NAME (please print): Steve

Please write out and sign the following statement:
I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

________________________________________

INSTRUCTIONS:

(1) Circle your professor's name.

(2) Check that you have all 7 problems and 9 pages (including this cover).

(3) The examination is 75 minutes long.

(4) Show all your work. You must provide complete, clear answers and reasons for all problems.

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1. (10 pts) Suppose that $f$ is continuous on $[a, b]$ and set $F(x) = \int_a^x f(t) \, dt$. Prove that for any $x \in (a, b)$, we have

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x).$$

(Do not just quote one of the fundamental theorems of calculus, prove it from scratch.)

We want

$$\lim_{h \to 0^+} \frac{\int_x^{x+h} f(t) \, dt}{h} = f(x).$$

Let

$$M(h) = \max \{ f(t) : x \leq t \leq x+h \},$$

$$m(h) = \min \{ f(t) : x \leq t \leq x+h \}.$$ (Since $f$ is int on $[x, x+h]$, these values are attained.)

Since $f$ is int, \( \lim_{h \to 0^+} M(h) = \lim_{h \to 0^+} m(h) = f(x) \)

Then

$$\frac{m(h) \cdot h}{h} \leq \frac{\int_x^{x+h} f(t) \, dt}{h} \leq \frac{M(h) \cdot h}{h},$$

so

$$m(h) \leq \frac{\int_x^{x+h} f(t) \, dt}{h} \leq M(h).$$

Since the upper & lower lims $\to f(x)$ as $h \to 0^+$, so does the middle expression. Q.E.D.
5. (12 pts)

(a) State the Second Fundamental Theorem of Calculus

\[ \int_{a}^{b} f(x) \, dx \quad \text{and} \quad g \text{ is a function } g' = f \text{ on } [a, b]. \]

Then

\[ \int_{a}^{b} f = g(b) - g(a). \]

(b) Use the Second Fundamental Theorem of Calculus to calculate

\[ \lim_{x \to \infty} \int_{1}^{x} \frac{1}{t^3} \, dt. \]

Let \( g(t) = \frac{1}{t^3} \)

Then \( g'(t) = -\frac{1}{2t^2} \) and \( g'(t) = \frac{1}{t^3} \).

By the Fundamental Theorem,

\[ \int_{1}^{x} \frac{1}{t^3} \, dt = g(x) - g(1) = -\frac{1}{2x^2} - \left( -\frac{1}{2} \right) \]

\[ \to \frac{1}{2} \quad \text{as} \quad x \to \infty. \]
6. (20 pts) Calculate:

(a) \[
\frac{d}{dx} \int_{2}^{x^2} (1 + 2^t) \, dt.
\]

\[
\left(1 + 2^x\right) \cdot 2x
\]

(b) \[
\frac{d}{dx} \int_{2}^{4} (1 + 2^t) \, dt.
\]
\[
\frac{d}{dx}(\sin x) = e^{\ln x} \frac{d}{dx}(\sin x) = e^{\ln x} \left( \cos x \frac{d}{dx}(x) + \frac{\sin x}{x} \right) = e^{\ln x} \left( \cos x \cdot \frac{e^x}{x} + \frac{\sin x}{x} \right) = x \left( \cos x \cdot \frac{e^x}{x} + \frac{\sin x}{x} \right)
\]

\[
\frac{d}{dx} \left( \arctan x + \log \sqrt{1 + x^2} \right) = \frac{1}{1 + x^2} + \frac{1}{\sqrt{1 + x^2}} \cdot \frac{1}{2} (1 + x^2)^{-\frac{1}{2}} \cdot x \times \frac{1}{1 + x^2} + \frac{x}{1 + x^2} = \frac{1 + x}{1 + x^2}
\]
7. (15 pts) Determine whether the following improper integrals converge or diverge. You must justify your answer.

(a) \[ \int_{8}^{\infty} \frac{1}{x^{5/3} - 1} \, dx \]

\[ \frac{1}{x^{5/3} - 1} \geq \frac{1}{x^{5/3}} = \frac{1}{x^{5/6}} \quad x \geq 8. \]

Since \( \int_{8}^{\infty} \frac{dx}{x^{5/3}} \) with \( p < 1 \) diverges, so does our \( \int \) by comp test.

(b) \[ \int_{1}^{\infty} \frac{1}{x^{2/3} + x} \, dx \]

\[ \frac{1}{x^{2/3} + x} \leq \frac{1}{x^{2/3}} \]

\( \frac{2}{3} > 1 \) so \( \int_{1}^{\infty} \frac{dx}{x^{2/3}} \) converges.

By comp test, so does our int.

(c) \[ \int_{1}^{\infty} \frac{x}{e^x} \, dx \]

\[ \frac{e^x}{x} \geq 1 \quad \text{for} \quad x > e^{x_0} \]

(since \( \frac{e^x}{x} \to \infty \) and \( \int_{x_0}^{\infty} \frac{dx}{e^x} \) diverges). So \( \int_{1}^{\infty} \frac{x}{e^x} \, dx \) diverges by comp test.