MATH 143
Midterm 1
February 27, 2014

NAME (please print legibly): 
Your University ID Number: 
Instructor: Mihai Bailesteanu

- No calculators are allowed on this exam.
- Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.
- The presence of cellphones is strictly forbidden during the exam.
- You are not allowed to communicate with any other student during the exam.

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| TOTAL    | 100   |
1. (20 points)

a) Eliminate the parameter to find a Cartesian equation of the curve: \( x = 5t - t^3, \ y = 5 - 5t. \)

b) Find the equation of the tangents to the curve \( x = 3t^2 + 1, \ y = 2t^3 + 1 \) that pass through the point (4,3).

c) Find \( dy/dx \) and \( d^2y/dx^2 \) and determine the values of \( t \) for which the following parametric curve is concave up: \( x = \cos(2t), \ y = \cos t, \ t \in (0, \pi). \)

\[
\begin{align*}
\text{a)} \quad & x = \frac{5t - t^3}{5} \Rightarrow x = 5 - \left( \frac{5-3}{5} \right) \frac{t^3}{3^3} = \frac{25 - 15t^2 + 50t + 500}{125} \\
\text{b)} \quad & \frac{dx}{dt} = 6t - 2t^3 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6t^3}{6t^3} = t \\
& \frac{d^2x}{dt^2} = 6t^2 - 6t^3 \quad \Rightarrow \quad m = \frac{dx}{dt} = t (dy/dx) \\
& m = 4 \Rightarrow t = \pm 1 \Rightarrow t = -1 \quad \text{at} \ (4,3) \\
& y = 4 + b \quad \Rightarrow \quad b = -1 \quad \text{at} \ (4,3) \\
& \frac{dy}{dx} = \pm \pm 1 \\
& \Rightarrow \quad \pm = 0 \\
\text{c)} \quad & \frac{dx}{dt} = -2 \sin(2t) \quad \Rightarrow \quad \frac{dy}{dt} = -4 \sin t \cos t \\
& \frac{d^2x}{dt^2} = -2 \cos t \\
& \therefore \quad \frac{1}{(\cos t)^3} = \frac{1}{4} \quad \sin t \cos t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{-2 \cos t}{(\cos t)^3} \\
& \therefore \quad \frac{d}{dt} \left( \frac{x}{dt} \right) = \frac{-2 \cos t}{(\cos t)^3} \\
& \Rightarrow \quad \frac{d^2x}{dt^2} = \frac{-2 \cos t}{(\cos t)^3} \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{-2 \cos t}{(\cos t)^3} \\
& \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{1}{4} \sin t \cos t \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{1}{16} (\cos t)^3 \\
& \therefore \quad \frac{dx}{dt} > 0 \quad \text{at} \ (0, \pi) \quad \text{and} \quad \frac{dy}{dt} > 0 \quad \text{at} \ (0, \pi) \\
& \therefore \quad \frac{d^2y}{dt^2} > 0 \quad \text{at} \ (0, \pi) \\
& \therefore \quad \text{the graph is concave down,} \\
& \therefore \quad \text{the graph is concave up.}
2. (20 points)

a) Find the length of the contour of the area enclosed by the curve \( x = t^2 - 2t, y = \frac{2}{3} t^{\frac{3}{2}} \) and the y-axis by following these steps:

i) Solve for \( t \): \( x(t) = 0 \). You should find two values \( t_1 \) and \( t_2 \).

ii) Find the length by evaluating an integral between \( t_1 \) and \( t_2 \). Then add the portion of the contour contained in the y-axis.

\[
\int_0^2 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_0^2 \sqrt{4t - 4 + \frac{4}{9} t} \, dt
\]

b) Consider the following parametric equation: \( x(t) = e^t \cos t \) and \( y(t) = e^t \sin t \). Determine the length of the curve between \( t = 0 \) and \( t = \pi \).

\[
L = \int_0^\pi \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

\[
= \int_0^\pi \sqrt{e^{2t} \cos^2 t + e^{2t} \sin^2 t} \, dt
\]

\[
= \int_0^\pi e^t \sqrt{\cos^2 t + \sin^2 t} \, dt
\]

\[
= \int_0^\pi e^t \, dt
\]

\[
= \left[ e^t \right]_0^\pi
\]

\[
= e^\pi - 1
\]

\[
L = \sqrt{e^{2\pi} - 1}
\]
3. (20 points)

a) The Cartesian coordinates of a point are given. Find the polar coordinates \((r, \theta)\), first when \(r > 0\) and \(0 \leq \theta < 2\pi\) and second, when \(r < 0\) and \(0 \leq \theta < 2\pi\).
   
   i) \((3\sqrt{3}, 3)\)
   
   ii) \((1, -2)\)

b) The following points are given in polar coordinates. Find the Cartesian coordinates for each of them.

   i) \((3, \pi/2)\)

   ii) \((-1, \pi/3)\)

c) A curve with polar equation

\[
r = \frac{11}{9 \sin \theta + 55 \cos \theta}
\]

represents a line. Write this line in the Cartesian form \(y = mx + b\).

\[
\begin{align*}
\text{(i)} & \quad x = \sqrt{x^2 + y^2} = \sqrt{2x + 9} = \sqrt{5} \\
& \quad \tan \theta = \frac{y}{x} = \frac{1}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \\
& \quad \begin{cases} r > 0, & 0 \leq \theta < 2\pi \quad \left( \frac{6}{5}, \frac{\pi}{2} \right) \\
& \quad \left( r < 0, & 0 \leq \theta < 2\pi \right) \left( -6, \frac{\pi}{2} \right) \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{(ii)} & \quad x = 1, \quad y = -2 \quad \Rightarrow \quad x = \sqrt{1 + y^2} = \sqrt{5} \\
& \quad \tan \theta = -2 \quad \Rightarrow \quad \theta = \tan^{-1}(-2) \\
& \quad \begin{cases} \frac{5}{\tan^{-1}(-2)} \\
& \quad \left( -5, \frac{\pi}{2} \right) \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{(iii)} & \quad x = 3, \quad y = 0 \\
& \quad \left( 0, 3 \right)
\end{align*}
\]

\[
\begin{align*}
\text{(iv)} & \quad x = 3 \cos \left( \frac{\pi}{2} \right) = 3 \\
& \quad \left[ 3, \quad 3 \cos \left( \frac{\pi}{2} \right) = 3 \right]
\end{align*}
\]

\[
\begin{align*}
\text{(v)} & \quad x = \left( -\frac{1}{2}, \quad \frac{\sqrt{3}}{2} \right) \\
& \quad \left[ \left( -\frac{1}{2}, \quad \frac{\sqrt{3}}{2} \right) \right]
\end{align*}
\]

\[
\begin{align*}
\text{(vi)} & \quad \frac{1}{9} \frac{\cos \theta}{6} + 55 \frac{\cos \theta}{8} = 11 \\
& \quad \frac{1}{9} \frac{x}{6} + 55 \frac{x}{8} = 11 \\
& \quad \frac{y}{6} + 55 \frac{x}{8} = 11 \\
& \quad \Rightarrow \quad y = \frac{55}{9} x + \frac{11}{9}
\end{align*}
\]
4. (20 points)

a) Find the area enclosed by the curve \( r = \sqrt{1 + \cos^2(5\theta)}, \ 0 \leq \theta < 2\pi \).

b) Consider the following polar curve: \( r = \sin(5\theta) \), with \( t \in [0, 2\pi] \). Find the area enclosed by one of the five loops by following these steps:

i) Find the values of \( \theta \) for which \( r = 0 \). There should be 10 values.

ii) Set up an integral for the area, by making sure you integrate between the first two values determined above.

\[
A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos(5\theta))^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ 1 + \cos^2(5\theta) \right] \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ 1 + \frac{1}{2} \left( 3 + \cos(10\theta) \right) \right] \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[ 3\theta + \frac{1}{10} \sin(10\theta) \right] \, d\theta = \frac{6\pi^2}{4} - \frac{3\pi^2}{2}, \ \forall \ A = \frac{3\pi^2}{2}.
\]

\[ (\text{I}) \] \( x = 0 \quad \Rightarrow \quad \sin(5\theta) = 0 \quad \Rightarrow \quad 5\theta = \pi n, \quad n \in \mathbb{Z} \quad \Rightarrow \quad \theta = \frac{\pi n}{5}, \quad n \in \mathbb{Z} \] but \( n \) can only be \( 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \) \( \Rightarrow \) the 10 solutions are: \( 0, \frac{\pi}{5}, \frac{2\pi}{5}, \ldots, \frac{9\pi}{5}, \frac{10\pi}{5}, \frac{11\pi}{5} \)

\[ (\text{II}) \] \( A \text{ (one loop)} = \int_0^{2\pi} \frac{1}{2} \sin^2(5\theta) \, d\theta = \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos(10\theta)}{2} \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[ 1 - \cos(10\theta) \right] \, d\theta = \frac{1}{4} \left( \theta \bigg|_0^{2\pi} - \frac{1}{10} \sin(10\theta) \bigg|_0^{2\pi} \right) = \frac{\pi}{20}.
\]

\( A \text{ (one loop)} = \frac{\pi}{20} \).
5. (20 points)

a) Find the limit of the following sequences (for some you may want to use L'Hôpital rule):

\[
an = \frac{\ln(1/n)}{\sqrt{2n}}
\]
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln(1/n)}{\sqrt{2n}} = 0
\]
\[
an = \ln(2n^2 + 1) - \ln(n^2 + 1)
\]
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln \left(\frac{2n^2 + 1}{n^2 + 1}\right) = \ln 2
\]
\[
an = \frac{n}{n+1} \cdot \frac{3}{n+2} \cdot \frac{5}{n+3} \cdots \frac{2n-1}{2n}
\]
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{2k - 1}{2k} = \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n}
\]
\[
an = \left(1 + \frac{3}{n}\right)^n
\]
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n = e^3
\]
\[
an = \frac{n!}{e^n}
\]
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n!}{e^n} = 0
\]

b) (BONUS 5 points) Let \( a_n = \frac{1}{2} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} \). Decide if the sequence is monotonic and bounded by following these steps:

i) Show that \( \frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2} \cdot \left(1 - \frac{2}{n+1}\right)^n \). Is this a number less than 1? If yes, then is the sequence decreasing or increasing?

ii) Write \( a_n = \frac{1}{2} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} \). Is each term in the product less than 1? If yes, then what is \( a_n \) bounded from above by? What is \( a_n \) bounded from below by (notice that \( a_n \) is a quotient of natural numbers)?

iii) Conclude that \( a_n \) is convergent. What do you think it converges to?

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} < \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{1}{n!} < \frac{1}{n^2}
\]

But \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \) no by squeeze theorem, \( \lim a_n = 0 \).

4) Notice that \( |a_n| = \left|\sin \left(\frac{2n}{3}\right)\right| \). But \( \lim_{n \to \infty} \left|\sin \left(\frac{2n}{3}\right)\right| = 0 \), so \( \lim a_n = 0 \), which implies that \( \lim a_n = 0 \).

5) \( a_n = \left(1 + \frac{3}{n}\right)^n = e^{\ln \left(1 + \frac{3}{n}\right)} \): Now \( \lim_{n \to \infty} \ln \left(1 + \frac{3}{n}\right) = \lim_{n \to \infty} \frac{\ln \left(1 + \frac{3}{n}\right)}{\frac{1}{n}} = \lim_{n \to \infty} \left(\frac{\ln \left(1 + \frac{3}{n}\right)}{\frac{1}{n}}\right) = \lim_{n \to \infty} \ln \left(1 + \frac{3}{n}\right) = \ln 1 = 0 \). So \( \lim a_n = e^0 = 1 \).

6) (BONUS) 1) This is straightforward. \( \frac{\ln \left(1 + \frac{3}{n}\right)}{\frac{1}{n}} \) is decreasing.

2) Yes, no \( 0 < a_n < 1 \) \( (a_n) \)-bounded

3) Monotone & bounded \( \Rightarrow (a_n) \) is convergent \( \Rightarrow (\text{in fact } \lim a_n = 0) \).