MTH 235: HOMEWORK 4

Scoring. Each problem is worth 3 points, where 3 points are awarded if the problem is entirely correct, including a sufficient amount of justification; 2 points are awarded if the student shows a general understanding of the main ideas of the problem, but their argument is flawed or not sufficiently justified; 1 point otherwise if the problem is at all attempted; 0 points are only given if the student does not attempt the problem. Also, if you award 1 or 2 points for a problem be sure to let the student know exactly what they did wrong and how they can fix their solutions.

Problem (Section 2.1; Problem 4)**.** *Let* $T : M_{2\times 3}(F) \to M_{2\times 2}(F)$ *be the map defined by*

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

Prove that T *is a linear transformation, and find bases for both* $N(T)$ *and* $R(T)$*. Then compute the nullity and rank of* T*, and verify the Dimension Theorem. Finally, use the appropriate Theorems in this section to determine whether* T *is one-to-one or onto.*

Proof. Let $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \in M_{2\times 3}(F)$ and $c \in F$. Then

$$\begin{aligned} T\begin{pmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{pmatrix} &= \begin{pmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12} & (ca_{13} + b_{13} + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c(2a_{11} - a_{12}) + (2b_{11} - b_{12}) & c(a_{13} + 2a_{12}) + (b_{13} + 2b_{12}) \\ 0 & 0 \end{pmatrix} \\ &= c\begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{pmatrix} \\ &= cT\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + T\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}, \end{aligned}$$

so T is linear.

We now find bases for $N(T)$ and $R(T)$, the null space and range of T respectively. First,

$$0 = T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

implies that

$$2a_{11} - a_{12} = 0$$

and

$$a_{13} + 2a_{12} = 0.$$

The first equation implies that $a_{12} = 2a_{11}$. Substituting this into the second equation and solving for a_{13} yields that $a_{13} = -4a_{11}$. Then

$$N(T) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in M_{2\times 3}(F) : a_{12} = 2a_{11}, a_{13} = -4a_{11} \right\}$$

$$= \left\{ \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right\}.$$

Then we see that a basis for $N(T)$ is given by

$$\left\{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

For grading purposes, the other two natural choices for bases are:

$$\left\{ \begin{pmatrix} \frac{1}{2} & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

and

$$\left\{ \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

From this, we see that the dimension of the null space of T is 4. We now use Theorem 2.2 to compute a basis for $R(T)$. First, we let T act on the standard basis $\{e_{ij}\}$ for $M_{2\times 3}(F)$ where e_{ij} is the matrix with 1 in the ij^{th} entry and 0 elsewhere.

$$T(e_{11}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}; T(e_{12}) = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}; T(e_{13}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$T(e_{21}) = T(e_{22}) = T(e_{23}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 2.2 then tells us that

$$R(T) = span \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

from which we may extract a linearly independent set. Clearly we must remove the zero matrix. Further, it is clear that the third matrix is a linear combination of the first two, so that in fact, the span of the above set is equal to the span of $\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, which is clearly a linearly independent set. Thus the dimension of the range of T is 2. Then $\dim(M_{2\times 3}(F)) = 6 = \dim(N(T)) + \dim(R(T))$. Finally, T is not one-to-one since the null space is not trivial nor is T onto since $\dim(R(T)) = 2 \neq 4 = \dim(M_{2\times 2}(F))$. ☐

Problem (Section 2.1; Problem 14). *Let V and W be vector spaces and $T : V \to W$ be linear.*

1. *Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.*
2. *Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.*
3. *Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis for W.*

Proof. (1) Assume that T is one-to-one, and let $S = \{v_1, \ldots, v_k\}$ be a linearly independent subset of V. We show that $\{T(v_1), \ldots, T(v_k)\}$ is a linearly independent subset of W. To that end, assume

$$0 = \sum_{i=1}^{k} a_i T(v_i) = T\left(\sum_{i=1}^{k} a_i v_i \right),$$

where the second equality follows from the linearity of T. That T is one-to-one implies

$$\sum_{i=1}^{k} a_i v_i = 0,$$

but S is linearly independent implies $a_i = 0$ for $i = 1, \ldots, k$ proving that $T(S)$ is linearly independent.

Conversely, assume that T carries linearly independent subsets of V to linearly independent subsets of W; assume that T is not one-to-one, and we will derive a contradiction. Since T is not one-to-one there exists $x \in V$ with $x \neq 0$ such that $T(x) = 0$. Note that any singleton set consisting of a nonzero vector is trivially linearly independent, so by assumption $\{T(x)\}$ must be linearly independent; however, $\{0\}$ is linearly dependent which gives us our contradiction. Thus, T must be one-to-one.

(2) Assume first that S is linearly independent. Since T is one-to-one, that $T(S)$ is linearly independent follows from the first part. Now assume that $T(S)$ is linearly independent, $S = \{v_1, \ldots, v_k\}$ and $0 = \sum_{i=1}^{k} a_i v_i$. Applying T to both sides of this equation and using the linearity of T, we find that

$$0 = T\left(\sum_{i=1}^{k} a_i v_i\right) = \sum_{i=1}^{k} a_i T(v_i).$$

Since $T(S)$ is linearly independent, we conclude that $a_i = 0$ for $i = 1, \ldots, k$ and hence, S is linearly independent.

(3) Since T is one-to-one, we immediately get that $T(\beta)$ is linearly independent by the previous part. Using Theorem 2.2 and the definition of "onto" we find that $W = R(T) = \mathit{span}(T(\beta))$. Then $T(\beta)$ is a basis for W by definition of a basis.

□

Problem (Section 2.1; Problem 20). *Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $S = \{x \in V : T(x) \in W_1\}$ is a subspace of V.*

Proof. We first show that $T(V_1)$ is a subspace of W. Since V_1 is a subspace of V, $0_V \in V_1$. Then $0_W = T(0_V) \in T(V_1)$. Let $w, \tilde{w} \in T(V_1)$ and $c \in F$, so there exists $v, \tilde{v} \in V_1$ such that $w = T(v)$ and $\tilde{w} = T(\tilde{v})$. Note that since V_1 is a subspace, $v + \tilde{v} \in V_1$ and $cv \in V_1$. Then $w + \tilde{w} = T(v) + T(\tilde{v}) = T(v + \tilde{v}) \in T(V_1)$ and $cw = cT(v) = T(cv) \in T(V_1)$. Thus, $T(V_1)$ is a subspace of W.

We now prove that S is a subspace of V. Since W_1 is a subspace of W, $0_W \in W_1$; moreover, since $T(0_V) = 0_W$, $0_V \in S$. Let $x, y \in S$ and $c \in F$. Because W_1 is a subspace and $T(x + y) = T(x) + T(y) \in W_1$, $x + y \in S$; further, again because W_1 is a subspace and $T(cx) = cT(x) \in W_1$, $cx \in S$. Hence S is a subspace of V. □

Problem (Section 2.2; Problem 3). *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[T]_\alpha^\gamma$.*

Proof. First, we calculate

$$T(1,0) = (1,1,2)$$

and

$$T(0,1) = (-1,0,1).$$

In order to find $[T]_\beta^\gamma$, we need to write these two vectors as linear combinations of the vectors of γ. Setting up the dependence relations, we find the following:

$$\begin{aligned}(1,1,2) &= b_1(1,1,0) + b_2(0,1,1) + b_3(2,2,3)\\ &= (b_1 + 2b_3, b_1 + b_2 + 2b_3, b_2 + 3b_3)\end{aligned}$$

and

$$(-1,0,1) = (c_1 + 2c_3, c_1 + c_2 + 2c_3, c_2 + 3c_3).$$

Solving the two associated systems of equations yields the following:

$$b_1 = -\frac{1}{3}; b_2 = 0; b_3 = \frac{2}{3}$$

and

$$c_1 = -1; c_2 = 1; c_3 = 0.$$

These then become the entries in the 3×2 matrix:

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

We now repeat this process to find $[T]_\alpha^\gamma$.

$$T(1,2) = (-1,1,4)$$

and

$$T(2,3) = (-1,2,7).$$

Then

$$\begin{aligned}(-1,1,4) &= d_1(1,1,0) + d_2(0,1,1) + d_3(2,2,3)\\ &= (d_1 + 2d_3, d_1 + d_2 + 2d_3, d_2 + 3d_3)\end{aligned}$$

and

$$(-1,2,7) = (f_1 + 2f_3, f_1 + f_2 + 2f_3, f_2 + 3f_3).$$

Solving the associated systems of equations yields

$$d_1 = -\frac{7}{3}; d_2 = 2; d_3 = \frac{2}{3}$$

and

$$f_1 = -\frac{11}{3}; f_2 = 3; f_3 = \frac{4}{3}.$$

This then yields the 3×2 matrix:

$$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

Note: In order to receive full credit, the student must show their calculations; just writing the two matrices is not enough, and should only be awarded 2 *points.* ☐

Problem (Section 2.2; Problem 15)**.** *Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V,W) : T(x) = 0 \forall x \in S\}$. Prove the following statements:*

1. *S^0 is a subspace of $\mathcal{L}(V,W)$.*
2. *If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.*

(3) *If* V_1 *and* V_2 *are subspaces of* V*, then* $(V_1+V_2)^0 = V_1^0 \cap V_2^0$.

Proof. (1) Let $\mathbf{0}$ denote the 0 map, i. e. $\mathbf{0}(x) = 0$ for all $x \in V$. Then by definition, $\mathbf{0} \in S^0$. Now, let $T, U \in S^0$ and $c \in F$. Then for all $x \in S$ we have the following:

$$(T+U)(x) = T(x) + U(x) = 0 + 0 = 0$$

and

$$(cT)(x) = c * T(x) = c * 0 = 0.$$

Thus, $T+U$ and cT are in S^0, so that S^0 is a subspace of $\mathcal{L}(V, W)$.

(2) Let $T \in S_2^0$, so that $T(x) = 0$ for all $x \in S_2$. Since $S_1 \subseteq S_2$, $x \in S_1$ implies that $x \in S_2$, so that $T(x) = 0$ for each $x \in S_1$. Hence $T \in S_1^0$, so $S_2^0 \subseteq S_1^0$.

(3) Let $T \in (V_1+V_2)^0$, so that $T(v_1+v_2) = 0$ for each $v_1+v_2 \in V_1+V_2$. Since V_i is a subspace of V for $i = 1, 2$, $0 \in V_i$ for $i = 1, 2$, so $v_1+0, 0+v_2 \in V_1+V_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Then for each $v_1 \in V_1$ and each $v_2 \in V_2$

$$T(v_1) = T(v_1+0) = 0$$

and

$$T(v_2) = T(0+v_2) = 0$$

giving that $T \in V_1^0$ and $T \in V_2^0$, i. e., $T \in V_1^0 \cap V_2^0$, so that $(V_1+V_2)^0 \subseteq V_1^0 \cap V_2^0$. Conversely, if $T \in V_1^0 \cap V_2^0$, then $T(v_1) = 0$ and $T(v_2) = 0$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Thus, for any $v_1+v_2 \in V_1+V_2$, we have that

$$T(v_1+v_2) = T(v_1) + T(v_2) = 0 + 0 = 0.$$

Hence, $T \in (V_1+V_2)^0$, so that $V_1^0 \cap V_2^0 \subseteq (V_1+V_2)^0$ and consequently, $(V_1+V_2)^0 = V_1^0 \cap V_2^0$ as desired.

Alternatively, to show that $(V_1+V_2)^0 \subseteq V_1^0 \cap V_2^0$ we can reason as follows: Let $T \in (V_1+V_2)^0$, so that $T(v_1+v_2) = 0$ for each $v_1+v_2 \in V_1+V_2$. Since V_i is a subspace of V for $i = 1, 2$, $0 \in V_i$ for $i = 1, 2$, so $v_1+0, 0+v_2 \in V_1+V_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. This shows that $V_1 \subseteq V_1+V_2$ and that $V_2 \subseteq V_1+V_2$. From part (2), we then have that $(V_1+V_2)^0 \subseteq V_1^0$ and that $(V_1+V_2)^0 \subseteq V_2^0$ from which we conclude $(V_1+V_2)^0 \subseteq V_1^0 \cap V_2^0$.

□