2 particles are positioned at the vertices of a cube. If at a given time $t$ the particles are in the same edge, then they remain in the same position up to time $t + 1$. Otherwise, one of them is chosen at random with equal probability, and at time $t + 1$ it is moved to one of the 3 neighboring vertices with equal probability. If the particles start at opposite vertices of the same face, what is the expected value of the least time they are in the same edge?

Problem E. Given an integer $N \geq 1$ consider a random walk $X_0, X_1, \ldots$ with state space $\{0, 1, \ldots, 3N\}$, such that 0 and 3N are absorbing states, and such that for each $j$ in $\{1, \ldots, 3N - 1\}$ we have

\[ \Pr[X_{t+1} = j - 1 | X_t = j] = \Pr[X_{t+1} = j + 1 | X_t = j] = \frac{1}{2}. \]

1. Defining the absorbing time $T := \min\{X_t = 3N : t \geq 0\}$, compute $E[X_T = 0 | X_0 = N]$ in terms of $N$;

2. What is the behavior of $E[X_T = 0 | X_0 = N]$ as $N \to \infty$?

Problem F (Bonus points). Given a real number $\epsilon$ in $(0, 1)$, consider a Markov chain $X_0, X_1, \ldots$ with states in $\{A, B, C\}$, whose transition probabilities are determined by

\[ P_{AB} = P_{BC} = 1, P_{CA} = \epsilon, \text{ and } P_{CB} = 1 - \epsilon. \]

Compute the limit

\[ \lim_{\epsilon \to 0} \Pr[X_n = A | X_0 = A]. \]