Math 143 Spring 2005
Practice Final Exam - Solutions

1. (a) This series is convergent by the alternating series test. Let \( a_n = \frac{e^{1/n}}{n^3} \).

\[ \lim_{n \to \infty} a_n = 0 \]
since the numerator converges to 1 and the denominator goes to infinity.

• We check that the sequence \( a_n \) is decreasing by looking at the behaviour of the function \( f(x) = \frac{e^{1/x}}{x^3} \) as \( x \) increases. This function will be a decreasing function if its derivative is negative. Using the quotient rule, we get
\[
f'(x) = -\frac{x e^{1/x} - 3x^2 e^{1/x}}{x^6}
\]
for all \( x > 0 \). Therefore \( a_n \) are decreasing to zero. Hence the Alternating Series Test is applicable, and the series is convergent.

(b) The series converges absolutely. Let \( b_n = \frac{e}{n^3} \). Then, \( a_n \leq b_n \) and
\[
\sum_{n} a_n \leq \sum_{n} b_n = e \sum_{n} \frac{1}{n^3} < \infty
\]
by the p-series Test. Therefore by the comparison test, the original series is absolutely convergent.

2. The series diverges by the Integral Test. Let \( f(x) \) be the following function:
\[
f(x) = \frac{1}{x \ln x}
\]
Then, \( f(x) \) is positive for \( x > 1 \) and is a decreasing function. (The denominator increases with increasing \( n \), while the numerator remains fixed at 1). Therefore, the Integral Test is applicable.

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \int_{\ln(2)}^{\infty} \frac{1}{u} \, du
\]
(Using the substitution \( u = \ln x \))
\[
= \ln(u) \bigg|_{\ln(2)}^{\infty}
= \infty - \ln \ln 2
= \infty
\]
Since the improper integral diverges, so does the original series.

3. (15 points) Determine the interval and radius of convergence of the following power series. You have to also determine convergence/divergence at the endpoints of the interval of convergence.

\[ \sum_{n=0}^{\infty} \frac{(-2)^n (x + 3)^n}{\sqrt{n}} \]

Let \( a_n \) be the general term in the power series:
\[
a_n = \frac{(-2)^n (x + 3)^n}{\sqrt{n}}
\]
Using the Ratio Test to check convergence, we get

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-2)^n(x+3)^n} \right| = \lim_{n \to \infty} 2|x+3| \frac{n}{n+1} = 2|x+3|
\]

For convergence, we require \(2|x+3| < 1\) which is the same as requiring

\[-3 - \frac{1}{2} < x < -3 + \frac{1}{2}\]

We need to check convergence at the endpoints individually by plugging the endpoints into the power series.

- At the left endpoint, \(x = -3 - 1/2\). Plugging this into the power series and simplifying the general term algebraically leads to the numerical series

\[
\sum_n \frac{1}{\sqrt{n}}
\]

which diverges by the p-series test.

- At the right endpoint, \(x = -3 + 1/2\). Plugging this into the power series and simplifying the general term algebraically leads to the numerical series

\[
\sum_n (-1)^n \frac{1}{\sqrt{n}}
\]

which converges by the Alternating Series Test.

Therefore, the interval of convergence is \((-7/2, -5/2]\), the radius of convergence is 1, and the center of convergence is \(-3\).

4. **Method 1: Using the formula for \(c_n\)**

The general formula for \(c_n\) is

\[
c_n = \frac{f^{(n)}(a)}{n!}
\]

Compute the first few derivative of \(\ln(x)\) to see the pattern in the derivatives

\[
\begin{align*}
  f^{(0)}(x) &= \ln x & c_0 &= \ln(2) \\
  f^{(1)}(x) &= \frac{1}{x} & c_1 &= \frac{1}{2} \\
  f^{(2)}(x) &= -\frac{1}{x^2} & c_2 &= -\frac{1}{2^2} \frac{1}{2!} \\
  f^{(3)}(x) &= \frac{1.2}{x^3} & c_3 &= \frac{1.2}{2^3} \frac{1}{3!} \\
  f^{(4)}(x) &= -\frac{1.2.3}{x^4} & c_4 &= -\frac{3!}{2^4} \frac{1}{4!} \\
  f^{(n)}(x) &= (-1)^{n-1}(n-1)! \frac{1}{x^n} & c_n &= (-1)^{n-1}(n-1)! \frac{1}{2^n} \frac{1}{n!}
\end{align*}
\]

Simplifying the expression for \(c_n\) we get

\[
c_n = (-1)^{n-1} \frac{1}{n2^n}
\]

Therefore, the Taylor series is

\[
\ln(2) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n} (x-2)^n
\]
Note the the constant term has to be written separately, since it does not fall under the same pattern as the subsequent terms.

Method 2: Using the derivatives and integrals of power series

\[
\ln(x) = \int \frac{1}{x} \, dx = \int \frac{1}{2 + (x - 2)} \, dx = \int \frac{1}{2} \frac{1}{1 + \left(\frac{x - 2}{2}\right)} \, dx
\]

\[
= \left(\frac{1}{2}\right) \int \sum_{n=0}^{\infty} \left(\frac{x - 2}{2}\right)^n \, dx
\]

\[
= \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \int (x - 2)^n \, dx
\]

\[
= C + \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n (x - 2)^{n+1}}{2^n (n + 1)}
\]

\[
= C + \left(\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x - 2)^{n}}{2^{n-1} n}
\]

\[
= C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x - 2)^n}{2^{n-1} n}
\]

We find that \( C = \ln(2) \) by plugging in \( x = 2 \) above.

5.

\[
\int \frac{x}{1 - x^8} \, dx = \int x \left(\sum_{n=0}^{\infty} x^n\right) \, dx = \int \left(\sum_{n=0}^{\infty} x^{8n+1}\right) \, dx = \sum_{n=0}^{\infty} \int x^{8n+1} \, dx = C + \sum_{n=0}^{\infty} \frac{x^{8n+2}}{8n + 2}
\]

Plugging in \( x = 0 \), we get \( C = 0 \). Therefore,

\[
\int \frac{x}{1 - x^8} \, dx = \sum_{n=0}^{\infty} \frac{x^{8n+2}}{8n + 2}
\]
1. (10 points)
(a) Compute the arclength along the curve \( y = \frac{x^2}{2} - \frac{\ln(x)}{4} \) between \( x = 2 \) and \( x = 4 \).

Then,

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( x - \frac{1}{4x} \right)^2 \\
= 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} \\
= x^2 + \frac{1}{2} + \frac{1}{16x^2} \\
= \left( x + \frac{1}{4x} \right)^2
\]

Therefore, the arclength integral is

\[
L = \int_2^4 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \\
= \int_2^4 x + \frac{1}{4x} \, dx \\
= \frac{x^2}{2} + \frac{\ln(x)}{4} \bigg|_2^4 \\
= \frac{4^2}{2} + \frac{\ln(4)}{4} - \frac{2^2}{2} - \frac{\ln(2)}{4}
\]

2. (a)

\[
x = r \cos(\theta) = -2 \cos(\pi/6) = -\sqrt{3} \\
y = r \sin(\theta) = -2 \sin(\pi/6) = -1
\]

(b) Multiplying both sides of the given equation by \( r \) we get

\[r^2 = 3r \sin(\theta)\]

But \( r^2 = x^2 + y^2 \). So, the equatiopn becomes

\[x^2 + y^2 = 3y\]

Completing the square, we get

\[x^2 + \left( y - \frac{3}{2} \right)^2 = \left( \frac{3}{2} \right)^2\]

which the the equation of a circle centered at \( (0, 3/2) \) with radius \( 3/2 \).

3. Plug in \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \):

\[r \cos(\theta) + r^2 \sin^2(\theta) = 0\]

If \( r \neq 0 \), we can cancel \( r \) on both sides to get

\[\cos(\theta) + r \sin^2(\theta) = 0\]

which implies

\[r = -\cot(\theta) \csc(\theta).\]

4. (10 points) Find the points on the following curve where the tangent line is vertical or horizontal:

\[r = \sin(\theta) + \cos(\theta)\]
\[ x = r \cos(\theta) = (\cos(\theta) + \sin(\theta)) \cos(\theta) \]
\[ y = r \sin(\theta) = (\cos(\theta) + \sin(\theta)) \sin(\theta) \]

Therefore, taking derivatives with respect to \( \theta \) and simplifying, we get
\[ \frac{dx}{d\theta} = \cos^2 \theta - \sin^2 \theta + 2 \cos \theta \sin \theta = \cos(2\theta) + \sin(2\theta) \]

Similarly, we get
\[ \frac{dx}{d\theta} = \cos(2\theta) \]

The tangent is horizontal is \( \frac{dy}{d\theta} = 0 \) and \( \frac{dx}{d\theta} \neq 0 \). Setting \( \frac{dy}{d\theta} = 0 \) and solving for theta, we get
\[ \cos(2\theta) = 0 \]

This is satisfied if \( 2\theta = \pi/2 \) or \( 3\pi/2 \). Therefore, \( \theta = \pi/4 \) or \( 3\pi/4 \). This only gives the solutions for \( \theta \) in the first and second quadrants. Since \( \cos(x) = \cos(-x) \) there are two more symmetry solutions: \( \theta = -\pi/4 \) and \( -3\pi/4 \). It can be checked that \( \frac{dx}{d\theta} \) is nonzero at each of these points.

For vertical tangents, we find the values of \( \theta \) such that \( \frac{dx}{d\theta} = 0 \neq \frac{dy}{d\theta} \). We solve
\[ \cos(2\theta) - \sin(2\theta) = 0 \]

which is equivalent to \( \tan(2\theta) = 1 \). Therefore, \( 2\theta = \pi/4 \) and \( 5\pi/4 \), and hence \( \theta = \pi/8 \) and \( 5\pi/8 \). It is again easy to check that \( \frac{dy}{d\theta} \) is nonzero at these points.