MTH 235: HOMEWORK 5

Scoring. Each problem is worth 3 points, where 3 points are awarded if the problem is entirely correct, including a sufficient amount of justification; 2 points are awarded if the student shows a general understanding of the main ideas of the problem, but their argument is flawed or not sufficiently justified; 1 point otherwise if the problem is at all attempted; 0 points are only given if the student does not attempt the problem. Also, if you award 1 or 2 points for a problem be sure to let the student know exactly what they did wrong and how they can fix their solutions.

Problem (Section 2.3; Problem 11)**.** *Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$, where T_0 is the zero map, if and only if $R(T) \subseteq N(T)$.*

Proof. Assume $T^2 = T_0$, and let $w \in R(T)$. Then by definition of $R(T)$, there exists $v \in V$ such that $w = T(v)$. Letting T act on both sides of this equation yields $T(w) = T(T(v))$, but since $T^2 = T_0$ we find that $T(w) = T_0(v) = 0$ implying that $w \in N(t)$. Hence $R(T) \subseteq N(T)$.

Conversely, assume $R(T) \subseteq N(T)$. Then for all $v \in V$, $T(v) \in N(T)$. Hence, for each $v \in V$, $T(T(v)) = 0$ by definition of $N(T)$. Thus $T^2(v) = T_0(v)$ for each $v \in V$, so indeed $T^2 = T_0$. □

Problem (Section 2.3; Problem 12)**.** *Let V, W and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.*

1. *Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?*
2. *Prove that if UT is onto, then U is onto. Must T also be onto?*
3. *Prove that if U and T are one-to-one and onto, then UT is also.*

Proof.

1. Let $v \in N(T)$, so that $T(v) = 0$. Then $U(T(v)) = U(0) = 0$ yielding that $v \in N(UT)$; however, UT is one-to-one, so $N(UT) = \{0\}$. This proves that $v = 0$, so that in fact $N(T) = \{0\}$, i. e. T is one-to-one.

   U does not need to be one-to-one. For this problem, the student needs to provide a counterexample. This involves finding explicit vector spaces V, W and Z and maps $T : V \to W$ and $U : W \to Z$ such that UT is one-to-one but U is not one-to-one. One possible counterexample is the following:

   Let $V = \mathbb{R}$ and $W = Z = \mathbb{R}^2$. Let $T : \mathbb{R} \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x) = (x, 0)$ and $U(x, y) = (x, 0)$. T is linear since $T(cx+y) = (cx + y, 0) = c(x, 0) + (y, 0) = cT(x) + T(y)$; U is linear since $U(cx_1 + x_2, cy_1 + y_2) = (cx_1 + x_2, 0) = c(x_1, 0) + (x_2, 0) = cU(x_1, y_1) + U(x_2, y_2)$. Further, UT is one-to-one because of the following: Let $x, y \in \mathbb{R}$ such that $UT(x) = UT(y)$; then $UT(x) = UT(y)$ if and only if $(x, 0) = (y, 0)$ if and only if $x = y$. U however is not one-to-one since for any $x \in \mathbb{R}$, $U(x, 0) = U(x, 1)$.
2. Assume UT is onto. Let $z \in Z$. Then there exists $v \in V$ such that $UT(v) = z$. Since $T(V) \in W$, this shows that U is onto.

T does not need to be onto. For this problem, the student needs to provide a counterexample. This involves finding explicit vector spaces V, W and Z and maps $T : V \to W$ and $U : W \to Z$ such that UT is onto but T is not onto. One possible counterexample is the following:

Let $V = Z = \mathbb{R}$ and let $W = \mathbb{R}^2$. Define $T : \mathbb{R} \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}$ by $T(x) = (x, 0)$ and $U(x, y) = x$. T is the same map as above, so it is linear by that discussion; U is linear because $U(cx_1 + x_2, cy_1 + y_2) = cx_1 + x_2 = cU(x_1, y_1)+U(x_2, y_2)$. UT is onto since for any $x \in \mathbb{R}$, $UT(x) = x$; however, T is not onto since there does not exist $x \in \mathbb{R}$ such that $T(x) = (1, 1)$.

(3) Let $v \in V$ such that $UT(v) = 0$. Since U is one-to-one, $T(v) = 0$, and since T is one-to-one, $v = 0$. Hence UT is one-to-one. Finally, let $z \in Z$. Since U is onto, there exists $w \in W$ such that $U(w) = z$. Further, since T is onto, there exists $v \in V$ such that $T(v) = w$. Hence, $UT(v) = U(T(v)) = U(w) = z$. Thus, UT is onto.

$\square$

Problem (Section 2.4; Problem 7). *Let A be an $n \times n$ matrix.*

(1) *Suppose that $A^2 = 0$. Prove that A is not invertible.*
(2) *Suppose that $AB = 0$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.*

Proof. (1) Assume to the contrary that A is indeed invertible. Then A^{-1} exists with $A^{-1}A = AA^{-1} = I$. Then $A^2 = 0$ implies $A = IA = (A^{-1}A)A = A^{-1}A^2 = A^{-1}0 = 0$. This is a contradiction since the zero matrix is not invertible.

(2) A cannot be invertible since if it were then $AB = 0$ would imply that $B = IB = (A^{-1}A)B = A^{-1}AB = A^{-1}0 = 0$ which is a contradiction to our hypothesis that B is nonzero.

$\square$

Problem (Section 2.4; Problem 16). *Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.*

Proof. We first show that Φ is linear.

$$\Phi(cA + D) = B^{-1}(cA + D)B = cB^{-1}AB + B^{-1}DB = c\Phi(A) + \Phi(D).$$

Now, define $\Phi^{-1}(A) = BAB^{-1}$. Then

$$\Phi(\Phi^{-1}(A)) = \Phi(BAB^{-1}) = B^{-1}(BAB^{-1})B = A$$

and

$$\Phi^{-1}(\Phi(A)) = \Phi^{-1}(B^{-1}AB) = B(B^{-1}AB)B^{-1} = A,$$

proving that Φ is one-to-one and onto. Hence, Φ is an isomorphism. $\square$

Problem (Section 2.4; Problem 25). *Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. Let $C(S, F)$ denote the vector space of all functions $f \in F(S, F)$ such that $f(s) = 0$ for all but a finite number of vectors in S. Let $\Psi : C(S, F) \to V$ be defined by $\Psi(f) = 0$ if f is the zero function, and*

$$\Psi(f) = \sum_{s\in S, f(s)\neq 0} f(s)s,$$

otherwise. Prove that Ψ is an isomorphism.

Proof. First,

$$\Psi(cf+g)=\sum(cf+g)(s)s=\sum(cf(s)s+g(s)s)=c\sum f(s)s+\sum g(s)s=c\Psi(f)+\Psi(g).$$

Now, let $f\in C(S,F)$ such that $\Psi(f)=0$. Then $\sum f(s)s=0$, but S is linearly independent. Thus, $f(s)=0$ for each $s\in S$, that is, f is the zero function. Hence Ψ is one-to-one. Finally, let $v\in V$. Then $v=\sum\alpha_s s$. Define $f\in C(S,F)$ by $f(s)=\alpha_s$. Then $\Psi(f)=\sum f(s)s=\sum\alpha_s s=v$, and so Ψ is an isomorphism. □