MTH 235: HOMEWORK 6

Scoring. Each problem is worth 3 points, where 3 points are awarded if the problem is entirely correct, including a sufficient amount of justification; 2 points are awarded if the student shows a general understanding of the main ideas of the problem, but their argument is flawed or not sufficiently justified; 1 point otherwise if the problem is at all attempted; 0 points are only given if the student does not attempt the problem. Also, if you award 1 or 2 points for a problem be sure to let the student know exactly what they did wrong and how they can fix their solutions.

Problem (Section 2.5; Problem 7)**.** *In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$ where*

(1) *T is the reflection of $\mathbb{R}^2$ about L.*
(2) *T is the projection on L along the line perpendicular to L.*

Proof. (1) Let L be the line $y = mx$ and let $L^{\perp}$ be the line perpendicular to L, given by $y = -\frac{1}{m}x$. Consider the vectors $(1, m)$, which lies on L, and $(-m, 1)$, which lies on $L^{\perp}$. If T is the reflection of $\mathbb{R}^2$ about L, then we have the following:

$$T(1, m) = (1, m)$$

and

$$T(-m, 1) = (m, -1) = -(-m, 1).$$

Letting

$$\beta' = \{\begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix}\}$$

be a basis for $\mathbb{R}^2$, the matrix representing T in this basis is given by

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now need to calculate the change-of-coordinate matrix $[I]_{\beta'}^{\beta}$, changing β'-coordinates into β-coordinates, as well as its inverse $[I]_{\beta}^{\beta'}$ changing β-coordinates into β'-coordinates, where β is the standard basis for $\mathbb{R}^2$. Immediately we get that

$$Q := [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}.$$

To find $[I]_{\beta}^{\beta'}$, we need to solve the following systems of equations

$$(1, 0) = \alpha_1(1, m) + \alpha_2(-m, 1) = (\alpha_1 - m\alpha_2, m\alpha_1 + \alpha_2)$$

and

$$(0, 1) = \alpha_3(1, m) + \alpha_4(-m, 1) = (\alpha_3 - m\alpha_4, m\alpha_3 + \alpha_4)$$

We find that

$$\alpha_1 = \frac{1}{1 + m^2}; \alpha_2 = \frac{-m}{1 + m^2};$$

and

$$\alpha_3 = \frac{m}{1+m^2}; \alpha_4 = \frac{1}{1+m^2}.$$

Hence we find that

$$Q^{-1} = [I]_\beta^{\beta'} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}.$$

Using Theorem 2.23 we have that

$$\begin{aligned}
[T]_\beta &= Q[T]_{\beta'}Q^{-1} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}.
\end{aligned}$$

Since β is the standard basis for $\mathbb{R}^2$, T is just left multiplication by $[T]_\beta$, so for all $(x,y) \in \mathbb{R}^2$, we have that

$$\begin{aligned}
T(x,y) &= \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} (1-m^2)x + 2my \\ 2mx + (m^2-1)y \end{pmatrix}.
\end{aligned}$$

(2) Let T be the projection on L along $L^\perp$. Since

$$\beta' = \{\begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix}\}$$

is a basis for $\mathbb{R}^2$, every vector v in $\mathbb{R}^2$ can be written as $v = l + l^\perp$ where $l \in L$ and $l^\perp \in L^\perp$. Then $T(v) = l$. Hence

$$T(1,m) = (1,m)$$

and

$$T(-m,1) = (0,0),$$

so that

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the coordinate change matrix Q and its inverse Q^{-1} are as above. We calculate

$$\begin{aligned}
[T]_\beta &= Q[T]_{\beta'}Q^{-1} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}.
\end{aligned}$$

Then for any $(x,y) \in \mathbb{R}^2$ we have

$$\begin{aligned} T(x,y) &= \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{1}{1+m^2}\begin{pmatrix} x+my \\ mx+m^2y \end{pmatrix}. \end{aligned}$$

□

Problem (Section 2.5; Problem 9)**.** *Prove that "is similar to" is an equivalence relation on $M_{n\times n}(F)$.*

Proof. We first show reflexive. Let $A \in M_{n\times n}(F)$. Then the identity matrix I is invertible and $A = I^{-1}AI$; hence A is similar to itself for each A.

Next, let $A, B \in M_{n\times n}(F)$ such that A is similar to B. Then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Since Q is invertible, Q^{-1} is also invertible with inverse $(Q^{-1})^{-1} = Q$. From the above equation, we have that $A = (Q^{-1})^{-1}BQ^{-1}$, so that B is similar to A.

Finally, let $A, B, C \in M_{n\times n}(F)$ such that A is similar to B and B is similar to C. Then there exist invertible matrices Q, Q' such that $B = Q^{-1}AQ$ and $C = (Q')^{-1}BQ'$. Now, since Q, Q' are invertible, QQ' is also invertible with inverse $(QQ')^{-1} = (Q')^{-1}Q^{-1}$. The above equations then imply that $C = (Q')^{-1}BQ' = (Q')^{-1}(Q^{-1}AQ)Q' = (QQ')^{-1}A(QQ')$. Hence A is similar to C, and we conclude that "is similar to" is an equivalence relation. □

Problem (Section 2.6; Problem 3)**.** *For each of the following vector spaces V and bases β, find explicit formulas for vectors of the dual basis β^* for V^*.*

(1) $V = \mathbb{R}^3$; $\beta = \{(1,0,1),(1,2,1),(0,0,1)\}$
(2) $V = P_2(\mathbb{R})$; $\beta = \{1, x, x^2\}$

Proof. (1) Let f_1 be the dual vector to $(1,0,1)$, that is, $f_1 : V \to \mathbb{R}$ such that

$$f_1(1,0,1) = 1; f_1(1,2,1) = 0; f_1(0,0,1) = 0.$$

Using the linearity of f_1 to expand the left sides of these equations, we get the following system:

$$\begin{aligned} 1 &= f_1(1,0,1) = f_1(e_1) + f_1(e_3); \\ 0 &= f_1(1,2,1) = f_1(e_1) + 2f_1(e_2) + f_1(e_3); \\ 0 &= f_1(0,0,1) = f_1(e_3), \end{aligned}$$

where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. The last equation gives that $f_1(e_3) = 0$ which from the first equation, yields that $f_1(e_1) = 1$. Plugging this into the second equation yields that $f_1(e_2) = -\frac{1}{2}$. Thus $f_1(x,y,z) = x - \frac{1}{2}y$.

Similarly we have the following system for f_2, the dual vector to $(1,2,1)$:

$$\begin{aligned} 0 &= f_2(1,0,1) = f_2(e_1) + f_2(e_3); \\ 1 &= f_2(1,2,1) = f_2(e_1) + 2f_2(e_2) + f_2(e_3); \\ 0 &= f_2(0,0,1) = f_2(e_3). \end{aligned}$$

Again, the last equation gives that $f_2(e_3) = 0$, which by the first equation, we have that $f_2(e_1) = 0$ as well. Hence from the second equation we find that $f_2(e_2) = \frac{1}{2}$, and so $f_2(x,y,z) = \frac{1}{2}y$.

Finally, we have the following system for f_3, the dual vector to $(0,0,1)$:

$$0 = f_3(1,0,1) = f_3(e_1) + f_3(e_3);$$

$$0 = f_3(1,2,1) = f_3(e_1) + 2f_3(e_2) + f_3(e_3);$$

$$1 = f_3(0,0,1) = f_3(e_3).$$

The last equation yields that $f_3(e_3) = 1$, so that by the first equation $f_3(e_1) = -1$. Plugging this into the second equation we find that $f_3(e_2) = 0$. Thus, $f_3(x,y,z) = -x + z$.

(2) If $\{g_1, g_x, g_{x^2}\}$ denotes the dual basis to $\{1, x, x^2\}$, then for an arbitrary element $a + bx + cx^2$ in $P_2(\mathbb{R})$ we must have

$$g_1(a + bx + cx^2) = ag_1(1) + bg_1(x) + cg_1(x^2) = a * 1 + b * 0 + c * 0 = a,$$

$$g_2(a + bx + cx^2) = ag_2(1) + bg_2(x) + cg_2(x^2) = a * 0 + b * 1 + c * 0 = b,$$

$$g_3(a + bx + cx^2) = ag_3(1) + bg_3(x) + cg_3(x^2) = a * 0 + b * 0 + c * 1 = c.$$

□

Problem (Section 2.6; Problem 8)**.** *Show that every plane through the origin in $\mathbb{R}^3$ may be identified with the null space of a vector in $(\mathbb{R}^3)^*$.*

Proof. Let P denote a plane through the origin in $\mathbb{R}^3$. Then P is a subspace of $\mathbb{R}^3$, and so has a basis $\beta = \{p_1, p_2\}$, that is $P = \{\alpha p_1 + \gamma p_2 : \alpha, \gamma \in \mathbb{R}\}$. Extend this to a basis $\beta' = \{p_1, p_2, v\}$ for $\mathbb{R}^3$, and consider the dual basis $(\beta')^* = \{p_1^*, p_2^*, v^*\}$. We now show that $P = \{(x,y,z) \in \mathbb{R}^3 : v^*(x,y,z) = 0\}$.

Let $(x,y,z) \in P$. Then $(x,y,z) = \alpha p_1 + \gamma p_2$ for some $\alpha, \gamma \in \mathbb{R}$, and $v^*(x,y,z) = v^*(\alpha p_1 + \gamma p_2) = \alpha v^*(p_1) + \gamma v^*(p_2) = \alpha * 0 + \gamma * 0 = 0$ since v^* is the dual vector to $v \in \beta$. Conversely, let $(x,y,z) \in \mathbb{R}^3$ such that $v^*(x,y,z) = 0$. Now, $(x,y,z) = ap_1 + bp_2 + cv$ since β' is a basis for $\mathbb{R}^3$; however,

$$0 = v^*(x,y,z) = v^*(ap_1 + bp_2 + cv) = av^*(p_1) + bv^*(p_2) + cv^*(v) = a*0 + b*0 + c*1 = c,$$

so that $(x,y,z) = ap_1 + bp_2 \in P$. Thus we may identify P with the null space of v^* as desired. □

Problem (Section 2.6; Problem 13)**.** *Let V be a finite-dimensional vector space over F. For every subset $S \subseteq V$, define the annihilator S^0 of S as*

$$S^0 = \{f \in V^* : f(x) = 0 \textit{ for all } x \in S\}.$$

1. *Prove that S^0 is a subspace of V^*.*
2. *If W is a subspace of V and $x \notin W$, prove that there exists $f \in W^0$ such that $f(x) \neq 0$.*
3. *Prove that $(S^0)^0 = span(\psi(S))$, where ψ is defined as in Theorem 2.26.*
4. *For subspaces W_1 and W_2, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.*

Proof. (1) First, the zero map is clearly in S^0. Let $f, g \in S^0$, $c \in F$. Then $(f+g)(x) = f(x) + g(x) = 0 + 0 = 0$ for all $x \in S$; $(cf)(x) = c * f(x) = c * 0 = 0$ for all $x \in S$. Hence $f + g \in S^0$ and $cf \in S^0$. Thus S^0 is a subspace of V^*.

(2) Let $\beta_W = \{w_1, \ldots, w_k\}$ be a basis for W. Since $x \notin W$, $\beta_W \cup \{x\}$ is a linearly independent set in V, and hence can be extended to a basis β_V for V. Let β_V^* denote the dual basis to β_V, and let x^* denote the dual vector to x, i. e., $x^*(v) = 1$ if $v = x$ and $x^*(v) = 0$ for all $v \in \beta_V \setminus \{x\}$. In particular, $x^*(x) \neq 0$; it is left to show that $x^* \in W^0$. Let $w \in W$. Then $w = \sum_{i=1}^{k} \alpha_i w_i$ for some $\alpha_i \in F$. Since $w_i \in \beta_V \setminus \{x\}$ for all $i = 1, \ldots, k$, we have the following:

$$x^*(w) = x^*(\sum_{i=1}^{k} \alpha_i w_i) = \sum_{i=1}^{k} \alpha_i x^*(w_i) = \sum_{i=1}^{k} \alpha_i * 0 = 0.$$

Hence $x^* \in W^0$ and $x^*(x) \neq 0$.

(3) Let $x \in span(\psi(S))$. Then $x = \sum_{i=1}^{n} \alpha_i \psi(s_i)$ for some $s_i \in S$ and $\alpha_i \in F$. Let $f \in S^0$. Then

$$\begin{aligned} x(f) &= \sum_{i=1}^{n} \alpha_i \psi(s_i)(f) = \sum_{i=1}^{n} \alpha_i \widehat{s_i}(f) \\ &= \sum_{i=1}^{n} \alpha_i f(s_i) = \sum_{i=1}^{n} \alpha_i * 0 = 0. \end{aligned}$$

Hence $x \in (S^0)^0$.

Now, since $span(\psi(S))$ is a subspace of V^{**}, a finite-dimensional vector space, there exists a basis $\{\widehat{s_1}, \ldots, \widehat{s_m}\}$ for $span(\psi(S))$. Extend this to a basis $\beta = \{\widehat{s_1}, \ldots, \widehat{s_m}, \widehat{x_{m+1}}, \ldots, \widehat{x_n}\}$ for V^{**}. Using ψ, this gives a basis $\beta_V = \{s_1, \ldots, s_m, x_{m+1}, \ldots, x_n\}$ for V. Finally, from this we can get the dual basis $\beta_{V^*} = \{s_1^*, \ldots, s_m^*, x_{m+1}^*, \ldots, x_n^*\}$ for V^*. Note that by definition of the dual basis, $x_i^* \in S^0$ for each $i = m+1, \ldots, n$.

Let $x \in (S^0)^0$. We will show that $x \in span(\psi(S))$. Since $(S^0)^0 \subseteq V^{**}$, we have that $x = \sum_{i=1}^{m} \alpha_i \widehat{s_i} + \sum_{i=m+1}^{n} \alpha_i \widehat{x_i}$. Then $0 = x(x_i^*) = \alpha_i$ for each $i = m+1, \ldots, n$. Hence $x = \sum_{i=1}^{m} \alpha_i \widehat{s_i} \in span(\psi(S))$.

(4) First note that $f \in W_1^0$ if and only if $f(w) = 0$ for all $w \in W_1 = W_2$ if and only if $f \in W_2^0$. Thus $W_1^0 = W_2^0$. Now assume $W_1 \neq W_2$. We show that $W_1^0 \neq W_2^0$. Without loss of generality, assume $w \in W_1$ and $w \notin W_2$. By part (2) there exists $f \in W_2^0$ such that $f(w) \neq 0$. Hence, $f \notin W_1^0$, and so $W_1^0 \neq W_2^0$.

□