MTH 255 : Differential Geometry I
Fall 2009

Instructor: Ibrahim Unal

Midterm Exam

10/30/2009, Friday

Name : ______________________________ **I.D.** ____________

Problems	1	2	3	4	5	6	Bonus	Total
Score								

MTH 255 : Differential Geometry I
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Take-Home Midterm Exam
Due 10/30/2009, Friday 5:00pm

You should do it by yourself and write your answers clearly. Please write your name and ID number and staple your exam papers with the cover page otherwise it will not be accepted. Problem 1 and Problem 2 are 15 points, the others are 20 points each.

Problem 1.a) Prove the following theorem which is the Theorem 7.3 in the book

Theorem : If $A = (a_{ij})$ is the attitude matrix and $\omega = (\omega_{ij})$ the matrix of connection forms of a frame field E_1, E_2, E_3, then $\omega = dA \cdot A^t$ (matrix multiplication)

b) Calculate the $\omega = (\omega_{ij})$ for the spherical frame field on $\mathbb{R}^3$

Problem 2. Given a k-form ω in $\mathbb{R}^n$, we will define an (n-k)-form $*\omega$ by setting

$$*(dx_{i_1} \wedge ... \wedge dx_{i_k}) = (-1)^{\sigma}(dx_{j_1} \wedge ... \wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1 < ... < i_k, j_1 < ... < j_{n-k}, (i_1, ...i_k, j_1, ...j_{n-k})$ is a permutation of $(1, 2, ..., n)$, and σ is 0 or 1 according to the permutation is even or odd, respectively. Show that :

a) If $\omega = a_1 dx_1 + a_2 dx_2$ in $\mathbb{R}^2$, then

$$*\omega = a_1 dx_2 - a_2 dx_1$$

b)If $\omega = a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{23} dx_2 \wedge dx_3$ is a 2-form in $\mathbb{R}^3$, then

$$*\omega = a_{12} dx_3 - a_{13} dx_2 + a_{23} dx_1$$

c) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function. Calculate $d * df$.

Problem 3. Let $\alpha(t) = (4cos(t), 1 + 5sin(t), 3cos(t) - 1)$ for $t \in \mathbb{R}$.
a) Compute the speed $v(t)$ of α.
b) Compute the Frenet apparatus of α.
c) Describe the shape of the image of α. How do you know this?
d) Find a unit-speed reparametrization of α.

Problem 4. The total curvature of a unit-speed curve $\alpha : I \longrightarrow \mathbb{R}^3$ is $\int_I \kappa(s)ds$. If α is merely regular, the formula becomes $\int_I \kappa(t)\nu(t)dt$. Prove that for a closed plane curve whose curvature is positive has total curvature 2π. (*Hint :* Consider its spherical image)

Problem 5. Suppose that γ is a smooth closed curve that lies on the unit sphere in $\mathbb{R}^3$

a) Show that the curvature of γ is always greater or equal to 1.

b) Show that at a point of maximum curvature, the torsion of γ vanishes.

Problem 6. Let $\gamma : (a, b) \longrightarrow \mathbb{R}^3$ be a regular unit-speed curve. If A is a rotation matrix for $\mathbb{R}^3$ (that is, A is an orthogonal matrix with positive determinant), and if q is a vector in $\mathbb{R}$, then the curve $\hat{\gamma} : (a, b) \longrightarrow \mathbb{R}^3$ given by

$$\hat{\gamma}(s) = A\gamma(s) + q$$

(You can think $\gamma(s)$ as a 3×1 matrix)

is the result of rotating and translating the image of γ by A and q respectively. Show that the curvature and torsion functions of the curve $\hat{\gamma}$ are the same as for γ. *Hint :* Rotation matrices preserve the dot product and the cross product.)

Bonus For X and Y vector fields on $\mathbb{R}^3$, the Lie Bracket of X and Y is the operator $[X, Y]$ defined by

$$[X, Y]_p[f] = X_p[Y[f]] - Y_p[X[f]]$$

a)Show that $[X, Y]_p$ is a derivation. (Hence, $[X, Y]$ is a vector field on $\mathbb{R}^3$.)

b)If ϕ is a 1-form on $\mathbb{R}^3$, show that

$$d\phi(X, Y) = Y[\phi(X)] - X[\phi(Y)] - \phi([X, Y])$$

c) Calculate $[U_i, U_j]$ for the natural frame field on $\mathbb{R}^3$