Math 165: Linear Algebra with Differential Equations

Final Exam
December 15, 2015

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Schedule</th>
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<tr>
<td>Herman</td>
<td>MWF 10:25</td>
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<tr>
<td>Lubkin</td>
<td>MW 2:00</td>
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<td>Madhu</td>
<td>TR 2:00</td>
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<td>McTague</td>
<td>MWF 9:00</td>
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<td>Rivera-Letelier</td>
<td>TR 3:25</td>
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2. Read the notes below:

- The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.
- Notes of any kind are strictly forbidden.
- Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 12 pages.

3. Read the following Academic Honesty Statement and sign:

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________________________


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Part A
1. (10 points)

(a) Solve the initial value problem:

$$\frac{dx}{dt} + \frac{2}{4 - t} x = 5, \quad x(0) = 4.$$

(b) Solve the initial value problem

$$(x^2 + 1)y' + y^2 = -1, \quad y(0) = 1.$$
2. (10 points)

(a) Compute the determinant in terms of $a, b, c,$ and $d$ showing all your steps:

$$
\text{det} \begin{bmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
a & b & 0 & 0 \\
c & d & 0 & 0
\end{bmatrix}
$$
(b) Let $k \in \mathbb{R}$ and a $3 \times 3$ matrix $A$ be given by

$$A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Suppose the row vectors satisfy

$$r_3 = 4r_1 + kr_2.$$ 

Find all values of $k$ for which the system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution. If there are no values of $k$, explain why not.
3. (10 points)

(a) Which of the following are subspaces of $P_3$? Circle all that apply, or write “NONE”. Answer only; no partial credit will be awarded. Your answer must be perfect to receive credit.

1. span($\{t, 2t - 6\}$)
2. $\{p(t)|p(1) = 0\}$
3. $\{p(t)|\int_0^1 p(t)dt = 1\}$

(b) Which of the following subsets of $M_2(\mathbb{R})$ span $M_2(\mathbb{R})$? Circle all that apply, or write “NONE”. Answer only; no partial credit will be awarded. Your answer must be perfect to receive credit.

1. $\begin{Bmatrix}
\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}
\end{Bmatrix}$
2. $\begin{Bmatrix}
\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}
\end{Bmatrix}$
3. $\begin{Bmatrix}
\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}
\end{Bmatrix}$
(c) Which of the following subsets of $\mathbb{R}^3$ are linearly independent? Circle all that apply, or write “NONE”. Answer only; no partial credit will be awarded. Your answer must be perfect to receive credit.

1. $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$
2. $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$
3. $\{(1, 2, 3), (4, 5, 6), (2, 1, 0)\}$

(d) Which of the following subsets of $P_2$ form a basis for $P_2$? Circle all that apply, or write “NONE”. Answer only; no partial credit will be awarded. Your answer must be perfect to receive credit.

1. $\{t^2 + 2t + 3, 4t^2 + 5t + 6, 7t^2 + 8t + 9\}$
2. $\{t - 1, 2t - 3, t\}$
3. $\{2t + 3, t^2 + 2t, 2t^2 + 2t - 3\}$
4. (10 points) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation given by $T(v) = Av$, where

$$A = \begin{bmatrix} 1 & 0 & 5 & 0 \\ -2 & 4 & 0 & 5 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis for the kernel of $T$.

(b) Find a basis for the range of $T$. 
Part B
5. (10 points) Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 & 3 \\
5 & 2 & 6 \\
-2 & -1 & -3
\end{bmatrix}.
\]

(a) Find all the eigenvalues of the matrix \( A \).

(b) For each eigenvalue of \( A \), find all the corresponding eigenvectors of \( A \).
6. (10 points)

(a) Find the general solution to the differential equation:

\[ y^{(4)} + 2y'' + y = 0 \]

[Hint: A polynomial \( P(r) \) having only even powers can be factored in stages by first considering it as a polynomial in \( r^2 \).]

(b) Find the general solution to the differential equation:

\[ y^{(5)} + 2y''' + y' = x \]
7. (10 points) Suppose an $m = 1 \text{ kg}$ mass is attached to a spring and dropped. What is the strongest the spring can be—that is, how large can the spring constant $k > 0$ be, measured in $N/m = kg/s^2$—to ensure that the mass does not pass through the equilibrium position $y = 0$ more than once? Assume the damping constant $c = 2 \text{ kg/s}$.

[Hint: Recall that the spring mass system is modeled by the differential equation $my'' + cy' + ky = 0$.]
8. (10 points) Find all solutions of the following system of differential equations,

\[ x' = 3x + 4y + 2z; \]
\[ y' = x + 2y + z; \]
\[ z' = -7x - 10y - 5z. \]