MTH 165: Linear Algebra with Differential Equations

Final Exam
May 4, 2015

NAME (please print legibly): ____________________________________________
Your University ID Number: _____________________________________________
Indicate your instructor with a check in the box:

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<td>Dummit</td>
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<td>Friedmann</td>
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<td>Petridis</td>
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<td>Rice</td>
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- You have 3 hours to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 14 pages.

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1. **(10 points)** Find a solution (implicit solutions are acceptable) for the following initial value problems on the domain \((0, \infty)\):

   (a) \(2y + xy' = x^{-1}, \ y(1) = A\).

   (b) \(2x + yy' = x^{-1}, \ y(1) = B\).
2. (10 points) Find a basis for the nullspace of each matrix.

(a) \( A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(b) \( B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

(c) \( C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \).
3. **(10 points)** Let \( M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix} \), where \( k \) is a parameter.

(a) Find \( \det(M) \).

(b) Find all value(s) of \( k \) such that \( M \) is not an invertible matrix.
We continue taking \( M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix} \), where \( k \) is a parameter.

(c) Find all value(s) of \( k \) such that \( \lambda = 2 \) is an eigenvalue of \( A \).
4. **(10 points)** Determine whether each given set $S$ is a subspace of the given vector space $V$. If so, give a proof; if not, explain why not.

(a) $V = \mathbb{R}^3$ and $S = \{(x, y, z) \in V \mid x + y = z\}$.

(b) $V = M_2(\mathbb{R})$, the set of $2 \times 2$ matrices, and $S = \{A \in V \mid A^2 = 0\}$.
5. (10 points) Let

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \]

(a) Find the eigenvalues of \( A \), and determine (with justification) whether \( A \) is a defective matrix. (In other words, determine whether \( \mathbb{R}^4 \) has a basis consisting of eigenvectors of \( A \).)

(b) Find the eigenvalues of \( A^2 \), and determine (with justification) whether \( A^2 \) is a defective matrix. (In other words, determine whether \( \mathbb{R}^4 \) has a basis consisting of eigenvectors of \( A^2 \).)
6. (10 points) Let $M_{2 \times 2}(\mathbb{R})$ be the vector space of $2 \times 2$ real matrices. Consider the linear transformation $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a + d & b - c \\ a - c & b + d \end{bmatrix}.$$ 

(a) Find a basis for the kernel of $T$, and the dimension of the kernel.
Recall that
\[ T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a + d & b - c \\ a - c & b + d \end{bmatrix}. \]

(b) Find the dimension of the range of \( T \).

(c) Is the identity matrix \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) in the range of \( T \)? Justify why or why not.
7. (10 points) Find the general solution for each differential equation:

(a) \( y'' + 4y' + 4y = 0 \).

(b) \( y^{(4)} - y = 0 \).

(c) \( y''' - 2y'' + 5y' = 0 \).
8. (10 points) Solve the equation

\[ y'' + 4y = 4 \cos(2x) + 8e^{2x} \]

with initial conditions \( y(0) = 3, \ y'(0) = 4. \)
9. (10 points) Consider a spring-mass system with spring constant $k = 4 \text{ N/m}$ and a mass $m = 1 \text{ kg}$.

(a) Suppose there is no friction (or damping), and an external driving force of $6 \sin(4t) \text{ N}$ is applied to the mass (in the positive direction). If at time $t = 0$ the mass is at rest in the equilibrium position, find the position $y(t)$ of the mass at time $t$ for $t \geq 0$. 
Continue to consider the spring-mass system with spring constant \( k = 4 \text{ N/m} \), a mass \( m = 1 \text{ kg} \), and an external driving force of \( 6 \sin(4t) \text{ N} \) and no friction (or damping).

(b) What is the earliest time that the mass returns to its equilibrium position? (Hint: you may need to use the identity \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \).)
10. (10 points) Solve the system of differential equations

\[ x'_1 = 2x_1 + 2x_2 \]
\[ x'_2 = -x_1 + 4x_2 \]

subject to the initial conditions \( x_1(0) = 1 \) and \( x_2(0) = 1 \).