1. (16 points)

Suppose that Cruz and Sanders are the nominees for president, but Trump runs as an independent. Further suppose that Sanders has a 47% chance of winning the presidency, Cruz has a 28% chance, and Trump has a 25% chance. Also suppose that if elected, Trump would have a 95% chance of building a wall at the Mexican border, Cruz would have a 10% chance, and Sanders would have a 2% chance. What is the overall probability that a wall would get built at the Mexican border?

Note: Since calculators are not allowed, you do not have to simplify your final answer to this question.

Answer:

Let \( W \) be the event that a wall gets built at the Mexican border. Let \( T \) be the event that Trump is elected president, \( S \) be the event that Sanders wins, and \( C \) be the event that Cruz wins. Note that \( T, S, C \) are mutually exclusive events whose union is the entire sample space. Using the conditional probability formula,

\[
P(W) = P(W|T)P(T) + P(W|C)P(C) + P(W|S)P(S)
\]

\[
= 0.95 \times 0.25 + 0.10 \times 0.28 + 0.02 \times 0.47
\]

\[
= 0.2749.
\]

2. (17 points)

Suppose that, in anticipation of an exam, you roll a six-sided die. You commit to the following plan: if the die rolls 6, you will study hard for the exam, if the die roll is 4 or 5, you will study a little, and if the roll is 1, 2, or 3 you will not study at all. Suppose that by studying hard for the exam, you secure a 95% chance of passing the exam, that by studying a little you have a 70% chance of passing, and that by not studying you have a 10% chance of passing. Now suppose you wake up after the exam with no memory of what happened, to find out that you miraculously passed the exam. What is the probability that you did not study for the exam?
Let $E$ be the event that you pass the exam, $F_1$ the event that you study hard, $F_2$ the event that you study a little, and $F_3$ the event that you do not study. We are given that

$$P(F_1) = \frac{1}{6}, \quad P(F_2) = \frac{1}{3}, \quad P(F_3) = \frac{1}{2},$$

$$P(E|F_1) = 0.95, \quad P(E|F_2) = 0.7, \quad P(E|F_3) = 0.1.$$

Note that $F_1$, $F_2$ and $F_3$ are mutually exclusive sets whose union is the entire sample space. Using Bayes’ formula we compute

$$P(F_3|E) = \frac{P(E|F_3)P(F_3)}{P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + P(E|F_3)P(F_3)} = \frac{(0.1)(1/2)}{(0.95)(1/6) + (0.7)(1/3) + (0.1)(1/2)} \approx 0.113.$$

Conditioned on having passed, the probability that you did not study is 0.113.

3. (17 points)

Let $X_1, X_2, X_3, X_4$ be i.i.d. (independent identically distributed) random variables, and assume

$$X_i = \begin{cases} 
0 & \text{with probability } \frac{1}{2} \\
1 & \text{with probability } \frac{1}{3} \\
2 & \text{with probability } \frac{1}{6}
\end{cases}$$

Let $S = X_1 + X_2 + X_3 + X_4$. Find $P(S = 3)$.

Answer:

There are two ways in which we could get $S = 3$. Let $A$ be the event that the sequence $(X_1, X_2, X_3, X_4)$ has three 1’s and one 0. Let $B$ be the event that the sequence has one 2, one 1, and two 0’s. These events are disjoint, and their union is the event that $S = 3$.

For the event $A$, there are four positions for the 0, and the rest of the sequence has to be 1. For an individual sequence like this, the probability is $(1/3)^3 \times (1/2) = 1/54$. Since there are 4 such sequences in the event $A$, it follows that

$$P(A) = \frac{4}{54} = \frac{2}{27}.$$

For the event $B$, there are four positions for the 2, three remaining positions for the 1, and the rest are 0. So there are $4 \times 3 = 12$ sequences with one 2, one 1, and two 0’s. The
probability of such a sequence is \((1/6) \times (1/3) \times (1/2)^2 = 1/72.\) It follows that
\[
P(B) = \frac{12}{72} = \frac{1}{6}.
\]

Adding these probabilities, we get
\[
P(S = 3) = P(A) + P(B) = \frac{2}{27} + \frac{1}{6} = \frac{13}{54}.
\]

4. (17 points)
You are dealt 5 cards from a standard deck of 52. Recall that a standard deck of playing cards consists of 4 suits (hearts, diamonds, clubs and spades) of 13 cards each, with different face values \((2,3,4,5,6,7,8,9,10,J,Q,K,A)\). Find the probability of drawing a “full house”, which refers to 3 cards of one face value, and 2 cards of another face value (e.g. 3 kings and 2 fours).

Answer:
There are 13 choices for the face value of the triple, and \(\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4\) distinct possibilities for the suits in the triple. For each triple in the full house, there are 12 remaining choices for the face value of the double, and \(\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6\) possibilities for its suits. We have counted \(13 \times 4 \times 12 \times 6\) distinct full house hands. There are \(\binom{52}{5} = \frac{52!}{5!47!} = 2598960\) distinct hands altogether, each of which are equally likely to be drawn, and conclude that a full house is drawn with probability
\[
\frac{13 \times 4 \times 12 \times 6}{2598960} \approx 0.14\%.
\]

5. (17 points)
Three unhappily married couples (6 guests altogether) attend a dinner party. They sit at a round table randomly in such a way that each outcome is equally likely. What is the probability that at least one person sits as far away as possible from his or her spouse? That is, there are 2 seats separating them in either direction around the table.

Hint: Use the inclusion-exclusion principle, and let \(A_i\) be the event that the \(i^{th}\) couple sits at opposite ends of the table.

Answer:
We use the hint. Note that there are 6! assignments of guests to seats. Let \(A\) be the event that at least one person sits as far away as possible from his or her spouse. We use the
inclusion-exclusion formula,

\[
P(A) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3).
\]

The probabilities \(P(A_i)\) are all equal. The first partner can sit anywhere, and the other has only one place to sit out of 5, and the other people can sit anywhere. This gives \(6 \times 1 \times 4!\) out of \(6!\), for a probability of \(1/5\).

For \(A_i \cap A_j\), the probabilities are again equal. For the first couple, as above, there are \(6 \times 1\) possibilities. For the second couple, the first partner can sit in any of the remaining 4 seats, and the other has only one place to sit. The third couple can sit in any of the two remaining seats, giving \(2!\) possibilities. Altogether this gives \(6 \times 1 \times 4 \times 1 \times 2!\) out of \(6!\), for a probability of \(1/15\).

For \(A_1 \cap A_2 \cap A_3\), we notice that if couples 1 and 2 have partners sitting at opposite ends of the table, there are only 2 spots left, and they are at opposite ends of the table. Therefore,

\[
P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2) = \frac{1}{15}.
\]

Finally, substituting into the inclusion-exclusion formula, we get

\[
P(A) = 3 \times \frac{1}{5} - 3 \times \frac{1}{15} + \frac{1}{15} = \frac{7}{15}.
\]

6. (16 points)

Let \((X,Y)\) denote a uniformly chosen random point inside the square 

\[
[0,2]^2 = [0,2] \times [0,2] = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 2\}.
\]

What is the probability \(P(X + Y < \frac{1}{2})\)?

**Hint:** Saying that \((X,Y)\) is uniformly distributed on a set \(S\) means that \(P((X,Y) \in A)\) is the ratio of the area of \(A\) to the area of \(S\), provided that \(A\) is a subset of \(S\).

**Answer:**

We use the hint. In this case \(S = [0,2]^2\) is a unit square of area 4. The set \(A\) is the subset of \(S\) for which \(x+y < 1/2\). Thus, \(A\) is the region in the square \([0,2]^2\) below the line \(x+y = 1/2\).
We can rewrite the equation of the line as $y = -x + 1/2$. Drawing a picture, we see that $A$ is a right triangle with vertices at $(0, 0), (0, 1/2), (1/2, 0)$. This triangle has base $1/2$ and height $1/2$, so its area is $1/8$. Therefore,

$$P(A) = \frac{1/8}{4} = \frac{1}{32}.$$