1. (10 points)

(a) (3 points) The statement ‘all non-negative real numbers are squares’ can be expressed as the following logical expression:

$$\forall x \geq 0 \ [\exists y \ (x = y^2)].$$

Write the negation of the above statement using logical expressions so that the negation symbol $\neg$ does not appear.

**Solution:**

$$\neg[\forall x \geq 0 \ [\exists y \ (x = y^2)]] \equiv \exists x \geq 0 \ [\exists y \ (x = y^2)] \equiv \exists x \geq 0 \ [\forall y \ (x \neq y^2)].$$

(b) (3 points) The implication ‘if $x$ is a non-negative real number, then it is a square’ can be expressed as the following logical expression:

$$(x \geq 0) \rightarrow [\exists y \ (x = y^2)].$$

Write the contrapositive of the above implication using logical expressions so that the negation symbol $\neg$ does not appear.

**Solution:** The negation of $(x \geq 0)$ is $(x < 0)$; the negation of $[\exists y \ (x = y^2)]$ is $[\forall y \ (x \neq y^2)]$. Therefore the contrapositive is

$$[\forall y \ (x \neq y^2)] \rightarrow (x < 0).$$

(c) (4 points) Let $S = \{+, \ast\}$. Write down the Cartesian product $S \times S$ and the power set $\mathcal{P}(S)$.

**Solution:**

$$S \times S = \{(+, +), (+, \ast), (\ast, +), (\ast, \ast)\}.$$

$$\mathcal{P}(S) = \{\emptyset, \{+\}, \{\ast\}, \{+, \ast\}\}.$$
2. (10 points)

(a) (6 points) Let $x$ be a positive integer. Prove that the following statements are equivalent:

(i) $x$ is odd.

(ii) $x^2 + 1$ is even.

**Solution:**

(i) $\rightarrow$ (ii): We suppose that $x = 2k + 1$ for some positive integer $k$ and deduce that $x^2 + 1 = 2\ell$ for some positive integer $\ell$. Indeed

$$x^2 + 1 = (2k + 1)^2 + 1 = 2(2k^2 + 2k + 1) \quad [\ell = 2k^2 + 2k + 1].$$

(ii) $\rightarrow$ (i): We prove the contrapositive: If $x$ is even, then $x^2 + 1$ is odd. We suppose that $x = 2k$ for some positive integer $k$ and deduce that $x^2 + 1 = 2\ell + 1$ for some positive integer $\ell$. Indeed

$$x^2 + 1 = (2k)^2 + 1 = 2(2k^2) + 1 \quad [\ell = 2k^2].$$

(b) (2 points) Let $x$ be a positive integer. Prove that if $x^2$ is even, then $x^3$ is even.

**Solution:** We suppose that $x^2 = 2k$ for some positive integer $k$ and deduce that $x^3 = 2\ell$ for some positive integer $\ell$.

$x^2 = 2k$ and so

$$x^3 = x \cdot x^2 = 2(kx) \quad [\ell = kx].$$

(c) (2 points) Let $x$ be a rational number. Is it true that $x(x - 1)$ is a rational number? Provide a proof or an explicit counter example.

**Solution:** The proposition is true. $x$ is rational and so $x = \frac{a}{b}$ for some integers $a$ and $b$ with $b \neq 0$. Then

$$x(x - 1) = \frac{a}{b} \left(\frac{a}{b} - 1\right) = \frac{a(a - b)}{b^2}$$

is the ratio of two integers, with the denominator being non-zero. Formally we used the fact that the difference and product of two integers is an integer.
3. **(10 points)** For each of the following functions \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \) prove or disprove (by an explicit counter example) each of the following:

- \( f \) is injective (one-to-one),
- \( f \) is surjective (onto),
- \( f \) is bijective.

**(a) (5 points)** \( f(x, y) = (y, x + y) \).

**Solution:** \( f \) is injective because if \( f(x, y) = f(u, v) \) then

\[
(y, x + y) = (v, u + v).
\]

This implies both \((y = v)\) and \((x + y = u + v)\). It follows that \(x = u\) and \(y = v\), which is equivalent to \((x, y) = (u, v)\).

\( f \) is surjective because for each \((x, y) \in \mathbb{R} \times \mathbb{R}\) we have

\[
f(y - x, x) = (x, (y - x) + x) = (x, y).
\]

Is \( f \) bijective? YES because it is both injective and surjective.

**(b) (5 points)** \( f(x, y) = (y^2, x) \).

**Solution:** \( f \) is not injective because \( f(0, 1) = (1, 0) = f(0, -1) \).

\( f \) is not surjective because for each \((x, y) \in \mathbb{R} \times \mathbb{R}\) we have

\[
f(x, y) \neq (-1, 0)
\]

because \(y^2 \neq -1\) for all \(y \in \mathbb{R}\).

Is \( f \) bijective? NO because it is not injective.
4. (10 points)

(a) (3 points) Write a pseudocode or give an accurate description of the “bubble sort” algorithm, which puts \( n \) real numbers \( a_1, \ldots, a_n \) in increasing order.

Solution: The pseudocode can be found on p.197 of the book.

A brief summary: The algorithm successively places the largest, second largest etc real number at the bottom. To place the \( j \)th largest \( a_i \) at the bottom, when the first \( (j - 1) \) largest \( a_i \) are already ordered at the bottom, the algorithm performs pairwise comparisons between \( a_1 \) and \( a_2 \), \( a_2 \) and \( a_3 \), \ldots, \( a_{j-1} \) and \( a_j \), interchanging the larger with the smaller if necessary.

(b) (1 point) The algorithm requires \( \Theta(n^k) \) comparisons, for what integer value of \( k \)?

Solution: \( k = 2 \) because the first step requires \( (n - 1) \) comparisons, the second \( (n - 2) \) and the last 1. So overall there are

\[
(n - 1) + \cdots + 1 = \sum_{i=1}^{n-1} = \frac{(n - 1)n}{2} = \Theta(n^2).
\]

(c) (4 points) For each of the following functions \( f(x) \) find the smallest \( k \in \{0, 1, \ldots\} \) and the largest \( \ell \in \{0, 1, \ldots\} \) such that \( f(x) = O(x^k) \) and \( f(x) = \Omega(x^\ell) \).

(i) \( f(x) = 10x^3 \ln(x) + \ln(x)^4 \).

Solution: \( k = 4 \) and \( \ell = 3 \) (the first term dominates).

(ii) \( f(x) = \frac{3x + x^2 \ln(x)}{10 + x/\ln(x)} \).

Solution: \( k = 2 \) and \( \ell = 1 \) (\( f(x) \) is roughly equal to \( x \ln(x)^2 \) when \( x \) is large).

(d) (2 points) For each of the following functions \( f(x) \) find a “simple” function \( g(x) \) so that \( f(x) = \Theta(g(x)) \). You are not allowed to use \( g(x) = f(x) \) nor should you write \( x = \Theta(2x) \).

(i) \( f(x) = 1/x + 5x2^x - 3x^{100} \).

Solution: \( g(x) = x2^x \) (exponentials grow faster than powers).

(ii) \( f(x) = (x - 9 \cdot 4^x)(-2 \ln(x) + 5) \).

Solution: \( g(x) = \ln(x)4^x \) because \( f(x) \) is roughly equal to \( 18 \ln(x)4^x \) when \( x \) is large.