MTH 235: HOMEWORK 3

Scoring. Each problem is worth 3 points, where 3 points are awarded if the problem is entirely correct, including a sufficient amount of justification; 2 points are awarded if the student shows a general understanding of the main ideas of the problem, but their argument is flawed or not sufficiently justified; 1 point otherwise if the problem is at all attempted; 0 points are only given if the student does not attempt the problem. Also, if you award 1 or 2 points for a problem be sure to let the student know exactly what they did wrong and how they can fix their solutions.

Problem (Section 1.6; Problem 13)**.** *The set of solutions to the system of linear equations*

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$*. Find a basis for this subspace.*

Proof. Note that

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

yields the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

by multiplying the first equation by -2 and adding it to the second. The second equation then gives that $x_2 = x_3$, so that the first equation then reduces to $x_1 = 2x_2 - x_3 = 2x_3 - x_3 = x_3$. That is, the set of solutions to the above system is given by

$$\begin{aligned} &\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_3, x_2 = x_3\} \\ =&\{(t, t, t) \in \mathbb{R}^3 : t \in \mathbb{R}\} \\ =&\{t(1, 1, 1) : t \in \mathbb{R}\} \end{aligned}$$

Thus, a basis for the subspace is given by $\{(1, 1, 1)\}$. *Note:* On this problem, the calculations shown above are the necessary justifications. Just writing $\{(1, 1, 1)\}$ is *not* enough, and should only be awarded 2 points. □

Problem (Section 1.6; Problem 25)**.** *Let* V*,* W *and* Z *be as in Exercise* 21 *of Section* 1.2*: that is,* V *and* W *are vector spaces over* F *and* $Z = \{(v, w) : v \in V \text{ and } w \in W\}$ *where addition and scalar multiplication are defined componentwise. If* V *and* W *are vector spaces over* F *of dimensions* m *and* n *respectively, determine the dimension of* Z*.*

Proof. Let $\beta_V = \{v_1, \ldots, v_m\}$ and $\beta_W = \{w_1, \ldots, w_n\}$ be bases for V and W respectively. We claim that

$$\beta_Z = \{(v_i, 0_W) : i = 1, \ldots, m\} \cup \{(0_V, w_j) : j = 1, \ldots, n\},$$

where 0_V and 0_W are the zero vectors in V, W respectively, is a basis for Z.

First we show that β_Z is a linearly independent set. Assume

$$\begin{aligned}(0,0) &= \alpha_1(v_1,0)+\cdots+\alpha_m(v_m,0)+\gamma_1(0,w_1)+\cdots+\gamma_n(0,w_n)\\ &= (\alpha_1 v_1+\cdots+\alpha_m v_m, \gamma_1 w_1+\cdots+\gamma_n w_n).\end{aligned}$$

for some scalars $\alpha_1,\ldots,\alpha_m,\gamma_1,\ldots,\gamma_n$ in F. This implies that

$$\alpha_1 v_1+\cdots+\alpha_m v_m = 0$$

and that

$$\gamma_1 w_1+\cdots+\gamma_n w_n = 0.$$

Since β_V and β_W are bases, we conclude that $\alpha_i = 0$ for each $i = 1,\ldots,m$ and $\gamma_j = 0$ for $j = 1,\ldots,n$. Hence β_Z is a linearly independent set.

We now show that β_Z is a spanning set. Let $(v,w)\in Z$. Then, since $v\in V$ and $w\in W$, we have that

$$v = a_1 v_1+\cdots+a_m v_m$$

and

$$w = b_1 w_1+\cdots+b_n w_n$$

for scalars $a_1,\ldots,a_m,b_1,\ldots,b_n$. Using the definitions and properties of addition and scalar multiplication on Z, we find that

$$\begin{aligned}(v,w) &= (a_1 v_1+\cdots+a_m v_m, b_1 w_1+\cdots+b_n w_n)\\ &= a_1(v_1,0)+\cdots+a_m(v_m,0)+b_1(0,w_1)+\cdots+b_n(0,w_n).\end{aligned}$$

Hence β_Z is a spanning set, so that β_Z is a basis by definition. Counting the elements of β_Z shows that $\dim Z = m+n$. □

Problem (Section 1.6; Problem 28)**.** *Let V be a finite-dimensional vector space over $\mathbb{C}$ with dimension n. Prove that if V is now regarded as a vector space over $\mathbb{R}$, then* $\dim V = 2n$.

Proof. Let $\beta_{\mathbb{C}} = \{v_1,\ldots,v_n\}$ be a basis for V over $\mathbb{C}$. Then for any $v\in V$, there exist $\alpha_1,\ldots,\alpha_n\in\mathbb{C}$ such that

$$v = \alpha_1 v_1+\cdots+\alpha_n v_n.$$

Since each of the α_j are complex numbers, we can write them as $\alpha_j = \gamma_j + i\delta_j$ where γ_j,δ_j are real numbers. Thus we have

$$v = (\gamma_1+i\delta_1)v_1+\cdots+(\gamma_n+i\delta_n)v_n$$

$$= \gamma_1 v_1+\cdots+\gamma_n v_n+\delta_1(iv_1)+\cdots+\delta_n(iv_n).$$

This shows that the set $\beta_{\mathbb{R}} = \{v_1,\ldots,v_n,iv_1,\ldots,iv_n\}$ is a spanning set for V over $\mathbb{R}$. It is left to show that $\beta_{\mathbb{R}}$ is a linearly independent set, so assume

$$0 = a_1 v_1+\cdots+a_n v_n+b_1(iv_1)+\cdots+b_n(iv_n)$$

for real numbers $a_1,\ldots,a_n,b_1,\ldots,b_n$. Then we have

$$0 = (a_1+ib_1)v_1+\cdots+(a_n+ib_n)v_n,$$

but since $\beta_{\mathbb{C}}$ is a basis, we have that $a_j+ib_j = 0$ for each $j=1,\ldots,n$ which implies that $a_j = 0$ and $b_j = 0$ for each $j = 1,\ldots,n$. Thus, $\beta_{\mathbb{R}}$ is linearly independent, and hence, a basis for V over $\mathbb{R}$. Counting the elements in this set yields that $\dim V = 2n$ over $\mathbb{R}$. □

Problem (Section 1.6; Problem 29). (1) *Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and* $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$*.* Hint: *Start with a basis $\{u_1, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_m\}$ for W_2.*

(2) *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if* $\dim(V) = \dim(W_1) + \dim(W_2)$*.*

Proof. (1) Using the notation of the hint from above, let $\beta_{W_1 \cap W_2} = \{u_1, \ldots, u_k\}$ be a basis for $W_1 \cap W_2$ and extend this set to a basis $\beta_{W_1} = \{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and to a basis $\beta_{W_2} = \{u_1, \ldots, u_k, w_1, \ldots, w_m\}$ for W_2. We can do this by part c) of Corollary 2 to Theorem 1.10, the Replacement Theorem. Then for $w_1 \in W_1$ and $w_2 \in W_2$, we can write $w_1 = a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m$ and $w_2 = c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_nw_n$. Hence, using the properties of a vector space, we can write the following:

$$\begin{aligned} w_1 + w_2 &= (a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m) \\ &\quad + (c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_nw_n) \\ &= (a_1 + c_1)u_1 + \cdots + (a_k + c_k)u_k + b_1v_1 + \cdots + b_mv_m \\ &\quad + d_1w_1 + \cdots + d_nw_n. \end{aligned}$$

This shows that the set $\beta_{W_1+W_2} = \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a spanning set for $W_1 + W_2$.

It is left to show that this set is linearly independent. Thus assume that

$$0 = \alpha_1u_1 + \cdots + \alpha_ku_k + \delta_1v_1 + \cdots + \delta_mv_m + \gamma_1w_1 + \cdots + \gamma_nw_n,$$

for scalars $\alpha_1, \ldots, \alpha_k, \delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_n$. Now, if δ_j is nonzero for some $1 \le j \le m$ then v_j can be written as a nontrivial linear combination of elements of W_1 and W_2 so that $v_j \in W_1 \cap W_2$; since $\beta_{W_1 \cap W_2}$ is a basis for the intersection, we can write v_j as a nontrivial linear combination of elements of $\beta_{W_1 \cap W_2}$ contradicting the fact that β_{W_1} is a basis. We arrive at a similar contradiction if γ_j is nonzero for some $1 \le j \le n$; hence, so far, we must have $\delta_1 = \cdots = \delta_m = \gamma_1 = \cdots = \gamma_n = 0$. Substituting this into the above equality, we get that

$$0 = \alpha_1u_1 + \cdots + \alpha_ku_k,$$

but $\beta_{W_1 \cap W_2}$ is linearly independent, so in fact, we must also have $\alpha_1 = \cdots = \alpha_k = 0$. Thus, $\beta_{W_1+W_2}$ is a linearly independent set, and hence, a basis for $W_1 + W_2$. Finally, counting the elements in $\beta_{W_1+W_2}$ we see that in fact $W_1 + W_2$ is finite dimensional, and

$$\begin{aligned} \dim(W_1 + W_2) &= k + m + n = (k + m) + (k + n) - k \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2). \end{aligned}$$

(2) Now, assume that V is the direct sum of W_1 and W_2. Then, by definition of the direct sum, $W_1 \cap W_2 = \{0\}$, i. e., $\dim(W_1 \cap W_2) = 0$. By the formula from above, we immediately find that $\dim(V) = \dim(W_1) + \dim(W_2)$. Conversely, if $\dim(V) = \dim(W_1) + \dim(W_2)$, we find that, by the above formula, $\dim(W_1 \cap W_2) = 0$, which is only possible if $W_1 \cap W_2 = \{0\}$. Since,

by assumption $V = W_1 + W_2$, we conclude that V is the direct sum of W_1 and W_2 by definition.

□

Problem (Section 1.7; Problem 6)**.** *Prove the following generalization of Theorem* 1.9 *(p.* 44*): Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$. If S_1 is linearly independent and S_2 generates V, then there exists a basis β for V such that $S_1 \subseteq \beta \subseteq S_2$.* Hint: *Apply the maximal principle to the family of all linearly independent subsets of S_2 that contain S_1, and proceed as in the proof of Theorem* 1.13*.*

Proof. Define

$$\mathcal{F} = \{U : S_1 \subseteq U \subseteq S_2, U \text{ linearly independent}\}.$$

Note that $S_1 \in \mathcal{F}$, so that $\mathcal{F} \neq \emptyset$. Now, let C be a chain in $\mathcal{F}$, and define

$$S = \bigcup_{\tilde{C} \in C} \tilde{C}.$$

Clearly, S contains each element of C, so it is left to show that $S \in \mathcal{F}$. Note that for each $\tilde{C} \in C$, $S_1 \subseteq \tilde{C} \subseteq S_2$, so that in fact $S_1 \subseteq S \subseteq S_2$. We now show that S is linearly independent. Let $v_1, \ldots, v_n \in S$ such that

$$0 = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

for scalars $\alpha_1, \ldots, \alpha_n$. Then there exist elements C_j in the chain C such that $v_j \in C_j$. Assume that the indexing is such that

$$C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n.$$

Then we have that $v_j \in C_n$ for $j = 1, \ldots, n$, but C_n is itself an element of $\mathcal{F}$ and hence is linearly independent. Thus, $\alpha_1 = \cdots = \alpha_n = 0$, and we find that S is linearly independent, i. e., $S \in \mathcal{F}$. Finally, we can now apply the Maximal Principle to conclude that there exists a maximal linearly independent set β, i. e., a basis, for V such that $S_1 \subseteq \beta \subseteq S_2$. □