MIDTERM, MATH 471
OCTOBER 21, 2009

(1) (33 Points)
Find
$$\lim_{n\to\infty}\int_0^\infty \frac{\sin\left(\frac{x}{n}\right)}{x+x^2}dx$$
and give a proof.

Solution: Since
$$\begin{aligned}\left|\sin\left(\frac{x}{n}\right)\right| &\le \left(\frac{x}{n}\right)\wedge 1\\ &\le x\wedge 1\end{aligned}$$
for $x > 0$ and $n \ge 1$, it follows that for $x > 0$
$$\left|\frac{\sin\left(\frac{x}{n}\right)}{x+x^2}\right| \le \frac{x\wedge 1}{x+x^2}$$
which is integrable over $(0, \infty)$. Furthermore, since
$$\lim_{n\to\infty}\sin\left(\frac{x}{n}\right) = 0$$
for all $x > 0$, the dominated convergence theorem implies that
$$\lim_{n\to\infty}\int_0^\infty \frac{\sin\left(\frac{x}{n}\right)}{x+x^2}dx = 0$$

(2) (34 Points)
Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable, and consider the set
$$A = \Big\{(x, f(x)) : x \in \mathbf{R}\Big\} \subset \mathbf{R}^2$$
Show that $m(A) = 0$, where m is Lebesgue measure on $\mathbf{R}^2$.

Solution: There are at least two solutions,

Solution 1: Consider $\mathbf{1}_A(x,y)$. For x fixed, $\mathbf{1}_A(x,y)=0$ except at one point, namely $y=f(x)$ and so for each x we have $\int \mathbf{1}_A(x,y)dy=0$ and hence by Tonelli's theorem

$$\begin{aligned} m(A) &= \int_{\mathbf{R}^2} \mathbf{1}_A \\ &= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \mathbf{1}_A(x,y)dy \right) dx \\ &= \int_{\mathbf{R}} 0dx \\ &= 0 \end{aligned}$$

Solution 2: Let $R_k = A \cap ([k,k+1] \times \mathbf{R})$. Since $A = \cup_{k\in\mathbf{Z}} R_k$, we need only show that $m(R_k)=0$ for all k.

Thus we need only consider

$$A = \{(x, f(x)) : x \in [0,1]\}$$

Fix $N > 0$. For $k \in \mathbf{Z}$, let

$$\begin{aligned} B_k &= \left\{ x \in [0,1] : f(x) \in \left(\frac{k}{N}, \frac{k+1}{N} \right] \right\} \\ A_k &= B_k \times \left[\frac{k}{N}, \frac{k+1}{N} \right] \end{aligned}$$

Note that $A \subset \cup_{k\in\mathbf{Z}} A_k$. Thus,

$$\begin{aligned} m(A) &\le \sum_{k\in\mathbf{Z}} m(A_k) \\ &\le \frac{1}{N} \sum_{k\in\mathbf{Z}} m(B_k) \\ &\le \frac{1}{N} m(A) \end{aligned}$$

since A equals the disjoint union of the A_k.

(3) (33 Points)

Show that for all $a \in (0,1)$ we can construct a set $E \subset [0,1]$ with $m([0,1] \setminus E) > a$, such that for every nonempty interval $(x,y) \subset [0,1]$ we have $m((x,y) \cap E) > 0$.

Solution: Given $a > 0$, choose n such that $2\sum_{k=n}^{\infty} 2^{-k} < 1 - a$. Let $(q_1, q_2, \ldots)$ denote the set of rationals in $[0, 1]$, and let

$$E = [0,1] \cap \left[\bigcup_{k=1}^{\infty} \left(q_k - 2^{-(k+n)}, q_k + 2^{-(k+n)} \right) \right]$$

Since the rationals are dense in $[0, 1]$, it follows that for every nonempty interval $(x, y) \subset [0, 1]$ we have $m((x, y) \cap E) > 0$. Also,

$$m([0,1] \setminus E) \geq 1 - 2\sum_{k=n}^{\infty} 2^{-k} \geq 1 - (1 - a) = a$$