MATH 172Q
EXAM II – March 29, 2016

NAME (please print): ____________________________________________

Please write out and sign the following statement:
I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

PROFESSOR (please circle): GONEK TUCKER

INSTRUCTIONS:

(1) Check that you have 8 pages (including this cover).
(2) The examination is 75 minutes long.
(3) Show all your work and provide complete and clear reasons.

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1. (24 pts) Evaluate the following integrals.

(a) 
\[
\int x^2 \cos x \, dx = x^2 \sin x - \int \frac{\sin x \, dx}{x^2} \\
= x^2 \sin x - \frac{\sin x}{x} - \int \frac{\sin x \, dx}{x^2} \\
= x^2 \sin x + 2 \int \sin x \, dx - 2 \left( x \left( -\sin x \right) - \int \cos x \, dx \right) \\
= x^2 \sin x + 2 \cos x - 2x \sin x + C \\
\]

(b) 
\[
\int \frac{\sqrt{x}}{1 + x} \, dx = \int \frac{u \, 2u \, du}{1 + u^2} \\
= 2 \int \frac{u^2 \, du}{1 + u^2} \\
= 2 \int \frac{u^2 \, du}{1 + u^2} \\
= 2 \int \frac{du}{1 + u^2} - 2 \int \frac{1 \, du}{1 + u^2} \\
= 2u - 2 \arctan u + C \\
= 2\sqrt{x} - 2 \arctan \sqrt{x} + C \\
\]
\[
\int \frac{\sqrt{x}}{1 + x} = \sqrt{x} \left( \arctan \sqrt{x} \right) - \int \frac{\sqrt{x}}{1 + (\sqrt{x})^2} \\
\]
\[
\int \frac{1}{e^x + e^{-x}} \, dx
\]

\[
u = e^x
\]
\[
du = e^x \, dx
\]
\[
dx = \frac{du}{u}
\]
\[
\int \frac{1}{u^2 - 1} \, du = \int \frac{du}{u^2 + 1}
\]
\[
= \arctan u + C
\]
\[
= \arctan e^x + C
\]

\[
e^x = \tan u
\]
\[
e^x \, du = \sec^2 u \, du
\]
\[
\frac{\sec^2 u \, du}{\tan u + \frac{1}{\tan u}} = \frac{du}{\tan^2 u + 1}
\]
2. (10 pts)

(a) Prove that \( \int_1^\infty x^{-x} \, dx \) converges.

For \( x \geq e \), we have \( x^{-x} \leq e^{-x} \) (and for \( x > 30 \))

Now \( \int_1^e x^{-x} \, dx \) exists since \( x^{-x} \) is continuous on \([1, e]\)

And \( \int_e^\infty x^{-x} \, dx \) converges by comparison with \( \int_e^\infty e^{-x} \, dx \)

So \( \int_1^\infty x^{-x} \, dx \) converges.

(b) Determine (with an argument) whether the following integral converges or diverges

\[
\int_1^\infty \frac{1}{x \log x \log \log x} \, dx.
\]

Note that \( \log \log \log x \)

\[
= \frac{1}{x \log x \log \log x}
\]

This can be derived by letting \( u = \log \log x \) in

\[
S \frac{1}{x (\log x) (\log \log x)}
\]

Then

\[
\lim_{M \to \infty} \int_1^M x^{-x} \, dx.
\]

\[
= \lim_{M \to \infty} \log \log \log \frac{1}{10}
\]

which goes to \( \infty \) so \boxed{\text{diverges}}

You may wish to use the Lagrange form of the Remainder in Taylor’s Theorem in the next two problems. Here it is: If \( f(x) \) is \( n+1 \) times differentiable at \( a \), then there is some \( t \in [a, x] \) such that

\[
R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} (x-a)^{n+1}.
\]

3. (16 pts) Let \( f(x) = \cos x \). Let \( R_{n,a}(x) \) denote the \( n \)-th remainder term for \( f \), as usual.

(a) Write down \( R_{n,a}(x) \) explicitly.

For \( n = 2m \) or \( n = 2m+1 \), we have

\[
P_{n,a}(x) = 1 - \frac{x^2}{2!} + \ldots + (-1)^m \frac{x^m}{m!}.
\]

(b) Show that

\[
|R_{n,a}(x)| \leq \frac{|x|^{n+2}}{(2n+2)!}.
\]

From Lagrange, we have

\[
|R_{n,a}(x)| \leq \frac{1}{(2m+1)!} x^{m+1}
\]

for any \( m \).

Since any derivative of \( \cos \) is \( \pm \sin \) or \( \pm \cos \) so bounded by 1 in absolute value.

Now, from (a), \( P_{n+1,0}(x) = P_{n,0}(x) + R_{n+1,0}(x) \) and

\[
R_{n+1,0}(x) = R_{n,0}(x) \quad \text{and} \quad |R_{n+1,0}(x)| \leq \frac{1}{(2n+2)!} x^{n+2}.
\]
4. (8 pts) Show that if you use

\[ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \]

to approximate \( e^x \) on the interval \( 0 \leq x \leq 1 \), the error will be \( \leq .005 \). You may use the fact that \( e < 3 \).

Since \( e < 3 \), we have

\( e^t < 3 \) for all \( t \) in \( [0, 1] \).

Since \( e^t \) is the \( k \)-th derivative

of \( e^t \) any \( k \), Lagrange

Gives

\[ |e^x - P_{5,0}(x)| = |R_{5,0}(x)| \leq \frac{3}{6!} x^6 \]

\[ \leq \frac{1}{605} (1x)^6 \]

\[ \leq \frac{1}{605} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \]

And \( \frac{1}{605} < 0.005 \).

Since \( P_{5,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \)

We are done.
5. (8 pts) Let \( f(x) = \int_1^x \log t \, dt \). Write down the third Taylor polynomial \( P_3(x) \) for \( f \) centered at 1.

\[
\begin{align*}
\tau(0) &= \int \log t \, dt = 0 \\
\tau'(0) &= \frac{\ln x}{x} = 0 \\
\tau''(0) &= \frac{1}{x} = 1 \\
\tau'''(0) &= -\frac{1}{x^2} = -1
\end{align*}
\]

So, Taylor polygon is

\[
\frac{(x-1)^3}{3} - \frac{(x-1)^3}{2}
\]

This can also be seen by taking the integral

\[
\int_1^x (t-1)^2 - \frac{(t-1)^2}{2}
\]

6. (8 pts) Let \( G(\lambda) = \int_0^\lambda \cos \lambda t \, dt \). Show that \( \lim_{\lambda \to \infty} G(\lambda) = 0 \).

\[
\int_0^\lambda \cos 2\varepsilon = \frac{1}{2} (\sin 2 - \sin 0)
\]

\[
= \frac{1}{2} \sin 2
\]

Since \(-1 \leq \sin 2 \leq 1\), we have

\[
\lim_{\varepsilon \to \infty} \frac{1}{2} \leq \lim_{\varepsilon \to \infty} \frac{\sin 2}{2} \leq \lim_{\varepsilon \to \infty} \frac{1}{2}
\]

So

\[
\lim_{\varepsilon \to \infty} \frac{\sin 2}{2} = 0
\]
7. (20 pts)

(a) Find the surface area of the cone of radius 2 and length 4 by rotating the graph of \( y = 2x \) from \( x = 0 \) to \( x = 4 \) about the \( z \)-axis.

\[
\begin{align*}
\frac{ds}{dz} &= \sqrt{1 + (2x)^2} \\
&= \sqrt{1 + 4x^2}
\end{align*}
\]

\[
\begin{align*}
S &= \int_0^4 \sqrt{1 + 4x^2} \, dx \\
&= \pi \int_0^4 \sqrt{4 - x^2} \, dx
\end{align*}
\]

\[
\begin{align*}
&= \pi \left[ \frac{x}{2} \sqrt{4 - x^2} + \frac{1}{2} \ln \left( x + \sqrt{4 - x^2} \right) \right]_0^4 \\
&= \pi \left[ \frac{16}{2} + \frac{1}{2} \ln \left( 4 + \sqrt{4 - 4} \right) \right] \\
&= \pi \left[ 8 + \frac{1}{2} \ln (4) \right] \\
&= \pi \left[ 8 + \frac{1}{2} \ln (4) \right]
\end{align*}
\]

(b) Find the volume of the sphere of radius \( r \) by rotating the area under the graph of \( y = \sqrt{4 - x^2} \) (from \( x = -r \) to \( x = r \)) around the \( x \)-axis.

\[
\begin{align*}
V &= \int_{-r}^{r} \pi \left( \sqrt{4 - x^2} \right)^2 \, dx \\
&= \pi \int_{-r}^{r} 4 - x^2 \, dx \\
&= \pi \left[ 4x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left[ 4r - \frac{r^3}{3} - \left( -4r + \frac{r^3}{3} \right) \right] \\
&= \pi \left( \frac{8r}{3} \right)
\end{align*}
\]