NAME (please print legibly): _______________________________
Your University ID Number: _______________________________

Instructions:

1. Read the notes below:
   
   • The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.

   • Notes of any kind are strictly forbidden.

   • Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

   • You are responsible for checking that this exam has all 9 pages.

2. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature: ____________________________________________

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points)

(a) (3 points) The statement ‘all non-negative real numbers are squares’ can be expressed as the following logical expression:

\[ \forall x \geq 0 \exists y \ (x = y^2) \].

Write the negation of the above statement using logical expressions so that the negation symbol \( \neg \) does not appear.

(b) (3 points) The implication ‘if \( x \) is a non-negative real number, then it is a square’ can be expressed as the following logical expression:

\[ (x \geq 0) \rightarrow [\exists y \ (x = y^2)] \].

Write the contrapositive of the above implication using logical expressions so that the negation symbol \( \neg \) does not appear.
(c) (4 points) Let $S = \{+, *\}$. Write down the Cartesian product $S \times S$ and the power set $\mathcal{P}(S)$.

$S \times S = \{\}$

$\mathcal{P}(S) = \{\}$
2. (10 points)

(a) (6 points) Let $x$ be a positive integer. Prove that the following statements are equivalent:

(i) $x$ is odd.
(ii) $x^2 + 1$ is even.
(b) (2 points) Let $x$ be a positive integer. Prove that if $x^2$ is even, then $x^3$ is even.

(c) (2 points) Let $x$ be a rational number. Is it true that $x(x - 1)$ is a rational number? Provide a proof or an explicit counter example.
3. (10 points) For each of the following functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ prove or disprove (by an explicit counter example) each of the following:

- $f$ is injective (one-to-one),
- $f$ is surjective (onto),
- $f$ is bijective.

(a) (5 points) $f(x, y) = (y, x + y)$. 

Is $f$ bijective? YES NO
(b) (5 points) \( f(x, y) = (y^2, x) \).

Is \( f \) bijective?  YES  NO
4. (10 points)

(a) (3 points) Write a pseudocode or give an accurate description of the “bubble sort” algorithm, which puts \( n \) real numbers \( a_1, \ldots, a_n \) in increasing order.

(b) (1 point) The algorithm requires \( \Theta(n^k) \) comparisons, for what integer value of \( k \)?

\[ k = \____ \]
(c) (4 points) For each of the following functions \( f(x) \) find the smallest \( k \in \{0, 1, \ldots \} \) and the largest \( \ell \in \{0, 1, \ldots \} \) such that \( f(x) = O(x^k) \) and \( f(x) = \Omega(x^\ell) \).

(i) \( f(x) = 10x^3 \ln(x) + \ln(x)^4 \).

\[ k = \quad \ell = \]

(ii) \( f(x) = \frac{3x + x^2 \ln(x)}{10 + x/\ln(x)} \).

\[ k = \quad \ell = \]

(d) (2 points) For each of the following functions \( f(x) \) find a “simple” function \( g(x) \) so that \( f(x) = \Theta(g(x)) \). You are not allowed to use \( g(x) = f(x) \) nor should you write \( x = \Theta(2x) \).

(i) \( f(x) = \frac{1}{x} + 5x2^x - 3x^{100} \).

\[ g(x) = \]

(ii) \( f(x) = (x - 9 \cdot 4^x)(-2\ln(x) + 5) \).

\[ g(x) = \]